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GENERALIZED SYLOW TOWER GROUPS

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A well-known theorem of P. Hall states that a finite group G is solvable if, and only if G possesses a complete set of permutable Sylow subgroups. Our goal here is to investigate finite (solvable) groups whose Sylow subgroups are related by a normalizer condition (N). The presence of property (N) for a group G implies that G has structure similar to a Sylow tower group.

Some well-known classes of finite groups can be described in terms of Sylow structure. For example, a finite nilpotent group is characterized by the property that distinct Sylow subgroups centralize one another. And P. Hall [3] has shown that a finite solvable group is characterized by the existence of a complete set of pairwise permutable Sylow subgroups. More recently, Huppert [6] has investigated groups with property (V): *any pair of Sylow subgroups having coprime orders permute as subgroups.*

This work investigates groups in connection with the following normalizer condition (N). If $\mathscr{S}$ is a collection of subgroups of a finite group, we say $\mathscr{S}$ satisfies (N) provided: for any pair of subgroups in $\mathscr{S}$ having coprime orders, at least one of the subgroups normalizes the other. The known structures of solvable and nilpotent groups motivate our choices for the set $\mathscr{S}$. The main results are listed as 1.7 – 1.8, 2.5 and 3.1.

All groups considered here are finite and the following notation is used: If G is a group then

$\pi(G)$ is the number of prime divisors of the order of G,

$l(G)$ is the nilpotent (Fitting) length of G,

$Z(G)$ is the center of G,

$\phi(G)$ is the Frattini subgroup of G.

If H is a subgroup of G then

$N_G(H)$ is the normalizer of H in G,

$C_G(H)$ is the centralizer of H in G.

If p is a prime, G_p denotes a Sylow p-subgroup of G.

1. Generalized Sylow tower groups.

Let G be a group with order $p_1^{\alpha_1} \cdots p_r^{\alpha_r}$, where $p_1, \cdots, p_r$ are distinct primes and $\alpha_1, \cdots, \alpha_r$ are positive integers. For $i = 1, \cdots, r$ let G_i denote a Sylow p_i-subgroup of G. The collection of subgroups $\mathscr{S} = \{G_1, \cdots, G_r\}$ is called a complete set of Sylow subgroups of G. If the subgroups of $\mathscr{S}$ are pairwise permutable as subgroups (that is, if $G_iG_j = G_jG_i$, $1 \leqq i, j \leqq r$),

$\mathscr{S}$ is called a Sylow basis of G. The collection of 2^r subgroups of G formed by all products of subgroups in a Sylow basis of G is a Sylow system of G.

If a complete set $\mathscr{S}$ of Sylow subgroups of G satisfies (N), we call G a generalized Sylow tower group (GSTG). In this case, $\mathscr{S}$ is a Sylow basis of G and hence G is solvable. On the other hand, one can easily show that every Sylow tower group is necessarily a GSTG.

LEMMA 1.1. *If G is a GSTG, then every Sylow basis of G satisfies* (N).

Proof. Let $\mathscr{S} = \{G_1, \cdots, G_r\}$ be a Sylow basis of G satisfying (N) and $\mathscr{T}$ any Sylow basis of G. By [4, p. 320], there exists an element $g \in G$ with $\mathscr{T} = \{G_1^g, \cdots, G_r^g\}$. Then $\mathscr{T}$ clearly satisfies (N) since $\mathscr{S}$ does.

Let G be a generalized Sylow tower group and $\mathscr{S}$ a Sylow basis of G which satisfies (N). If R is a relation on the set of all primes and if the Sylow p-subgroup of G in $\mathscr{S}$ normalizes the Sylow q-subgroup of G in $\mathscr{S}$ whenever pRq, then we call G a generalized Sylow tower group of complexion R. The preceding lemma shows that the complexion of G is independent of the Sylow basis chosen. It should be noted that a given GSTG may have many complexions.

In the sequel, R denotes a relation on the set of all primes.

PROPOSITION 1.2. *Let G be a* GSTG *of complexion R. If K is a subgroup of G, then K is also a* GSTG *of complexion R.*

Proof. Let $\mathscr{S} = \{G_1, \cdots, G_r\}$ be a Sylow basis of G which satisfies (N). If K is a Hall subgroup of G, then K is conjugate to the subgroup of G formed by the product of those subgroups in $\mathscr{S}$ whose orders divide $|K|$. Suppose then $K = (G_{i_1} \cdots G_{i_n})^g$, some $g \in G$. Clearly $\{G_{i_1}^g, \cdots, G_{i_n}^g\}$ is a Sylow basis of K satisfying (N) and K has complexion R.

