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# DOMAIN-PERTURBED PROBLEMS FOR ORDINARY LINEAR DIFFERENTIAL OPERATORS

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**The variation of the eigenvalues and eigenfunctions of an ordinary linear self-adjoint differential operator $L$ is considered under perturbations of the domain of $L$. The basic problem is defined as a suitable singular eigenvalue problem for $L$ on the open interval $\omega_- < s < \omega_+$ and is assumed to have at least one real eigenvalue $\lambda$ of multiplicity $k$. The perturbed problem is a regular self-adjoint problem defined for $L$ on a closed subinterval $[a, b]$ of $(\omega_-, \omega_+)$. It is proved under suitable conditions on the boundary operators of the perturbed problem that exactly $k$ perturbed eigenvalues $\mu^i_{ab} \to \lambda$ as $a, b \to \omega_-, \omega_+$. Further, asymptotic estimates are obtained for $\mu^i_{ab} - \lambda$ as $a, b \to \omega_-, \omega_+$. The other results are refinements which lead to asymptotic estimates for the eigenfunctions and variational formulae for the eigenvalues.**

Let $L$ be the $n$-th order ordinary linear differential operator defined by

$$(1.1) \qquad Lx = \frac{1}{k(s)} \sum_{i=0}^{n} p_i(s) x^{(n-i)}(s)$$

on the open interval $\omega_- < s < \omega_+$, where $k$ and $p_i$, $i = 0, 1, \cdots, n$ are real-valued functions on this interval with the properties that

( i ) $p_i \in C^{n-i}(\omega_-, \omega_+)$, $i = 0, 1, \cdots, n$;

(ii) $k$ is piecewise continuous on $(\omega_-, \omega_+)$; and

(iii) $p_0$ and $k$ are positive-valued. Furthermore the operator $k \cdot L$ is assumed to be formally self-adjoint, i.e. $k \cdot L$ coincides with its Lagrangian adjoint $[k \cdot L]^+$ where

$$(1.2) \qquad [k \cdot L]^+ x = \sum_{i=0}^{n} (-1)^{n-i} [p_i x]^{(n-i)} .$$

The points $\omega_+$ and $\omega_-$ are in general singularities for $L$; the possibility that they are $\pm\infty$ is not excluded.

It will be convenient to use the following notations:

$$(1.3) \qquad (x, y)_s^t = \int_s^t x(u)\overline{y(u)}k(u)du,\ \omega_- \leqq s < t \leqq \omega_+ ;$$

$$(1.4) \qquad (x, y)_a = (x, y)_a^{\omega_+};\ (x, y)^b = (x, y)_{\omega_-}^b ;$$

$$(1.5) \qquad (x, y) = (x, y)_{\omega_-}^{\omega_+} ;$$

$$[xy](s) = \sum_{m=1}^{n} \sum_{\substack{j+k=m-1 \\ j\geqq 0, k\geqq 0}} (-1)^j x^{(k)}(s)[p_{n-m}(s)\overline{y(s)}]^{(j)} ; \tag{1.6}$$

$$[xy](\pm) = \lim_{s\to\omega\pm} [xy](s) . \tag{1.7}$$

Since the operator $k\cdot L$ is formally self-adjoint Green's symmetric formula has the form

$$(Lx, y)_s^t - (x, Ly)_s^t = [xy](t) - [xy](s) . \tag{1.8}$$

Let $H$, $H[a, b]$ denote the Hilbert spaces which are the Lebesgue spaces with respective inner products $(x, y)$, $(x, y)_a^b$ and norms $\|x\| = (x, x)^{1/2}$, $\|x\|_a^b = [(x, x)_a^b]^{1/2}$, $\omega_- \leqq a < b \leqq \omega_+$. For $c$ any intermediate point, $\omega_- < c < \omega_+$, the symbols $H(\omega_-, c]$, $H[c, \omega_+)$ will similarly denote the Lebesgue spaces with respective inner products $(x, y)^c$, $(x, y)_c$ and norms $\|x\|^c = [(x, x)^c]^{1/2}$, $\|x\|_c = [(x, x)_c]^{1/2}$. From (1.8) it is clear that $[xy](+)$ (or $[xy](-)$) exists provided $x$, $y$, $Lx$, $Ly$ are in $H[c, \omega_+)$ (or $x$, $y$, $Lx$, $Ly$ are in $H(\omega_-, c]$).

