ON AUTOMORPHISMS OF SEPARABLE ALGEBRAS II

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A Galois theory for separable projective algebras over a
class of commutative rings more general than Von Neumann
regular rings with includes the full group of algebra auto-
morphisms is presented. Fundamental facts concerning the
group of units and the automorphisms of these algebras are
also given.

A Galois theory for simple algebras over fields which included the
full group of automorphisms was presented by G. Hochschile [5]. After
the work of M. Auslander and O. Goldman [1] and S. Chase, D. K.
Harrison, and A. Rosenberg [3] it was possible to generalize Hochschild's
theory to semi-local rings with no idempotents other than 0 and 1 [4].
More recent results of O. Villamayor and D. Zelinsky [14], and H.
Kreimer [8] permit the extension of Hochschild's theory to a much
broader class of rings than in [4]. The context includes separable
projective algebras over commutative regular rings and is stated ex-
plicitly at the beginning of § 3. Theorem 3 and Theorem 5 of § 2
have also been obtained by A. Magid and will appear in the Illinois
Journal of Mathematics as part of an article entitled *Pierce's repre-
sentation and separable algebras.*

We first state some of the results of [14] without proof. The
Galois theory in [14] is then improved in a special case. Next, in
§ 2, several basic properties of separable algebras are developed. The
Galois theory which generalizes [4] is presented in § 3. Henceforth R
always is a commutative ring with identity. All rings have identity,
all subrings share the common identity, all modules are unitary, and
all algebra homomorphisms carry identity to identity. Also $\otimes$ will
mean $\otimes_R$. A familiarity with the basic properties of separable algebras
is assumed. In particular, we employ a result of Villamayor [10] in
the sequel which asserts that any separable projective R-algebra is
finitely generated as an R-module.

1. We first enumerate, with corresponding nomenclature, several
results in [14]. Let $B(R)$ be the Boolean algebra of idempotents of
the commutative ring R. If $e, f \in B(R)$ the Boolean operations are
given by $e \oplus f = e + f - ef$ and $e * f = ef$. Let spec $B(R)$ be the space
of maximal (= prime) ideals in $B(R)$ and call the subsets U_e of spec
$B(R)$ defined for each $e \in B(R)$ by

$$U_e = \{p \in \operatorname{spec} B(R) \mid e \in p\}$$

a base for the open sets of spec $B(R)$. In this topology spec $B(R)$ is a totally disconnected compact Hausdorff space. If $p \in \operatorname{spec} B(R)$ we let $R_p = R/p \cdot R$, then R_p is a homomorphic image of R. If M is an R-module then $R_p \otimes M = M/pM$ is denoted M_p and the natural image of $a \in M$ in M_p is denoted a_p. In most of what follows we will use the compactness of spec $B(R)$ to recover information true at R_p for each $p \in \operatorname{spec} B(R)$ to information about R.

The next seven results are proved in [14].

(2.7)   If M and N are finitely generated projective R-modules the natural map from $\operatorname{Hom}_R(M, N)$ to $\operatorname{Hom}_{R_p}(M_p, N_p)$ denoted $f \to f_p$ is an isomorphism.

(2.9)   If a and b are elements of the finitely generated R-module M and $a_p = b_p$ for one $p \in \operatorname{spec} B(R)$ then $a_q = b_q$ for all q in some neighborhood U_e of p.

(2.11)   If N is a submodule of the finitely generated R-module M and $N_p = M_p$ for all $p \in \operatorname{spec} B(R)$ (R_p is flat over R so $N_p c M_p$) then $N = M$.

(2.12)   If S is a finitely generated R-algebra and u is an idempotent in S_p for some $p \in \operatorname{spec} B(R)$, then there is an idempotent v in S such that $v_p = u$.

(2.13) R_p has no idempotents other than 0 and 1 for each $p \in \operatorname{spec} B(R)$.

(2.14)   Suppose S is an R-algebra that is finitely presented as an R-module, let F be a finite subset of S and let g be an R_p-algebra homomorphism of S_p which is the identity on F_p. Then there is an R-algebra homomorphism h of S such that h is the identity on F and $h_p = g$.