If $\pi(K) < \pi(G)$, the assertion follows by induction. Suppose now that $\pi(K) = \pi(G)$ and let $\mathscr{K} = \{K_1, \cdots, K_r\}$ be any Sylow basis of K. By a result of P. Hall [4, p. 321], there is an element $x \in G$ such that $K_i = K \cap G_i^x$ for $i = 1, \cdots, r$. It is now clear that $\mathscr{K}$ satisfies (N) and K has complexion R.

PROPOSITION 1.3. *A homomorphic image of a* GSTG *with complexion R is again a* GSTG *of complexion R.*

Proof. Let $\{G_1, \cdots, G_r\}$ be any Sylow basis of G. Let τ be a homomorphism of G onto H and denote the image of G_i under τ by

H_i (for $i = 1, \cdots, r$). Then the collection of the nontrivial H_i is a Sylow basis of H which satisfies (N). Clearly H has the same normalizing structure as G.

Let A be the symmetric group of degree 4 and B the normal subgroup of A having order 4. Then both B and A/B are GSTG's of the same complexion. But A is clearly not a GSTG since $\pi(A) = 2$ and A has no normal Sylow subgroup. In this respect, a GSTG has structure similar to a nilpotent group.

PROPOSITION 1.4. *Let Z_∞ denote the hypercenter of a group G. If G/Z_∞ is a* GSTG, *then G is a* GSTG.

Proof. Let Z denote the center of G and show that G is a GSTG whenever the factor group G/Z is a GSTG. Suppose G/Z is a GSTG and $\mathscr{S} = \{G_1, \cdots, G_r\}$ is a complete set of Sylow subgroups of G chosen so that $\mathscr{S}^* = \{G_1Z/Z, \cdots, G_rZ/Z\}$ satisfies (N). For integers i and j with $1 \leqq i, j \leqq r$, let G_iZ/Z normalize G_jZ/Z. Then G_i normalizes G_j and it follows that $\mathscr{S}$ satisfies (N).

It is easy to see that G need not have the same complexion as G/Z_∞. The importance of the complexion of a GSTG arises in connection with the direct product of GSTG's. If the direct product of groups A and B is a GSTG of complexion R, 1.2 shows that both A and B are GSTG's of complexion R. The converse is the following.

PROPOSITION 1.5. *If H and K are* GSTG's *of (the same) complexion R, the direct product $H \times K$ is also a* GSTG *of complexion R.*

Proof. Let $\mathscr{H}$ and $\mathscr{K}$ be (resp.) Sylow bases of H and of K. Construct a complete set of Sylow subgroups of $H \times K$, say $\mathscr{D}$, by forming the appropriate products of subgroups from $\mathscr{H}$ and $\mathscr{K}$. Then $\mathscr{D}$ satisfies (N) and $H \times K$ is a GSTG of complexion R.

PROPOSITION 1.6. *Let H and K be normal subgroups of G. If the factor groups G/H and G/K are both* GSTG's *of complexion R, then $G/H \cap K$ is a* GSTG *of complexion R.*

Proof. This follows immediately since $G/H \cap K$ is isomorphic to a subgroup of $G/H \times G/K$.

If R is any relation on the set of all primes, then 1.3 and 1.6 together show that the class of all GSTG's with complexion R is a formation in the sense of Gaschütz. However, since the direct product of GSTG's need be a GSTG, the class of all GSTG's is not a formation.

An interesting property of GSTG's and a connection between GSTG's and Sylow tower groups may now be shown.

THEOREM 1.7. *Let G be a* GSTG. *Then the nilpotent length of G is at most $\pi(G)$, the number of distinct prime divisors of the order of G.*

Proof. If $\pi(G) \leqq 2$, G has a normal Sylow subgroup and clearly $l(G) \leqq \pi(G)$. Suppose $\pi(G) = k \geqq 3$ and let $\mathscr{S}$ be a Sylow basis of G which satisfies (N). We proceed by induction on $|G|$.

Suppose first that G has a nontrivial, normal Sylow subgroup T. By induction, the factor G/T then has nilpotent length at most $k-1$. Hence G has nilpotent length at most $k = \pi(G)$.