Let $a_0$ and $b_0$ be fixed numbers satisfying $\omega_- < a_0 < b_0 < \omega_+$ and let $R_0$ be the rectangle in the $a - b$-plane described by the inequalities $\omega_- < a \leqq a_0$, $b_0 \leqq b < \omega_+$. Then every closed, bounded interval $[a, b]$, $\omega_- < a \leqq a_0$, $b_0 \leqq b < \omega_+$, can be associated in a one-to-one manner with a point of $R_0$. For $k = 0, 1, \cdots, n-1$, let $\alpha_{ik}(a)$, $i = 1, 2, \cdots, m$, and $\beta_{jk}(b)$, $j = 1, 2, \cdots, n-m$ be real-valued functions defined on the respective intervals $\omega_- < a \leqq a_0$, $b_0 \leqq b < \omega_+$, such that for every $[a, b] \in R_0$ the boundary operators

$$\begin{cases} U_a^i y = \sum_{k=0}^{n-1} \alpha_{ik}(a) y^{(k)}(a),\ i = 1, 2, \cdots, m \\ U_b^j y = \sum_{k=0}^{n-1} \beta_{jk}(b) y^{(k)}(b),\ j = 1, 2, \cdots, n-m \end{cases} \tag{1.9}$$

yield a linearly independent self-adjoint set of boundary conditions

$$\begin{cases} U_a^i y = 0,\ i = 1, 2, \cdots, m \\ U_b^i y = 0,\ i = 1, 2, \cdots, n-m \end{cases} \tag{1.10}$$

for $L$ (see [3] Chapter 11). Also for each $[a, b] \in R_0$ let $D[a, b]$ denote the set of all $y \in H[a, b]$ which have the properties that

(i) $y \in C^{n-1}[a, b]$, $y^{(n-1)}$ is absolutely continuous on $[a, b]$;

(ii) $Ly \in H[a, b]$; and

(iii) $y$ satisfies (1.10).

Then the self-adjoint eigenvalue problem

$$Ly = \mu y , \qquad y \in D[a, b] \tag{1.11}$$

is known to have a countable set of real eigenvalues with no finite cluster point and a corresponding set of (real) eigenfunctions complete in $H[a, b]$. Our problem is to obtain estimates for each eigenvalue $\mu = \mu_{ab}$ of (1.11) for $a, b$ near $\omega_-, \omega_+$ under hypotheses that will ensure that the limits of $\mu_{ab}$ as $a, b \to \omega_-, \omega_+$ will exist. Accordingly, eigenvalues $\lambda$ of suitable singular eigenvalue problems for $L$ on $(\omega_-, \omega_+)$ will be assumed to exist. If the eigenspace of $\lambda$ is $k$-dimensional the first theorem shows in particular that at least $k$ eigenvalues of (1.11) converge to $\lambda$ as $a, b \to \omega_-, \omega_+$. The other results are refinements of this which lead to asymptotic estimates for eigenfunctions. The method of estimation used is due to H. F. Bohnenblust [1]. Results like these have been previously obtained for second order cases by C. A. Swanson [8], [9]. See also [10] where he considers the biharmonic operator.

Let $l_0$ be any fixed complex number, $\operatorname{Im} l_0 \neq 0$, and let $\psi_i$, $i = 1, 2, \cdots, n$, denote linearly independent solutions (hereafter to be referred to as *basic solutions*) of $L_0 x = 0$ where $L_0 = L - l_0$. If basic solutions $\psi_i$, $i = 1, 2, \cdots, n$ exist such that the $\lim |\psi_i/\psi_j|$ is either 0 or $\infty$ as $s \to \omega_+$ for each pair $\psi_i, \psi_j$, $i, j = 1, 2, \cdots, n$, $i \neq j$, then $\omega_+$ will be referred to as a *class 1* singularity. On the other hand, $\omega_+$ will be called a *class 2* singularity when the behaviour of the basic solutions is essentially arbitrary as $s \to \omega_+$. In particular this includes cases where the basic solutions may oscillate as $s \to \omega_+$. Similar definitions also apply to the singularity $\omega_-$. The singularity $\omega_+$ (or $\omega_-$) is further characterized by the number of basic solutions in $H[c, \omega_+)$ (or in $H(\omega_-, c]$) where $c$ is any number satisfying $\omega_- < c < \omega_+$. For $n = 2$ this reduces to Weyl's well-known *limit circle, limit point* classification of singular points [3, p. 225].

For the present perturbation problems will be considered for which both $\omega_-$ and $\omega_+$ are *both* class 2 singularities and all basic solutions are in $H(\omega_-, c]$ and in $H[c, \omega_+)$. In another paper class 1 singularities (and mixed cases) as well as examples will be considered.