(2.17)   If S is an R-algebra and J is a finite group of R-algebra automorphisms of S; then for all $p \in \operatorname{spec} B(R)$, $(S^J)_p = (S_p)_p^J$ where S^J denotes the fixed ring under J.

Let K be a commutative ring containing R as a subring.

If K is a projective separable R-algebra and if $K^G = R$ where G is the group of all R-automorphisms of K, then K is called a normal separable extension of R with group G. If H is a subset of G, the closure $\bar{H}$ of H is the set of all automorphisms g of K such that for each $p \in \operatorname{spec} B(R)$ and each minimal idempotent f_p in K_p, $f_p \cdot g_p = f_p \cdot h_p$

for some $h \in H$. A closed subgroup of G is one equal to its own closure. Another result from [14] is

LEMMA 1. *Let K be a normal separable extension of R with group G, then the following statements are equivalent.*

(1) *For each $p \in \operatorname{spec} B(R)$, K_p is a normal separable extension of R_p with group G_p.*

(2) $\operatorname{Hom}_R (K, K)$ *is generated as a K-module by R-algebra automorphism of K.*

(3) R *is the fixed ring of some finite set (equivalently, finite group) of automorphisms of S.*

An example in [14] shows that when K is a normal separable R-algebra the equivalent conditions of Lemma 1 need not always hold. It is only in the situation where these conditions hold that the techniques we employ are effective. As a consequence of Lemma 1 we can restate Theorem 3.8 of [14].

THEOREM 3.8. *Let K be a normal separable R extension of R with group G, if there is a finite subset J of G with $K^J = R$ then there is a one to one correspondence between the set of all separable subalgebras of K and the set of all subgroups H of G which are the closure of some finite subgroup of G by $H \to K^H$.*

The result we require is in the situation where every idempotent in K belongs to R. In this case Theorem 3.8 can be improved.

COROLLARY 2. *Let K be a normal separable extension of R with group G. Assume R is the fixed ring of a finite subset of G and assume every idempotent in K belongs to R. Then there is a one-to-one correspondence between the separable subalgebras of K containing R and the closed subgroups H of G by $H \to K^H$. The closed subgroup H of G is a normal subgroup of G if and only if the corresponding subalgebra is a normal extension of R with group G/H.*

Proof. We apply Theorem 3.8 to give the subgroup-subring correspondence. Assume H is a closed normal subgroup of G, then G/H acts as a group of automorphisms of K^H in the usual way and $(K^H)^{G/H} = R$. Let σ be an R-automorphism of K^H. By hypothesis on the idempotents in K and Theorem 3.5 of [3] we have for each $p \in \operatorname{spec} B(R)$ an automorphism τ_p of K_p which extends σ_p on K_p^{Hp}. Use (2.14) to lift τ_p to an automorphism τ on K. Since K^H is finitely generated and projective over R (Proposition 1.5 of [4]), $\operatorname{Hom}_R (K^H, K)$ is finitely generated over R so there is by (2.9) and (2.7) a neighborhood

U of p such that for each $q \in U, \tau_q$ restricted to $(N^H)_q$ is σ_q. Cover spec $B(R)$ with such neighborhoods and use compactness of spec $B(R)$ to obtain a decomposition $R = Re_1 \oplus \cdots \oplus Re_n$ and automorphisms τ_i of Ke_i with τ_i restricted to $(K^H)e_i$ equal to σ on $(K^H)e_i$ (since $e_i \in R$, $\sigma(e_i) = \tau(e_i) = e_i$). The automorphism τ on K constructed from the τ_i extends σ and is in G so K^H is a normal separable extension of R with group G/H. Conversely, assume H is a closed subgroup of G and K^H is a normal separable extension of R. Then by Theorem 2.3 of [3] and (2.17) we know that for each $p \in \mathrm{spec}\, B(R)$ and each $\sigma \in G$

$$\sigma_p(K^H)_p = \sigma_p(K_p^{Hp}) = K_p^{Hp} = (K^H)_p \,.$$

Since K^H is finitely generated over R, (2.7) implies $\sigma(K^H) = K^H$. This is equivalent to the statement that H is a normal subgroup of G and completely proves the corollary.