Assume now that G has no nontrivial, normal Sylow subgroup. Since G is solvable, G has a nontrivial normal p-subgroup for some prime p. Let P denote the maximal normal p-subgroup of G and G_p the Sylow p-subgroup of G belonging to $\mathscr{S}$. Since $G_p \ntrianglelefteq G$, there is a prime q, $q \neq p$, and Sylow q-subgroup G_q belonging to $\mathscr{S}$ for which G_p normalizes G_q. Consequently $G_q \leqq G_G(P)$ and so $\pi(C_G(P)) \geqq 2$.

Let $P_0 = C_G(P) \cap P$, the maximal normal p-subgroup of $C_G(P)$, and let V/P_0 denote a nontrivial normal subgroup of $C_G(P)/P_0$ of prime power order r^α (r some prime). By the maximality of $P_0, r \neq p$. If W/P_0 denotes the maximal normal r-subgroup of $C_G(P)/P_0$, then W/P_0 is characteristic in $C_G(P)/P_0$ and hence $W \trianglelefteq G$. Since $W \leqq C_G(P)$ and W/P_0 is an r-group, W has normal Sylow r-subgroup R. Then R is characteristic in W, hence normal in G.

By induction, the factors G/P and G/R both have nilp. length at most k. Since $P \cap R = 1$, G is isomorphic to a subgroup of the direct product $G/P \times G/R$. Therefore $l(G) \leqq \max\{l(G/P), l(G/R)\} \leqq k = \pi(G)$.

THEOREM 1.8. *If G is a* GSTG *and $l(G) = \pi(G)$, then G is a Sylow tower group.*

For the proof of this, the following lemma is helpful.

LEMMA. *Let G be a Sylow tower group with $l(G) = \pi(G)$. If S and T are distinct Sylow subgroups of G, then S and T do not centralize one another.*

Proof. Suppose the assertion false and let G be a counterexample of minimal order. Then G is a Sylow tower group with $l(G) = \pi(G) = n \geqq 3$. Let

$$1 \triangleleft S_1 \triangleleft S_2S_1 \triangleleft \cdots \triangleleft S_nS_{n-1} \cdots S_1 = G$$

be a Sylow tower of G, where S_i denotes a Sylow p_i-subgroup of G for $i = 1, 2, \cdots, n$. The factor group G/S_1 is a Sylow tower group with $l(G/S_1) = n - 1 = \pi(G/S_1)$. Therefore, by the minimality of $|G|$, no distinct Sylow subgroups of G/S_1 centralize one another.

By assumption, some Sylow p_j-subgroup T of G centralizes some Sylow p_k-subgroup V of G (for some j, k distinct). Then TS_1/S_1 centralizes VS_1/S_1 and consequently one of these Sylow subgroups of G/S_1 is trivial. Suppose TS_1/S_1 is trivial. Then $j = 1$ and it follows that S_k centralizes S_1. Choose $m > 1$ to be the smallest integer for which p_m divides $|C_G(S_1)|$ and put $R = S_m \cdot S_{m-1} \cdots S_1 \cap C_G(S_1)$. Then R is normal in G and has order $p_1^\alpha p_m^\beta$. If W denotes the Sylow p_m-subgroup of R, then W is characteristic in R and normal in G.

We may assume W is not a Sylow subgroup of G. Otherwise WS_1 is a normal nilpotent subgroup of G and $l(G) \leqq n - 1$, which is a contradiction. Hence $S_k W/W$ is a nontrivial Sylow p_k-subgroup of G/W which centralizes $S_1 W/W$. Since $\pi(G/W) = n$, the minimality of G implies $l(G/W) < n$.

Since $W \cap S_1 = 1$, G is isomorphic to a subgroup of $G/W \times G/S_1$. Therefore $l(G) \leqq \max\{l(G/S_1), l(G/W)\} < n$, which is the desired contradiction.

Proof of 1.8. Let G be a GSTG with $l(G) = \pi(G)$. If $\pi(G) \leqq 2$, G has a normal Sylow subgroup and clearly G is a Sylow tower group. Take $l(G) = \pi(G) = n \geqq 3$ and proceed by induction.