**2. Basic and perturbed problems.** Rather than general spectral theory, one is interested in cases that the limits of $\mu_{ab}$ as $a, b \to \omega_-, \omega_+$ exist in an elementary sense. Thus, eigenvalues of suitable singular eigenvalue problems for $L$ on $(\omega_-, \omega_+)$ are supposed to exist. Such eigenvalue problems may be established by following basically the methods suggested by Kodaira [5] and Coddington [2]. Note that for the particular case $n = 2$, a theorem of Weyl [7] leads to singular "limit circle" problems which possess eigenvalues.

Let $D$ be the set of all $x \in H$ such that $x \in C^{n-1}(\omega_-, \omega_+)$ and $x^{(n-1)}$ is absolutely continuous on every closed bounded sub-interval of $(\omega_-, \omega_+)$. Let $\chi_i$, $i = 1, 2, \cdots, n$ be functions (to remain fixed) such that

(i) $L\chi_i \in H$, $i = 1, 2, \cdots, n$;

(ii) the end conditions $[x\chi_i](-) = 0$, $i = 1, 2, \cdots, m$ are linearly independent; and

(iii) the end conditions $[x\chi_i](+) = 0$, $i = m+1, m+2, \cdots, n$, are linearly independent.

Then the *basic problem* is the singular eigenvalue problem

$$Lx = \lambda x\ , \qquad x \in D_0 \tag{2.1}$$

where $D_0$ is the set of all $x \in D$ such that

$$\begin{cases} [x\chi_i](-) = 0,\ i = 1, 2, \cdots, m \\ [x\chi_i](+) = 0,\ i = m+1, \cdots, n\ . \end{cases} \tag{2.2}$$

Again (2.1) is to be a reasonable eigenvalue problem, i.e., at least one eigenvalue $\lambda$ is supposed to exist which is assumed to be real. Note that the methods used by Coddington [2] and Kodaira [5] ensure that all eigenvalues are real. The eigenvalue problem (1.11) is to be regarded as a perturbation of (2.1) and hence will be referred to as the *perturbed problem.*

For the class of perturbation problems to be considered, the basic solutions are not necessarily ordered according to their asymptotic behaviour at $\omega_+$ or at $\omega_-$. Consequently strong conditions have to be imposed on the limiting behaviour of the boundary operators $U_a^i$, $U_b^i$ as $a, b \to \omega_-, \omega_+$. In particular every $n-1$ times differentiable function y shall satisfy

$$\begin{cases} U_a^i y = [y\chi_i](a)[1 + o(1)] \text{ as } a \to \omega_-\ , & i = 1, 2, \cdots, m \\ U_b^i y = [y\chi_{m+i}](b)[1 + o(1)] \text{ as } b \to \omega_+ & i = 1, 2, \cdots, n-m\ . \end{cases} \tag{2.3}$$

Let $A$ denote the matrix $(A_{ij})$ where

$$A_{ij} = \begin{cases} [\psi_i\chi_j](-),\ i = 1, 2, \cdots, n;\ j = 1, 2, \cdots m \\ [\psi_i\chi_j](+),\ i = 1, 2, \cdots n;\ j = m+1, \cdots, n \end{cases}$$

and let $\Omega = \det A$. Then since $\Omega = \det A^t$, where $A^t$ is the transpose of $A$, and since $l_0$ is nonreal it follows immediately that $\Omega \neq 0$ (otherwise $l_0$ would be an eigenvalue of (2.1)). Also for each $j$, $j = 1, 2, \cdots, n$, $\psi_j$, $L\psi_j$, $\chi_j$, $L\chi_j$ are in $H$; hence (1.8) implies that each limit $[\psi_i\chi_j](\pm)$ exists (finite) for $i, j = 1, 2, \cdots, n$. This implies that $\Omega$ is equal to some nonzero constant.

Let $A(a, b)$ denote the matrix $(A_{ij}(a, b))$ where

$$A_{ij}(a, b) = \begin{cases} U_a^i\psi_j,\ i = 1, 2, \cdots, m;\ j = 1, 2, \cdots, n \\ U_b^{i-m}\psi_j,\ i = m+1, \cdots, n;\ j = 1, 2, \cdots, n \end{cases}$$

ane let $\Omega(a, b) = \det A(a, b)$. Since $[\psi_i\chi_j](a)$ and $[\psi_i\chi_j](b)$ are finite as $a \to \omega_-$ and $b \to \omega_+$ for $i, j = 1, 2, \cdots, n$, it follows from (2.3) that numbers $a_0$, $b_0$, can be selected (which may be pre-supposed to be the original choices) and a constant $C$ such that

$$(2.4)\quad | U_a^i\psi_j | \leqq C, | U_b^k\psi_j | \leqq C, i = 1, 2, \cdots, m, j = 1, 2, \cdots, n, \quad k = 1, 2, \cdots, n - m$$

whenever $\omega_- < a \leqq a_0, b_0 \leqq b < \omega_+$. Also by (2.3) the element in the $i$-th row and $j$-th column in $A(a, b)$ approaches the element in the $i$-th row and $j$-th column in $A^t$ as $a, b \rightarrow \omega_-, \omega_+$. This implies that

$$\Omega(a, b) \rightarrow \Omega \neq 0 \tag{2.5}$$

as $a, b \rightarrow \omega_-, \omega_+$ and hence by (2.4) and (2.5) the numbers $a_0, b_0$ previously chosen can be assumed to be such that $\Omega(a, b)$ is bounded above and away from zero whenever $\omega_- < a \leqq a_0, b_0 \leqq b < \omega_+$.