With the hypothesis of Corollary 2, if N is a normal separable extension of R in K which corresponds to the closed normal subgroup H of G observe that there is a finite subset of G/H on N leaving exactly R fixed. Therefore our Galois theory also applies to the extension N of R with group G/H.

2. We now derive some properties of separable algebras. The group of isomorphism classes of rank $= 1$ projective R-modules forms a group called the class group (Picard group) of R. Generalizing Theorem 3.6 of [1] we have

THEOREM 3. *Let A be a central separable R-algebra. Assume the class group of R_p is trivial for each $p \in \mathrm{spec}\, B(R)$, then every R-algebra automorphism of A is an inner automorphism.*

Proof. Let σ be an algebra automorphism of A. By Theorem 3.6 of [1] we know σ_p induces an inner automorphism of A_p for each $p \in \mathrm{spec}\, B(R)$. Let $\sigma_p = i_{\bar{x}}$ on A_p, then there exists a $\bar{y} \in A_p$ with $\bar{x}\bar{y} = 1$. Lift $\bar{x}$ and $\bar{y}$ to elements x' and y' in A. By (2.9) there is a neighborhood U_e of p so that for all $q \in U_e$ we have $x_q'y_q' = 1$. Therefore $(x'e)(y'e) = e$ and $(x'e)_p = \bar{x}$, $(y'e)_p = \bar{y}$. Let $x = (1 - e) + x'e$ and $y = (1 - e) + y'e$. Then $x, y \in A$ and $xy = 1$. Moreover i_x is an inner automorphism of A with $(i_x)_p = i_{\bar{x}}$. Now $\mathrm{Hom}_R(A, A)$ is a finitely generated projective R-module so by (2.9) there is a neighborhood U of p in spec $B(R)$ with $(i_x)_q = \sigma_q$ for all $q \in U$. Cover spec $B(R)$ with such neighborhoods, since spec $B(R)$ is compact one can find a finite subcover. Thus $R = Re_1 \oplus \cdots \oplus Re_n$ with $e_i^2 = e_i \in R$ and there are elements $x_j \in Ae_j (j = 1, \cdots n)$ such that x_j is a unit in Ae_j and σ on Ae_j is inner automorphism by x_j. Let $x = x_1 + \cdots + x_n$, then σ is inner automorphism by x on A.

A corollory of this result shows that we are in the context of [9].

COROLLARY 4. *Let A be a separable projective R-algebra with center K, and assume the class group of K_p is trivial for each $p \in \operatorname{spec} B(R)$, then every R-algebra automorphism of A which is not inner is represented nontrivially on K.*

Proof. For $p \in \operatorname{spec} B(R)$, K_p is a finitely generated projective separable R_p-algebra and R_p has no idempotents other than 0 and 1. Thus C_p is a finite direct sum of finitely generated projective separable R_p-algebras whose only idempotents are 0 and 1. The class group of K_p is the product of the class groups of the summands and the summands correspond to the elements of $\operatorname{spec} B(K)$ so Theorem 3 applies.

Our inability to extend the next result restricts the generality of the Galois theory in § 3. See the examples following Theorem 1.1 and Theorem 1.2 of [4].