Suppose first that G has a nontrivial normal Sylow subgroup K. Then G/K is a GSTG with $l(G/K) = n - 1 = \pi(G/K)$ and induction implies G/K is a Sylow tower group. Then G is also a Sylow tower group as asserted. Now assume G has no nontrivial normal Sylow subgroup. As in the proof of 1.7, G then has nontrivial normal subgroups M_1 and M_2 with $|M_1| = p^\alpha$, $|M_2| = q^\beta$ for p, q distinct primes. If both G/M_1 and G/M_2 have nilpotent length less than n, then $l(G) < n$, a contradiction. Therefore we may assume $l(G/M_1) = n$. Since $\pi(G/M_1)$ is also n, induction shows G/M_1 is a Sylow tower group. Let

$$\bar{1} = M_1/M_1 \lhd S_1M_1/M_1 \lhd S_2S_1M_1/M_1 \lhd \cdots \lhd S_n \cdots S_1M_1/M_1 = G/M_1$$

be a Sylow tower of G/M_1, where S_i is a Sylow p_i-subgroup of G for $i = 1, 2, \cdots, n$. In addition, choose S_n to be the Sylow p_n-subgroup of G belonging to $\mathscr{S}$, where $\mathscr{S}$ is a Sylow basis of G which satisfies (N) and contains S_1.

We first show that $M_1 \leqq S_n$. If this were not the case, then $M_1 \leqq S_k$ for some $k < n$. Then $H = S_k \cdots S_2S_1$ is a normal Hall subgroup of G and, by induction, H is a Sylow tower group. Therefore

H has a normal Sylow subgroup K. But then K is a normal Sylow subgroup of G, which contradicts our assumption about G.

By our choice of S_1 and S_n, either $S_1 \leqq N_G(S_n)$ or $S_n \leqq N_G(S_1)$. If $S_n \leqq N_G(S_1)$, $[S_1, M_1] = 1$ and consequently $S_1 \lhd G$ — which is impossible. Therefore we may assume $S_1 \leqq N_G(S_n)$. Then the Sylow p_1-subgroup S_1M_1/M_1 of G/M_1 centralizes the Sylow p_n-subgroup S_n/M_1 of G/M_1, which contradicts the lemma. Therefore G must have a normal Sylow subgroup and the theorem is proved.

The following construction shows the existence of a GSTG G with nilpotent length n and $\pi(G) = m + n$, where m and n are arbitrary nonnegative integers and $n \neq 0$. This shows in particular that the inequality of Theorem 1.7 cannot be improved.

If A and B are groups, let A or B denote the wreath product of A by B. For nonnegative integers m and n with $n \neq 0$, choose distinct primes $p_1, \cdots, p_{m+1}, q_1, \cdots, q_{n-1}$. Let D be the cyclic group of order $p_1 \cdots p_{m+1}$ and C_j the cyclic group of order q_j (for $j = 1, \cdots, n-1$). Then the repeated wreath product

$$G = [\cdots ((D \operatorname{wr} C_1) \operatorname{wr} C_2) \cdots] \operatorname{wr} C_{n-1}$$

is a GSTG with $l(G) = n$ and $\pi(G) = m + n$.

2. N-groups. In the preceding section groups for which some complete set of Sylow subgroups satisfies (N) were examined. Now we examine groups with the property that every complete set of Sylow subgroups satisfies (N). This is equivalent to demanding that the collection of all Sylow subgroups of a group satisfies (N). We call such groups N-groups.

Any nilpotent group extended by a p-group is an N-group. And clearly every N-group is a GSTG and hence solvable. An N-group need not be a Sylow tower group however, as the following example shows. For m and n any positive integers, let $\operatorname{wr}(m, n)$ denote the wreath product of a cyclic group of order m by a cyclic group of order n. Let p_1, p_2 and p_3 be distinct primes and put $A = \operatorname{wr}(p_1, p_2)$, $B = \operatorname{wr}(p_3, p_1)$ and $C = \operatorname{wr}(p_2, p_3)$. Then the direct product of A, B and C is an N-group which is not a Sylow tower group.

Since any two Sylow bases of a solvable group are conjugate [4, p. 321], the following holds.

PROPOSITION 2.1. *If G is an N-group and some Sylow p-subgroup of G normalizes some Sylow q-subgroup of G, then every Sylow p-subgroup of G normalizes every Sylow q-subgroup of G (p and q distinct primes).*

In many respects N-groups behave like GSTG's. Define an N-group G to have complexion R when G considered as a GSTG has complexion R. Then 1.2–1.6 remain valid when "GSTG" is replaced by "N-group". Therefore the class of all N-groups having complexion R is a formation. This formation will not necessarily be saturated, since G need not be an N-group whenever $G/\phi(G)$ is an N-group (see [6, p. 265]).