**3. Comparison of the basic and perturbed problems.** The two problems (1.11) and (2.1) will be compared, with (1.11) regarded as a perturbation of (2.1). An estimate will be obtained for the variation of the eigenvalues and eigenfunctions under the perturbation $D_0 \rightarrow D[a, b]$. In particular it will be shown that this variation has the limit 0 as $a, b \rightarrow \omega_-, \omega_+$. Let $\lambda$ be an eigenvalue of (2.1) and let $A_\lambda$ denote the eigenspace of dimension $k$ corresponding to $\lambda$. Let $x_j, j = 1, 2, \cdots, k$ be an orthonormal basis for $A_\lambda$ and define $\tau_a^i(x)$, $\tau_b^i(x)$, $\Gamma_a(x)$ and $\Gamma_b(x)$ by

$$\tau_a^i(x) = \sum_{j=1}^{k} | U_a^i x_j |; \tau_b^i(x) = \sum_{j=1}^{k} | U_b^i x_j | ; \tag{3.1}$$

$$\Gamma_a(x) = \sum_{i=1}^{m} \tau_a^i(x); \Gamma_b(x) = \sum_{i=1}^{n-m} \tau_b^i(x) . \tag{3.2}$$

Then (2.2) and (2.3) clearly imply that $\tau_a^i(x) = o(1), i =, 1, 2, \cdots, m$ and $\tau_b^i(x) = o(1), i = 1, 2, \cdots, n - m$ and hence

$$\Gamma_a(x) = o(1), \qquad \Gamma_b(x) = o(1) \tag{3.3}$$

as $a \rightarrow \omega_-, b \rightarrow \omega_+$. The following theorem proves the convergence of the eigenvalues of (1.11) to those of (2.1).

THEOREM 1. *Let $\omega_-$ and $\omega_+$ be singularities for $L$ as described in* § 1. *Let $\lambda$ be an eigenvalue of* (2.1) *possessing $k$ orthonormal eigenfunctions. Then under assumption* (2.3) *there exists a rectangle $R_0$, and a constant $C$ on $R_0$, such that at least $k$ perturbed eigenvalues $\mu_{ab}^j$ of* (1.11) *satisfy*

$$| \mu_{ab}^j - \lambda | \leqq C[\Gamma_a(x) + \Gamma_b(x)] \tag{3.4}$$

*whenever* $[a, b] \in R_0$.

*Proof.* Let $G_{ab}(s, t)$ be the Green's function for the operator $k \cdot L_0$

associated with (1.10) and let $G_{ab}$ be the linear transformation on $H[a, b]$ defined by

$$G_{ab}y = \int_a^b G_{ab}(s, t)y(t)k(t)dt , \qquad y \in H[a, b] .$$

It is well-known [3, Chapter 7], that for any function $y \in H[a, b]$, the function $w = G_{ab}y$ is the unique solution in $D[a, b]$ of the differential equation $L_0 w = y$. For $\lambda$ an eigenvalue and $x$ any corresponding normalized eigenfunction of (2.1), we define a function $f$ on $[a, b]$ by

$$f = x - \gamma G_{ab}x , \qquad \gamma = \lambda - l_0 . \tag{3.5}$$

It is easily verified because of the linearity of all the operators involved that $f$ is a solution of the boundary problem

$$\begin{aligned} L_0 f = 0,\ U_a^i f &= U_a^i x,\ i = 1, 2, \cdots, m ,\\ U_b^i f &= U_b^i x,\ i = 1, 2, \cdots, n - m . \end{aligned} \tag{3.6}$$

Let $K^j(a, b)$ denote the determinant of the matrix obtained from $A(a, b)$ by replacing the $j$-th column by

$$U_a^1 x,\ U_a^2 x,\ \cdots,\ U_a^m x,\ U_b^1 x,\ \cdots,\ U_b^{n-m} x .$$

Then Cramer's rule yields the following representation of $f$ in terms of the basic solutions:

$$f(s) = \frac{1}{\Omega(a, b)} \sum_{j=1}^{n} K^j(a, b)\psi_j(s) . \tag{3.7}$$

The solution $f$ of (3.6) is unique for if $g$ is any solution of (3.6) then the function $h = g - f$ satisfies $L_0 h = 0$, $U_a^i h = 0$, $i = 1, 2, \cdots, m$, $U_b^i h = 0$, $i = 1, 2, \cdots, n - m$. This implies that $h$ is the zero function or $g = f$.