THEOREM 5. *Let R be a commutative ring, assume for each $p \in \operatorname{spec} B(R)$ that R_p is a semi-local ring, (finite number of maximal ideals). Let A be a separable projective R-algebra with center K and let B be a separable R-subalgebra of A with center C containing K. Assume every idempotent in C belongs to R. Let σ be an R-algebra monomorphism from B into A leaving K fixed, then σ can be extended to an inner automorphism of A.*

Proof. We know for any $p \in \operatorname{spec} B(R)$ that A_p is a projective separable R_p algebra with center K_p and B_p is a separable subalgebra with center C_p containing K_p. By hypothesis on the idempotents in C it follows that the only idempotents in C_p are 0 and 1. By the lemma on page 25 of [4], B is projective over R so $\operatorname{Hom}_R(B, A)$ is a finitely generated projective R-module and (2.7) implies $\operatorname{Hom}_R(B, A)_p = \operatorname{Hom}_{R_p}(B_p, A_p)$. With this identification, Theorem 1.2 of [4] gives an inner automoróphism $i_{\bar{x}}$ on A_p which extends σ_p on B_p. As in the proof of Theorem 3 lift $\bar{x}$ to a unit x in A so that i_x is an inner automorphism of A with $(i_x)_p = i_{\bar{x}}$. Apply (2.9) to get a neighborhood U_e of p so that for all $q \in U_e$, $\sigma_q = (i_x)_q$ on B_q. Applying the same compactness argument as used in Theorem 3 we construct an inner automorphism i_y on A which extends σ.

If R is a regular ring (in the sense of Von Neumann) then R satisfies the hypothesis of Theorem 5 since every principal ideal in a regular ring is generated by an idempotent. Thus if R is regular, R_p is a field for each $p \in \operatorname{spec} B(R)$. The next result improves Theorem

1.5 of [4].

THEOREM 6. *Let A be a separable projective algebra over K and let J be a finite group of ring automorphisms of A with the property that J restricted to K is a group of automorphisms of K isomorphic to J. Let $R = K^J$ and assume K is a separable finitely generated R-algebra with every idempotent in K belonging to R. If $B = A^J$, then B is a separable projective R-algebra and $A = B \otimes K$.*

Proof. The hypothesis of Theorem 1.5 in [4] is satisfied at each $p \in \operatorname{spec} B(R)$. Therefore the natural homomorphism from $B \otimes K$ to A is an isomorphism at each $p \in \operatorname{spec} B(R)$. Applying (2.7) and (2.11) we conclude $B \otimes K = A$. Now K is a finitely generated, projective, separable R-algebra so $R \cdot 1$ is an R-direct summand of K (see Proposition A.4 of [1]). By ([2], IX, 9.1) B is separable over R. By the lemma on page 2.5 of [4] B is projective over R.

The next result includes a correction of Lemma 1.8 in [4]. Let Z be the ring of integers.

LEMMA 7. *Let A be a separable projective R-algebra, then A is generated as an algebra by its units if and only if A_p is generated as an R_p algebra by its units for all $p \in \operatorname{spec} B(R)$. Moreover, if R_p is semi-local then A_p is not generated as an algebra by its units if and only if $R_p/\operatorname{Rad}(R_p) = Z/(2) \oplus R'$ and the central component of $A_p/\operatorname{Rad}(A_p)$ over $Z/(2)$ contains a direct factor isomorphic to $Z/(2) \oplus Z/(2)$.*

Proof. Assume that A_p is generated by its units for all $p \in \operatorname{spec} B(R)$. As in the proof of Theorem 3, each unit in A_p lifts to a unit in A. Thus if N is the R-subalgebra of A generated by the units of A we know $N_p = A_p$ for all $p \in \operatorname{spec} B(R)$. By (2.11) this implies $N = A$. The converse is clear.

For the last statement observe that the proof of Lemma 1.8 in [4] implies A_p is generated by its units over R_p if and only if $A_p/\operatorname{Rad}(A_p)$ is generated by its units over $R_p/\operatorname{Rad}(R_p)$. Now $R_p/\operatorname{Rad}(R_p)$ is a finite direct sum of fields and $A_p/\operatorname{Rad}(A_p)$ is a finite direct sum of *semi*-simple algebras over fields. It is an elementary exercise to show that the only semi-simple algebras over a field not generated by units occurs when the algebra contains $Z/(2) \oplus Z/(2)$ as a direct summand over the field $Z/(2)$. This proves the lemma.

