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POSITIVE HOLOMORPHIC DIFFERENTIALS ON KLEIN SURFACES

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Let $\mathfrak{X}$ be a compact Klein surface with boundary ∂X, and let $\mathcal{O}$ be an orientation of ∂X. We conjecture that there is a holomorphic differential which is positive on $\mathcal{O}$ if and only if $\mathcal{O}$ is not induced by an orientation of X, and we prove this when $\mathfrak{X}$ is elliptic or hyperelliptic.

Let $\mathfrak{X}$ be a Klein surface, with underlying topological space X, and let η be a meromorphic differential on $\mathfrak{X}$ (for basic definitions and results see [1], [2]). If $\mathfrak{g} \in E(\mathfrak{X})$ is a nonconstant meromorphic function, then there is a unique $\mathfrak{f} \in E(\mathfrak{X})$ such that $\eta = \mathfrak{f} \cdot d\mathfrak{g}$.

Let B be an oriented component of ∂X, and let $-B$ be the same component with the opposite orientation. For $x \in B$ choose a local parameter $\mathfrak{g} \in E(\mathfrak{X})$ such that $\mathfrak{g}$ is increasing on B near x. We say that η is positive on B at x if $\eta = \mathfrak{f} \cdot d\mathfrak{g}$ with $0 < \mathfrak{f}(x) < \infty$, and that η is positive on B if it is positive at all $x \in B$. It is easily checked that this definition does not depend on the choice of local parameters. Further η is positive on B or $-B$ if and only if it has no zeros or poles on B, and if η is positive on B, then $-\eta$ is positive on $-B$.

By an orientation $\mathcal{O}$ of ∂X we mean an orientation of each component of ∂X. If ∂X has r components, then it has 2^r orientations. If X is orientable, then two of these are induced by the two possible orientations of X. If η is positive on each component of $\mathcal{O}$, we will say that it is positive on $\mathcal{O}$, and that $\mathcal{O}$ has a positive differential.

In this note we investigate the following question: if $\mathfrak{X}$ is a compact Klein surface and $\mathcal{O}$ is an orientation of ∂X, does $\mathcal{O}$ have a positive holomorphic differential. Our first result is in the negative direction.

THEOREM 1. *Let $\mathfrak{X}$ be a compact orientable Klein surface, and let $\mathcal{O}$ be an orientation of ∂X induced by an orientation of X. Then $\mathcal{O}$ has no positive holomorphic differentials.*

Proof. Let $\mathfrak{X}_1$ be the analytic structure which is contained in the dianalytic structure $\mathfrak{X}$ and which corresponds to the orientation of $\mathfrak{X}$ which induces $\mathcal{O}$. If η is a holomorphic differential on X, we can as well regard it as a differential on $\mathfrak{X}_1$, and we can then apply the Cauchy integral theorem to obtain $\int_{\mathcal{O}} \eta = 0$. If η were positive on $\mathcal{O}$, this integral would be strictly positive. Note that this proof extends to meromorphic differentials of the second kind which have no poles on ∂X.

We conjecture that if an orientation $\mathcal{O}$ is not induced by an orientation of X, then it has a positive holomorphic differential, but we can so far prove this only in the cases $\mathfrak{X}$ elliptic or $\mathfrak{X}$ hyperelliptic (i.e., when $\mathfrak{X}$ can be represented as a double cover of the compactified upper half plane $\mathfrak{D}$).

THEOREM 2. *Let $\mathfrak{X}$ be an elliptic or hyperelliptic Klein surface and let $\mathcal{O}$ be an orientation of ∂X not induced by any orientation of X. Then $\mathcal{O}$ has a positive holomorphic differential.*

Proof. Let $\mathfrak{X}$ be an elliptic or hyperelliptic with $r \geqq 1$ boundary components. We can find meromorphic functions $\mathfrak{f}, \mathfrak{g}$ which generate $E(\mathfrak{X})$ over the reals, with $\mathfrak{f}^2 = H(\mathfrak{g})$, where H is a real polynomial of degree n without multiple factors. Then the mapping associated with $\mathfrak{g}$ represents $\mathfrak{X}$ as a double cover of $\mathfrak{D}$, which is ramified at the zeros of H, and also at ∞ if n is odd. If H has no real zeros, then $r = 1$ or $r = 2$, depending on whether $n/2$, the number of ramified points in the interior of $\mathfrak{D}$, is odd or even. If H has $m \geqq 1$ real zeros, then $r = [(m + 1)/2]$.

The genus of $\mathfrak{X}$ is $\gamma = [(n - 1)/1]$, and the differentials $\{d\mathfrak{g}/\mathfrak{f}, \mathfrak{g}\cdot d\mathfrak{g}/\mathfrak{f}, \cdots, \mathfrak{g}^{\gamma-1}\cdot d\mathfrak{g}/\mathfrak{f}\}$ form a basis over R for the space of holomorphic differentials on $\mathfrak{X}$ (see [3], p. 293). $\mathfrak{X}$ may have two real points, one real point, or one complex point at infinity. The differential $d\mathfrak{g}/\mathfrak{f}$ has all of its zeros at infinity. In the first case it has zeros of order $\gamma - 1$ at each such point, in the second a zero of order $2\gamma - 2$, and in the third a zero of order $\gamma - 1$.

Assume now that H has no real zeros. Then X is orientable. If $r = 1$, then every orientation of ∂X comes from an orientation of X, so there is nothing to prove. If $r = 2$, then $\gamma - 1 = n/2 - 2$ is even. The differential $(\mathfrak{g}^{\gamma-1} + 1)\cdot d\mathfrak{g}/\mathfrak{f}$ has no zeros on ∂X and hence is positive with respect to some orientation $\mathcal{O}$, and its negative is positive on $-\mathcal{O}$.

Now assume that H has $m \geqq 1$ real zeros. By choosing, if necessary, a new generator for $R(\mathfrak{g})$, we may assume that $\mathfrak{X}$ has a single complex point at infinity. Then H has $2r$ real zeros, and $n = 2(r + s)$, where s is the number of irreducible quadratic factors of H. Let the real zeros of H, in increasing order, be $a_1, b_1, \cdots, a_r, b_r$, and pick c_j between b_j and $a_{j+1}, j = 1, \cdots, r - 1$. Then the components of ∂X lie over the intervals $[a_j, b_j], j = 1, \cdots, r$. Let $J \subset \{1, \cdots, r - 1\}$ be any set of cardinality at most $\gamma - 1$, and set

$$\eta_J = \prod_{j \in J} (\mathfrak{g} - c_j) \cdot d\mathfrak{g}/\mathfrak{f} \,.$$

Each of the differentials $\pm\eta_J$ is positive with respect to a different orientation of ∂X. Hence for $\gamma \geqq r$ we obtain positive differentials for all 2^r possible orientations of ∂X, and the theorem is proved. So assume that $\gamma < r$. Since $r + 1 = n/2 = r + s$, we must have $s = 0$ and $\gamma = r - 1$. Because $s = 0$, X is orientable, and because $\gamma = r - 1$ we can use all subsets J except $J = \{1, \cdots, r - 1\}$. We have thus obtained positive differentials for $2^r - 2$ different orientations of ∂X, and have completed the proof of the theorem.

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