It follows from (2.4), (3.1) and (3.2) that there exists a constant $C$ such that

$$|K^j(a, b)| \leqq C[\Gamma_a(x) + \Gamma_b(x)]$$

for each $j$, $j = 1, 2, \cdots, n$ whenever $[a, b] \in R_0$. This in addition to (2.5), (3.5), (3.7) and the fact that all the basic solutions are in $H$, enables one to deduce that there exists a constant $C$ such that

$$\| x - \gamma G_{ab}x \|_a^b \leqq C(\Gamma_a(x) + \Gamma_b(x)) \| x \|_a^b \tag{3.8}$$

whenever $[a, b] \in R_0$. The following fundamental lemma was obtained by H. F. Bohnenblust the proof of which is outlined in [8, p. 1554].

LEMMA 1. *Let $P(\delta)$ be the projection mapping from the Hilbert space $H[a, b]$ onto its subspace $H_\delta[a, b]$ $(\delta > 0)$ spanned by all the* *eigenfunctions* $y_j$ *of* (1.11) *such that the corresponding eigenvalues* $\mu^j$ *satisfy* $|\mu^j - \lambda| \leqq \delta$. *Then for any* $w \in H[a, b]$,

$$\| w - P(\delta)w \|_a^b \leqq \left(1 + \frac{|\gamma|}{\delta}\right) \| w - \gamma G_{ab} w \|_a^b \,.$$

It follows from (3.8) and Lemma 1 that there exists a constant $C$ on $R_0$ such that

$$\| x - P(\delta)x \|_a^b \leqq \frac{C}{2\delta}(\Gamma_a(x) + \Gamma_b(x)) \, \| x \|_a^b \,. \tag{3.9}$$

With the choice $\delta = C[\Gamma_a(x) + \Gamma_b(x)]$, we obtain

$$\| x - P(\delta)x \|_a^b \leqq \frac{1}{2} \| x \|_a^b \tag{3.10}$$

and conclude that $P(\delta)x = 0$ implies $x = 0$ on $[a, b]$. But $\dim A_\lambda = k$; hence there exists at least $k$ perturbed eigenvalues $\mu_{ab}^j$ (counting multiplicities) of (1.11) such that

$$|\mu_{ab}^j - \lambda| \leqq C[\Gamma_a(x) + \Gamma_b(x)]$$

for $[a, b] \in R_0$. This completes the proof of the theorem.

Theorem 1 and (3.3) show in particular that if $\lambda$ is a basic eigenvalue of multiplicity $k$ there exist at least $k$ perturbed eigenvalues $\mu_{ab}^j$ (counting multiplicities) such that $\mu_{ab}^j \to \lambda$ when $a, b \to \omega_-, \omega_+$. To obtain the stronger result that *exactly* $k$ perturbed eigenvalues $\mu_{ab}^j$ satisfy (3.4) in Theorem 1, we require the monotonicity property that the absolute value of the $n$-th eigenvalue of (2.1), $|\lambda_1| \leqq |\lambda_2| \leqq \cdots$, is not larger than the absolute value of the $n$-th eigenvalue of (1.11), $|\mu_1| \leqq |\mu_2| \leqq \cdots$. Then an inductive proof similar to that used in [8, p. 1554] yields the following result:

THEOREM 2. *If in addition to the hypotheses of Theorem* 1 *the above monotonicity property holds, then for every basic eigenvalue* $\lambda$ *of* (2.1), *of multiplicity* $k$, *there exists a rectangle* $R_0$ *and a constant* $C$ *on* $R_0$, *such that exactly* $k$ *eigenvalues* $\mu_{ab}^j$ *(counting multiplicities) of* (1.11) *satisfy* (3.4) *whenever* $[a, b] \in R_0$.

THEOREM 3. *Let the hypotheses of Theorem* 2 *be satisfied. Then corresponding to the eigenvalues* $\lambda$ *and* $\mu_{ab}^j$ *of Theorem* 2, *there are orthogonal eigenfunctions* $x^j$ *on* $[a, b]$ *associated with* $\lambda$ *and* $y^j$ *associated with the* $\mu_{ab}^j$ *such that*

$$\| y_{ab}^j - x^j \|_a^b \leqq C[\Gamma_a(x) + \Gamma_b(x)], \ \| x^j \|_a^b = \| y^j \|_a^b = 1 \,, \quad j = 1, 2, \cdots, k \,,$$

*whenever* $[a, b] \in R_0$.