More information about automorphisms of separable algebras is contained in

LEMMA 8. *Let A be a projective separable R-algebra with center K and assume every idempotent in K belongs to R, then any R-homomorphism σ from A to A is an automorphism of A.*

Proof. Observe that σ is an R-homomorphism from K to K. Consider the exact sequence

$$0 \longrightarrow \ker(\sigma) \longrightarrow K \xrightarrow{\sigma} \sigma(K) \longrightarrow 0 .$$

Now $\sigma(K)$ is a separable R-subalgebra of K so by the lemma on page 25 of [4] $\sigma(K)$ is projective over R. Therefore $\ker(\sigma)$ is an ideal direct summand and is generated by an idempotent in K. But every idempotent in K belongs to R and σ is one-to-one on R so σ is one-to-one on K. For each $p \in \operatorname{spec} B(R)$ we know $\sigma_p \colon K_p \to K_p$ is one-to-one. K_p has no idempotents other than 0 and 1 and K_p can be imbedded in a Galois (in the sense of [3]) extension of R_p by [6] so Theorem 2.3 of [3] and the fact that $\operatorname{Rank}_{R_p}(K_p) = \operatorname{Rank}_{R_p}(\sigma_p(K_p))$ implies σ_p is also onto K_p. By (2.11) and (2.7) this implies $\sigma(K) = K$. Since there is a one-to-one correspondence between the two sided ideals of A and those of K by $a \, cK$ corresponds to Aa we know $\ker(\sigma) = 0$ on A and σ is one to one on A.

Now $\sigma(A)$ is a central separable K-algebra (since $\sigma(K) = K$) and $\operatorname{Rank}_K(A) = \operatorname{Rank}_K(\sigma(A))$ since $\sigma(A)$ is one to one. By Theorem 3.3 of [1], $A = \sigma(A) \otimes_K A'$ where A' is a central separable K-algebra. By a rank argument, $A' = K$ and σ maps A onto A. This proves the lemma.

LEMMA 9. *Let K and L be commutative separable R-subalgebras of the finitely generated projective R-algebra A, then $K \cap L$ and $K \cdot L$ are separable projective R-subalgebras of A.*

Proof. First, $K \cdot L$ is a homomorphic image of $K \otimes L$ and therefore is separable over R. By the lemma on page 25 of [4] $K \cdot L$ is projective over R. To prove $K \cap L$ is separable over R we employ the construction in §3 of [12] to find a commutative projective separable R-algebra N containing $K \cdot L$ such that there is a finite group J of automorphisms of N with $N^J = R$. By Theorem 3.8 there are finite sets H, H' of R-automorphisms of N with $N^H = K$ and $N^{H'} = L$. Observe that $N^{H \cdot H'} = K \cap L$ which by Theorem 3.8 implies $K \cap L$ is projective and separable over R.

3. Throughout this section we shall adhere to the following notations and assumptions. R is a commutative ring with R_p with semi-local for each $p \in \operatorname{spec} B(R)$. A is a separable projective R-algebra with

center K, and every idempotent of K belongs to R. Let G be the group of all R-algebra automorphisms of A and assume $A^G = R$. Also we assume there is a finite subset of G whose restrictions give a finite set J of automorphisms of K with $K^J = R$.

If H is a subgroup of G we let H_0 be the normal subgroup of H consisting of those elements of H which are inner automorphisms of A. We let $R(H)$ be the subring of A generated as a K-algebra by all the units giving the inner automorphisms in H. If B is an R-subalgebra of A we let G_B be the group of all elements in G leaving B elementwise fixed.

Call a subgroup H of G complete if every inner automorphism of A by an element of $R(H)$ is in H and if when restricted to K, every element of G/H_0 which leaves K^H fixed is in H/H_0. We show later that this last condition implies that on K we have H a closed group of automorphisms in the sense defined in § 1. Call a subgroup H of G regular if H is complete, $R(H)$ is a separable K-algebra, and every central idempotent in $R(H)$ belongs to K.