Although G need not be an N-group whenever $G/\phi(G)$ is an N-group, the following holds.

PROPOSITION 2.2. *If G is a group of order $p^\alpha q^\beta$, where p and q are primes, and $G/\phi(G)$ is an N-group, then G is an N-group.*

Proof. We may assume that $G/\phi(G)$ has normal Sylow p-subgroup $G_p\phi(G)/\phi(G)$, where G_p is any Sylow p-subgroup of G. Let W be the normalizer of G_p in G. If $W = G$, G_p is normal in G and G is an N-group. Otherwise W lies in some maximal subgroup M of G. Let x be an element of G not in M. Since

$$G_p\phi(G)/\phi(G) \lhd G/\phi(G), \quad G_p^x \leqq G_p^x\phi(G) \leqq M$$

and there is an element $y \in M$ with $G_p^y = G_p^x$. Then $x \in M$, which is impossible.

A similar argument establishes the following.

PROPOSITION 2.3. *If $G/\phi(G)$ is an N-group and all maximal subgroups of G are N-groups, then G is an N-group.*

Unlike GSTG's, every N-group necessarily satisfies property (V) of Huppert. This observation leads to an upper bound for the nilpotent length of an N-group.

PROPOSITION 2.4. *If G is an N-group, then $l(G) \leqq 2$.*

Proof. Since G satisfies property (V), Satz 2 [6, p. 253] shows that $G/\phi(G)$ is isomorphic to a subgroup, H (say), of a direct product of groups $T_1, \cdots, T_n$, where $|T_i| = p_i^{a_i}q_i^{b_i}$ with p_i and q_i primes. For each integer k, $1 \leqq k \leqq n$, define the homomorphism π_k of H into T_k by $\pi_k(t_1 \cdots t_n) = t_k$; $t_1 \cdots t_n \in H$ and $t_i \in T_i$ for each i. Now $\pi_k(H)$ is an N-group, being a homomorphic image of G. And since $\pi_k(H)$ is a subgroup of T_k, the nilpotent length of $\pi_k(H)$ is at most 2. Therefore $l(G) = l(H) \leqq \max_{1 \leqq k \leqq n} \{l(\pi_k(H))\} \leqq 2$.

N-groups may now be described relative to property (V).

THEOREM 2.5. *Let G satisfy (V). Then G is an N-group if, and only if*

(i) *G is a partially complemented extension of a nilpotent group H by a nilpotent group K, and*

(ii) *for distinct primes p and q, the Sylow p-subgroup H_p of H normalizes the Sylow q-subgroup K_q of K or the Sylow q-subgroup H_q of H normalizes the Sylow p-subgroup K_p of K.*

Proof. Let G be an N-group with nilpotent length 2. Then [5, p. 211] there exist nilpotent subgroups H and K of G with $G = HK$, $H \lhd G$. To verify (ii), let p and q be any distinct prime divisors of $(|H|, |G/H|)$. $L_1 = H_pK_pK_q$ is an N-group of order $p^\alpha q^\beta$ and hence $K_q \leqq N_G(H_pK_p)$ or $H_pK_p \leqq N_G(K_q)$. We may assume $K_q \leqq N_G(H_pK_p)$ and $H_pK_p \not\leqq N_G(K_q)$. Since L_1 and G have the same complexion, this means H_qK_q normalizes H_pK_p. Similarly, considering $L_2 = H_qK_qK_p$ we may assume H_pK_p normalizes H_qK_q. Therefore H_pK_p and H_qK_q centralize one another and we have established (ii).

For the converse, let G satisfy (V) and both (i) and (ii). If G is a $p^\alpha q^\beta$ group, it easily follows that G must be an N-group.

Suppose now that $\pi(G) = k \geqq 3$ and let p and q be prime factors of the order of G. Let G_p and G_q be any Sylow p- and q-subgroups of G (respectively). Then by Sylow arguments

$$G_p = H_pK_p^x \quad \text{for some} \quad x \in G$$
$$G_q = H_qK_q^y \quad \text{for some} \quad y \in G\,.$$

Since G satisfies (V), $L = (H_pK_p)(H_qK_q^{yx^{-1}})$ is a subgroup of G. L satisfies both (i) and (ii) and hence, by induction, is an N-group. Therefore $H_pK_p \leqq N_G(H_qK_q^{yx^{-1}})$ or $H_qK_q^{yx^{-1}} \leqq N_G(H_pK_p)$. Then G_p normalizes G_q or G_q normalizes G_p and we have shown G is an N-group.