*Proof.* Let $\{y^j\}$ be a set of orthonormal eigenfunctions on $[a, b]$ corresponding to the set of eigenvalues $\{\mu_{ab}^j\}$ in Theorem 2. Then $H_\delta[a, b]$ is $k$-dimensional by Theorem 2 and $P(\delta)x = 0$ implies $x = 0$ by (3.10). Hence there exist $k$ unique linearly independent eigenfunctions $z^j$ corresponding to $\lambda$ which $P(\delta)$ maps into the orthonormal eigenfunctions $y^j$ and by (3.9)

$$(3.11) \qquad \| z^j - y^j \|_a^b = O[\Gamma_a(x) + \Gamma_b(x)] , \qquad [a, b] \in R_0 .$$

Since

$$| (z^i, z^j)_a^b - (y^i, y^j)_a^b | \leqq \| y^i \|_a^b \| z^j - y^j \|_a^b + \|z^j\|_a^b \| z^i - y^i \|_a^b$$

by the Schwarz inequality

$$(z^i, z^j)_a^b = \delta_{ij} + O[\Gamma_a(x) + \Gamma_b(x)]$$

for $i, j = 1, 2, \cdots, k$ where $\delta_{ij}$ denotes the Kronecker delta. Since the $z^j$ are linearly independent, an orthonormal sequence $x^j$ can be constructed by the Schmidt process as linear combinations of the $z^j$ and it is easily verified that

$$\| x^j - z^j \|_a^b = O[\Gamma_a(x) + \Gamma_b(x)] .$$

This combined with (3.11) gives the desired result.

**4. Uniform estimate for eigenfunctions.** For the class of singular problems under consideration, additional restrictions are needed on the basic solutions $\psi_j$, $j = 1, 2, \cdots, n$, to obtain uniform estimates for $y_{ab}^j(s) - x^j(s)$, $a \leqq s \leqq b$, in Theorem 3. In particular the requirement will be that all basic solutions are bounded on $(\omega_-, \omega_+)$.

LEMMA 2. *Let $G_{ab}(s, t)$ be the Green's function for $k \cdot L_0$ associated with* (1.10). *Then the positive function $g_{ab}(s)$ defined by*

$$(4.1) \qquad [g_{ab}(s)]^2 = \int_a^b | G_{ab}(s, t) |^2 k(t) dt$$

*is uniformly bounded on $a \leqq s \leqq b$ provided $a \leqq a_0$, $b_0 \leqq b$.*

*Proof.* The Green's function $G_{ab}(s, t)$ will be constructed first. From (1.6) it is clear that $[xy](s)$ may be written in the form

$$[xy](s) = \sum_{i,j=0}^{n-1} B_{ij}(s) x^{(i)}(s) \overline{y^{(j)}(s)}$$

with

$$(4.2) \qquad B_{ij}(s) = \begin{cases} (-1)^j P_0(s) , & i + j = n - 1 \\ 0 , & i + j > n - 1 . \end{cases}$$

Let $B$ denote the $n$-by-$n$ matrix which has $B_{ij} = B_{ij}(s)$ in the $i+1$-th row and $j+1$-th column, $i, j = 0, 1, 2, \cdots, n-1$. Then (4.2) implies that $B$ is nonsingular on $(\omega_-, \omega_+)$.

Considering now the basic solutions one obtains from Green's formula (1.8) that $[\psi_\alpha \bar{\psi}_\beta](s)$ is a constant $[\psi_\alpha \bar{\psi}_\beta]$ independent of $s$, $\alpha, \beta = 1, 2, \cdots, n$. With $S$ representing the matrix with element $[\psi_\alpha \bar{\psi}_\beta]$ in the $\alpha$-th row and $\beta$-th column, it is easily verified that

$$S = Y^t B Y \tag{4.3}$$

where $Y$ denotes the Wronskian matrix $(\psi_j^{(i-1)}(s))$, $i, j = 1, 2, \cdots, n$ and $Y^t$ the transpose of the matrix $Y$. Since $B$, $Y$ (and hence $Y^t$) are nonsingular it follows that $S$ is a nonsingular constant matrix. Let $S^{-1} = (\gamma_{\alpha\beta})$ denote the matrix inverse to $S$ and consider the function $K(s, t)$ defined by

$$K(s, t) = \sum_{\alpha,\beta=1}^{n} \gamma_{\alpha\beta} \psi_\alpha(t) \psi_\beta(s) \; . \tag{4.4}$$