If B is an R-algebra of A we call B regular if B is separable as an R-algebra, if every idempotent in the center of $B \otimes_{B \cap K} K$ is in K, and if the commutator of $K \cdot B$ in A is generated as a K-algebra by its units.

If R is a semi-local ring whose only idempotents are 0 and 1 these definitions reduce to [4]. For the same reasons as in the Galois theory for simple algebras over fields, it is the regular subgroups and the regular subalgebras which the Galois theory relates.

THEOREM 10. *If H is a regular subgroup of G, then A^H is a regular R-subalgebra of A, and H is the group of all automorphisms of A leaving A^H fixed.*

Proof. The commutator of $R(H)$ in A is A^{H_0}; thus by Theorem 2 of [7], A^{H_0} is a separable R-algebra. Now H leaves $R(H)$ setwise invariant so H leaves A^{H_0} setwise invariant. Restricting H to A^{H_0} yields a group of automorphisms H' of A^{H_0} leaving A^H fixed which is isomorphic to H/H_0. Similarly H leaves K setwise invariant. Since all automorphisms of A leaving K fixed are inner by Theorem 3, H restricted to K is also isomorphic to H/H_0 and can be viewed as the restriction of H' to K. By our assumptions there is a finite subset J of H' with $K^J = K^{H'} = K \cap A^H$. By Corollary 2, $K \cap H^H$ is separable over R. The center of A^{H_0}, being the same as the center of $R(H)$, is projective and separable over K. Furthermore K is projective and separable over R so A^{H_0} is projective and separable over R. By Theorem 6 we know $(A^{H_0})^J = B$ is separable over K^J and $A^{H_0} = B \otimes K$

(tensor over K^J). Since K^H is separable over R, B is separable over R. Every idempotent in the center of A^{H_0} is in K by assumption on $R(H)$. Now $A^H c B$ but if $\sigma \in H$ and $\sigma|_B \neq e$ then $\sigma \otimes 1$ is an automorphism of $B \otimes K = A^{H_0}$ which is the identity on K so $\sigma \otimes 1$ extends by Theorem 3 to an element of H_0. This is impossible since $B c A^{H_0}$. Thus $A^H = B$ is a regular subalgebra of A.

To prove the last statement of the theorem observe that H_0 contains all the inner automorphisms leaving A^H fixed. Also $H' = H/H_0$ contains all the automorphisms of A^{H_0} leaving A^H fixed since $A^{H_0} = A^H \otimes K$ (tensor over $A^H \cap K$). Moreover H/H_0 is closed when viewed as a group of automorphisms of K.

THEOREM 11. *Let B be a regular subalgebra of A containing R, then the group G_B is a regular subgroup of B and B is the fixed ring of G_B.*

Proof. If C is the center of B, then $C \otimes_{B \cap K} K$ is the center of $B \otimes_{B \cap K} K$, so $C \cdot K$ is the center of $B \cdot K$ and is separable over R. By Lemma 9 $C \cap K$ is separable over R, therefore $B \cap K$ is separable over R. By our assumption there is a finite subset of G whose restrictions to K leave exactly R elementwise fixed so by Theorem 6 we can write $A = A^G \otimes K$. By Corollary 2 there is a finite subset J' of the R-automorphism group of K with $K^{J'} = B \cap K$. The representation of A as $A^G \otimes K$ insures that every element of J' extends to an element in G/G_0. Next we show the natural map $j \colon B \otimes_{B \cap K} K \to B \cdot K$ is an isomorphism. Now $\ker j = I \cdot (B \otimes_{B \cap K} K)$ where I is the kernel of the restriction of j to $C \otimes_{C \cap K} K$. Consider the exact sequence of $C \otimes_{C \cap K} K$-modules

$$0 \longrightarrow I \longrightarrow C \otimes_{C \cap K} K \longrightarrow C \cdot K \longrightarrow 0 \ .$$

Now $Z \cdot K$ is a projective R-module, so since $C \otimes_{C \cap K} K$ is a separable R-algebra $Z \cdot K$ is a projective $C \otimes_{C \cap K} K$-module. Hence the sequence splits and I is generated by an idempotent. Since every idempotent in $C \otimes_{C \cap K} K$ belongs to K, $I = 0$ and j is an isomorphism.