In the previous theorem, if all Sylow subgroups of G are abelian, then "partial complement" can be replaced by "complement". An example can be given to show that "partial complement" cannot be improved to "complement" in general.

3. Strongly Sylow towered groups.

Since every finite solvable group possesses Sylow systems, it seems worth-while to consider property (N) in connection with these collections of subgroups. If G is a solvable group and some Sylow system of G satisfies (N), we call G a strongly Sylow towered group (SSTG). It is easily shown that every SSTG is necessarily a Sylow tower group. However a SSTG need not be an N-group. For example, the holomorph of the cyclic group of order 7 is an SSTG but not an N-group.

The inheritence properties of SSTG's resemble those of GSTG's. For instance, subgroups and homomorphic images of SSTG's are again SSTG's and the direct product of SSTG's with similar normalizing structures is again an SSTG.

THEOREM 3.1. *A group G is an* SSTG *if, and only if, G is a split extension of a nilpotent group A by a nilpotent group B where A and B have coprime orders and either A or B is a p-group.*

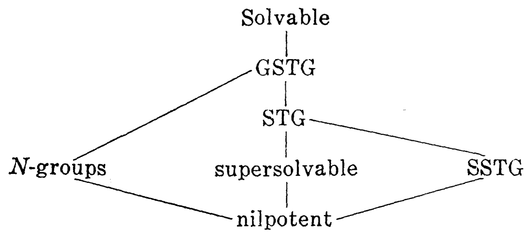
Proof. Let $\mathscr{S}$ be a Sylow system of G which satisfies (N). Let S and T denote any distinct nonnormal Sylow subgroups of G belonging to $\mathscr{S}$ and let S' and T' be (respectively) the complements of S and T in $\mathscr{S}$. Since $\mathscr{S}$ satisfies (N), both S' and T' are normal in G. Furthermore, we may assume S normalizes T. Then $[S, T] \leqq T' \cap T = 1$ and hence any distinct nonnormal Sylow subgroups of G in $\mathscr{S}$ centralize one another. Let A denote the product of all normal Sylow subgroups of G and B the product of the nonnormal Sylow subgroups of G in $\mathscr{S}$. Then A and B are nilpotent and $G = A \cap B = 1$ and $A \lhd G$.

If A or B is a p-group, we are done. Assume not and let $\pi(A) = k \geqq 2, \pi(B) = l \geqq 2$. Let $N_1, \cdots, N_k$ be the normal Sylow subgroups of G and $S_1, \cdots, S_l$ the nonnormal Sylow subgroups of G in $\mathscr{S}$. Since $S_1 \ntrianglelefteq G$, we may assume $N_1 \nleqq N_G(S_1)$. Then, for any positive integers $i, j \geqq 2$, $N_1S_j \leqq N_G(N_iS_1)$ or $N_iS_1 \leqq N_G(N_1S_j)$. Consequently we have $N_i \leqq C_G(S_j)$ whenever $i, j \geqq 2$. It follows that $N_1 \nleqq N_G(S_j)$; $j = 1, \cdots, l$.

Now suppose N_i does not normalize S_1, some $i \geqq 2$. Since N_1S_1 and N_iS_2 are subgroups of G belonging to f, either $N_1S_1 \leqq N_G(N_iS_2)$ or $N_iS_2 \leqq N_G(N_1S_1)$. Then $N_1 \leqq C_G(S_2)$ or $N_i \leqq C_G(S_1)$, a contradiction. Therefore we may assume N_i normalizes S_1 for all $i \geqq 2$. Then $H = N_2 \cdots N_kS_1 \cdots S_l$ is a nilpotent subgroup of G and G is the split extension of N_1 by H.

The converse follows easily. For, if G is the split extension of a p-group A by a nilpotent group B of coprime order and having Sylow subgroups $B_1, \cdots, B_n$, then the Sylow basis $\{A, B_1, \cdots, B_n\}$ generates a Sylow system of G which satisfies (N). Next, suppose G is the split extension of a nilpotent group A by a p-group B with $(|A|, p) = 1$. If $A_1, \cdots, A_n$ are the Sylow subgroups of A, then $\{A_1, \cdots, A_n, B\}$ generates a Sylow system of G which satisfies (N).

The diagram below illustrates the connections between the classes of solvable groups considered here and other well-known classes.



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