Since $YS^{-1}Y^t = B^{-1}$ by (4.3) one obtains by inspection that

$$K_s^{(i)}(s, t) = \begin{cases} 0 \, , & i = 1, 2, \cdots, n-2 \\ -1/p_0(s) \, , & i = n-1 \, . \end{cases}$$

Let

$$K_{ab}(s, t) = \begin{cases} K(s, t) \, , & a \leqq t \leqq s \leqq b \\ 0 \, , & a \leqq s \leqq t \leqq b \end{cases} \tag{4.5}$$

where $[a, b]$ is any closed sub-interval of $(\omega_-, \omega_+)$. Then from the above remarks it follows that

$$G_{ab}(s, t) = K_{ab}(s, t) + \sum_{k=1}^{n} A_k(t) \psi_k(s) \tag{4.6}$$

where $A_k(t)$, $k = 1, 2, \cdots, n$, is chosen in such a way that $G_{ab}(s, t)$, as a function of $s$, satisfies (1.10). Compare (4.6) with [4, Th. 8, p. 1319]. In particular, one obtains by Cramer's rule that

$$A_k(t) = \frac{\Omega_{ab}^k(t)}{\Omega(a, b)}$$

where $\Omega_{ab}^k(t)$ denotes the determinant of the matrix obtained from $A(a, b)$ by replacing the $k$-th column by the column whose $r$-th component $v_r$ is given by

$$v_r = \begin{cases} 0 \, , & r = 1, 2, \cdots, m \\ -\sum_{\alpha,\beta=1}^{n} \gamma_{\alpha\beta} \psi_\alpha(t) U_b^{r-m} \psi_\beta \, , & r = m+1, \cdots, n \, . \end{cases}$$

Since $\psi_k \in H$, $k = 1, 2, \cdots, n$ it follows immediately from (2.4) and (2.5) that there exists a constant $C$ such that

$$(4.7) \qquad \| A_k(t) \|_a^b \leqq C , \qquad k = 1, 2, \cdots, n$$

whenever $a \leqq a_0$, $b_0 \leqq b$.

It follows from (4.1) that for $a \leqq s \leqq b$

$$(4.8) \qquad g_{ab}(s) \leqq \left\{\int_a^s | G_{ab}(s, t) |^2 k(t)dt\right\}^{1/2} + \left\{\int_s^b | G_{ab}(s, t) |^2 k(t)dt\right\}^{1/2} .$$

By (4.4), (4.6) and the triangle inequality we obtain that

$$\left\{\int_a^s | G_{ab}(s, t) |^2 k(t)dt\right\}^{1/2} \leqq \sum_{i,j=1}^{n} | \gamma_{ij}\psi_j(s) | \, \| \psi_i(t) \|_a^s + \sum_{j=1}^{n} | \psi_j(s) | \, \| A_j(t) \|_a^s .$$

But $\psi_j$ is bounded on $(\omega_-, \omega_+)$ and $\psi_j \in H$, $j = 1, 2, \cdots, n$; hence by (4.7) the first quantity on the right in (4.8) is uniformly bounded on $a \leqq s \leqq b$ provided $a \leqq a_0$, $b_0 \leqq b$. A similar proof shows that the second integral on the right in (4.8) is also uniformly bounded on $a \leqq s \leqq b$ provided $a \leqq a_0$, $b_0 \leqq b$. This gives the desired result. The next result gives uniform estimates for the eigenfunctions of Theorem 3.

THEOREM 4. *If in addition to the hypotheses of Theorem 3, $\psi_j$ is bounded on $(\omega_-, \omega_+)$, $j = 1, 2, \cdots, n$, then the eigenfunctions $x^j$ corresponding to $\lambda$ and $y^j_{ab}$ corresponding to $\mu^j_{ab}$ of Theorem 3 are such that*

$$(4.9) \quad y^j_{ab}(s) = x^j(s) - f^j(s) + O[\Gamma_a(x)] + O[\Gamma_b(x)] , \quad j = 1, 2, \cdots, k ,$$

*where $f^j(s)$ is the unique solution of the boundary problem*

$$(4.10) \qquad \begin{aligned} Lf = l_0 f,\ U^i_a f &= U^i_a x^j , \qquad i = 1, 2, \cdots, m , \\ U^i_b f &= U^i_b x^j , \qquad i = 1, 2, \cdots, n - m . \end{aligned}$$

*Proof.* The Schwarz inequality for $H[a, b]$ yields

$$\begin{aligned} &| y^j_{ab}(s) - (\lambda - l_0)G_{ab}x^j(s) | \\ &= | G_{ab}[(\mu^j_{ab} - l_0)y^j_{ab}(s) - (\lambda - l_0)x^j(s)] | \\ &\leqq g_{ab}(s)\{| \mu^j_{ab} - l_0 | \, \| y^j_{ab} - x^j \|_a^b + | \mu^j_{ab} - \lambda | \, \| x^j \|_a^b\} . \end{aligned}$$

Hence Lemma 2 and Theorems 2 and 3 show that there exists a constant $C$ such that

$$(4.11) \qquad | y^j_{ab}(s) - (\lambda - l_0)G_{ab}x^j(s) | \leqq C[\Gamma_a(x) + \Gamma_b(x)]$$

on $a \leqq s \leqq b$, whenever $a \leqq a_0$, $b_0 \leqq b$.