Extend each σ' in J' to $B \cdot K = B \otimes_{B \cap K} K$ by

$$\sigma'(bk) = b\sigma'(k) \ .$$

Lift σ', viewed as an element of G/G_0, to an element σ of G. Then σ restricted to K is σ' so $\sigma^{-1}\sigma' \colon B \cdot K \to A$ leaves K fixed. Since every central idempotent in $B \cdot K$ belongs to K, $\sigma^{-1}\sigma'$ extends to an inner automorphism τ of A by Theorem 5. Now σ' is $\sigma\tau$ restricted to $B \cdot K$ so σ' extends to an automorphism of A. Each σ' in J' in this way extends to an element of G_B, and since J' defined on $B \cdot K$ leaves ex-

actly B fixed, G_B must leave exactly B fixed.

The commutator L in A of B is the commutator in A of $B \cdot K$, so is finitely generated and separable over K with every central idempotent of L in K. Clearly, any automorphism of G_B comes from a unit in L and every unit in L defines an inner automorphism in G_B. To show that $L = R(G_B)$ and consequently that G_B is regular, it suffices to show that L is generated as a K-algebra by its units. This follows from the definition of regularity for B.

Retain the notation of Theorem 11 in Theorem 12.

THEOREM 12. *If B is a regular subalgebra of A and G_B is a normal subgroup of G, then G/G_B is the group of all algebra automorphisms of B.*

Proof. Set $H = G_B$. The restriction map from G to the automorphism group of B has kernal H. Since H is normal in G, this implies every element in G/H can be viewed as an automorphism of B. The proof will be complete when we extend any R-automorphism σ' on B to an automorphism of A. Now $H' = H/H_0$ is a normal subgroup of $G' = G/G_0$ by the second isomorphism theorem so restricting our attention to K we know by Corollary 2 that $B \cap K$ is a normal separable extension of R with group G'/H' and G'/H' contains all automorphisms of $B \cap K$ leaving R fixed. View σ' as an element of G'/H' and extend to an element σ of G. Then $\tau' = \sigma^{-1}\sigma'$ is an automorphism of B leaving $B \cap K$ fixed. Define τ' on $B \otimes_{B \cap K} K = B \cdot K$ by $\tau'(bk) = \tau'(b)k$; then τ' is an automorphism of $B \cdot K$ leaving K fixed. By Theorem 5, τ' extends to an automorphism τ of G. For any $b \in B$, $\sigma\tau(b) = \sigma\tau'(b) = \sigma(\sigma^{-1}\sigma)(b) = \sigma'(b)$ so $\sigma\tau$ is the desired automorphism.

Following [5] we call a subalgebra B of A almost regular in case B is separable over R, every central idempotent in $B \cdot K$ belongs to K, and the commutator of $B \cdot K$ in A is generated by its units. Every regular algebra is almost regular and in this context the converse also holds.

THEOREM 14. *With A, K, R, and G as above, every almost regular R-subalgebra of A is regular.*

Proof. Let B be an almost regular R-subalgebra of A. Since every idempotent in $B \cdot K$ belongs to K it suffices to show that the natural map

$$j \colon B \otimes_{B \cap K} K \longrightarrow B \cdot K$$

is an isomorphism. As in the proof of Theorem 11 we can restrict our

attention to the natural map $j: C \otimes_{C \cap K} K \to C \cdot K$ where C is the center of B.

Since $C \cap K = L$ is separable over R by Lemma 9 and projective by the lemma on page 25 of [4] both C and K are projective when viewed as L-algebras. Thus $C \otimes_L K = C \cdot K \oplus \ker(j)$. By Theorem 2.3 of [4], $\ker(j) = 0$ for every $p \in \mathrm{spec}\, B(R)$ so $\ker(j) = 0$ and j is an isomorphism.

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