The solution $f^j(s)$ of the boundary problem (4.10) is given by (3.5) or (3.7) with $x$ replaced by $x^j$. The function $F^j$ defined by

$$F^j(s) = (\lambda - l_0)G_{ab}x^j(s) - x^j(s) + f^j(s)$$

satisfies

$$\begin{aligned} LF^j = l_0F^j,\ U_a^iF^j &= 0\ , \qquad i = 1, 2, \cdots, m\ , \\ U_b^iF^j &= 0\ , \qquad i = 1, 2, \cdots, n - m \end{aligned}$$

and hence $F^j$ is the zero solution on $a \leqq s \leqq b$ for $j = 1, 2, \cdots, k$. This with (4.11) immediately gives the uniform estimates (4.9).

**5. Asymptotic variational formulae for eigenvalues.** The purpose here is to derive formulae for the change $\mu_{ab}^j - \lambda$ of eigenvalues under the perturbation $D_0 \to D[a, b]$, valid for $a, b$ in neighbourhoods of $\omega_-$, $\omega_+$ respectively. Let $x^j$, $y^j$ denote the normalized eigenfunctions associated with $\lambda$ and $\mu^j = \mu_{ab}^j$ as described in Theorem 3 and let $f^j$ be the unique solution of (4.10). One obtains the following theorem:

THEOREM 5. *Under the assumptions of Theorem* 4 *the following asymptotic variational formulae for the eigenvalues* $\lambda$, $\mu_{ab}^j$ *are valid*:

$$\begin{aligned} \lambda - \mu_{ab}^j = {} & [f^jx^j](b) - [f^jx^j](a) \\ & + (l_0 - \lambda)(f^j, f^j)_a^b + [\Gamma_a(x) + \Gamma_b(x)](f^j, 1)_a^b O(1) \end{aligned} \tag{5.1}$$

*as* $a, b \to \omega_-, \omega_+$.

*Proof.* Let $Uy = 0$ denote the self-adjoint set of boundary conditions given by (1.10). Then by [3, Chapter 11] there exist boundary forms $U_c$, $U_c^+$ of rank $n$ such that

$$[uv](b) - [uv](a) = Uu \cdot U_c^+v + U_c^+u \cdot Uv$$

for any pair $u, v \in C^{n-1}[a, b]$, where $\cdot$ represents the scalar product.

Now $Uy^j = 0$ by (1.10) and (1.11) and $Ux^j = Uf^j$ by (4.10); hence (dropping the superscripts $j$)

$$\begin{aligned} [xy](b) - [xy](a) &= Ux \cdot U_c^+y \\ &= [fy](b) - [fy](a)\ . \end{aligned}$$

Then, application of Green's formula (1.8) to the differential equations $Lx = \lambda x$, $Lf = l_0 f$ and $Ly = \mu y$ on $[a, b]$, leads to

$$(\lambda - \mu)(x, y)_a^b = (l_0 - \mu)(f, y)_a^b\ ; \tag{5.2}$$

$$[fx](b) - [fx](a) = (l_0 - \lambda)(f, x)_a^b\ . \tag{5.3}$$

Hence one obtains as a consequence of Theorems 1, 2 and 3 that $\mu = \lambda + o(1)$ and

$$|(x, y)_a^b - (x, x)_a^b| \leqq ||x||_a^b \, ||y - x||_a^b = o(1)$$

as $a, b \to \omega_-, \omega_+$. Hence

$$(x, y)_a^b = 1 + o(1)\,, \qquad a, b \to \omega_-, \omega_+$$

and (5.2) yield

$$\lambda - \mu = (l_0 - \lambda)(f, y)_a^b[1 + o(1)]\,. \tag{5.4}$$

We now appeal to the uniform estimate (4.9) to obtain

$$(f, y)_a^b = (f, x)_a^b - (f, f)_a^b + [\Gamma_a(x) + \Gamma_b(x)](f, 1)_a^b O(1)\,. \tag{5.5}$$

Then applying (5.3) and (5.5) to (5.4) the result (5.1) follows easily.

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