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COMPACT INTEGRAL DOMAINS

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It is well known that if A is a compact integral domain and R is its Jacobson radical, then $A = R$ or A/R is a division ring and A has an identity. The object of this paper is to investigate some of the algebraic properties of A.

If A has an identity and finite characteristic, then there exists a maximal subfield F of A which is isomorphic to A/R. Furthermore A is topologically isomorphic to $F + R$. The existence of a subfield is a necessary and sufficient condition for A to have finite characteristic. If A does not have an identity but does have finite characteristic, then it can be openly embedded in a compact integral domain with an identity. Finally, the main result shows that if the center of A is open, then A is commutative.

1. Preliminaries. An integral domain is defined to be a ring with more than one element such that the nonzero elements form a multiplicative semigroup (which is not necessarily commutative). It is always assumed that a topological ring is Hausdorff.

Throughout this paper A will denote a compact integral domain and R will denote its Jacobson radical.

It has been shown [4, Lemma 2, p. 279] that A has a fundamental system of neighborhoods at zero consisting of open (two-sided) ideals. Kaplansky has shown that R is open [3, Th. 7, p. 161]. Also, that $A = R$ or A/R is a division ring and A has an identity [3, Th. 19, p. 168].

Because A is an integral domain, it can have no elements or ideals that are nilpotent in an algebraic sense, but nilpotency can be defined in a topological sense. We say that an element x is nilpotent if $\lim x^n = 0$, and that an ideal V is nilpotent if for every open set containing zero, there is an integer N such that for every $n \geqq N$, $V^n \subset W$ where V^n is the set of all finite sums of the product of n elements of V.

It has been shown that R is nilpotent [3, Th. 14, p. 163].

The following theorem, which may be thought of as an extension of Wedderburn's theorem, immediately follows from the above results.

THEOREM 1. *Any compact semi-simple integral domain is a finite field.*

2. General properties. In this section it is assumed unless otherwise stated that A is a compact integral domain with an identity. The units and nilpotent elements of A are characterized by the following lemma.

LEMMA 1. *If A is a compact integral domain with an identity and $x \in A$, then x has an inverse if and only if $x \notin R$, and x is nilpotent if and only if $x \in R$.*

Proof. If $x \notin R$, then there exists an $x' \in A$ such that $xx' - 1 \in R$. Let y be the right quasi-inverse of $xx' - 1$, then

$$xx' - 1 + y + (xx' - 1)y = 0 ,$$

which implies that $x[x'(1 + y)] = 1$.

If x has an inverse, then it can not be nilpotent. Since R is itself nilpotent, and by the above all elements not in R have inverses, R consists of all the nilpotent elements of A.

THEOREM 2. *If A is a compact integral domain with an identity and has characteristic p, then A contains a maximal subfield which is isomorphic to A/R.*

Proof. Let $a+R$ be a generator of A/R and let the number of elements of A/R be p^q. Now for every positive integer k, $a^{p^{kq}}$ belongs to $a + R$. Since A is compact and R is closed, $a + R$ is closed and compact. Hence there exists a subnet $\{a^{p^{k(i)q}}\}$ of the sequence $\{a^{p^{kq}}\}$ which converges to some a^* belonging to $a + R$. Hence $a^* + R$ generates A/R. If $n = p^q - 1$, then $a^n - 1 \in R$ and $0 = \lim (a^n - 1)^{p^{k(i)q}} = (a^*)^n - 1$. The ring F generated by $\{j1 \mid 0 \leqq j \leqq p - 1\}$ and $\{a^*\}$ is a finite field containing p^q elements.

Let D be any subfield of A. For every $d \in D$, let $\theta(d) = d + R$. Clearly θ is a homomorphism mapping D into A/R. If $\theta(d_1) = \theta(d_2)$, $d_1 - d_2 \in R$, and if $d_1 - d_2 \neq 0$, then $1 \in R$. Since this is impossible, θ is a monomorphism and D contains at most p^q elements.

The field F constructed above is hence a maximal subfield of A and is isomorphic to A/R.

Note that in the above Theorem, D could have been a subdivision ring of A, and hence any subdivision ring of A is a finite field.

Zelinsky has shown [7, Th. E, p. 321] that if A has finite characteristic, then $A = S + R$ (group direct sum) where S is a compact subring of A. If in addition A has an identity, then since R is open and F is finite, the following theorem immediately follows.

THEOREM 3. *If A is a compact integral domain with an identity having finite characteristic, and if F is a maximal subfield of A, then A is topologically isomorphic to $F + R$.*

Not all compact integral domains have finite characteristic as is seen in the following example. Let Q be the field of rational numbers and P be any prime divisor of Q which is nonarchimedean. Let A^* be the valuation ring at P with the usual topology. Now A^* is a topological integral domain. Let A be the completion of A^*. A is an integral domain [5] which is compact [6, Lemma 1, p. 434], but it does not have finite characteristic.

It should also be noted that A does not have a subfield. As is seen in the following theorem, the existence of a subfield of A is equivalent to A having finite characteristic.

THEOREM 4. *If A is a compact integral domain with an identity, then A has finite characteristic if and only if A has a subfield.*

Proof. Let F be a subfield of A. As in Theorem 2, F is finite, and hence it has finite characteristic. Since A can have only one nonzero idempotent, the identity of F must be the identity of A, and hence A has finite characteristic.

The other implication is obvious.

In the rest of this section let us assume that A is a compact integral domain with characteristic p which does not have an identity. We say that a topological ring B can be openly embedded in a topological ring C if there exists a continuous open monomorphism which maps B into C.

THEOREM 5. *If A is a compact integral domain with finite characteristic and does not have an identity, then A can be openly embedded in a compact integral domain with an identity.*

Proof. Let $K = \{j1 \mid 0 \leqq j \leqq p-1\}$ be the discrete field of integers mod p and let $A^* = A \times K$. For every $(x, i) \in A^*$ and $(y, j) \in A^*$ define

$$(x, i) + (y, j) = (x + y, i + j)\ , \quad \text{and}$$

$$(x, i)\cdot(y, j) = (xy + jx + iy, ij)\ .$$

With the usual product topology A^* is a compact ring.

Let $P = \{(x, i) \in A^* \mid xa + ia = 0 \text{ for every } a \in A\}$. Now P is a closed two sided ideal in A^*. Furthermore it is easily seen that A^*/P is an integral domain with an identity $(0, 1)$. For every $a \in A$, let $\theta(a) = (a, 0) + P$. Clearly θ is a continuous open monomorphism, and hence A can be openly embedded in A^*/P, a compact integral domain with an identity.

3. Commutativity of compact integral domain. If A has an identity, then A/R is of course a commutative ring, and it would be tempting to assert that A itself is commutative. However in general, this is not true as seen in the following example.

Let D be any finite discrete topological field containing p^q elements where $q > 1$. Let σ be any automorphism on D. For every integer $i \geqq 0$, let $D(i) = D$. Also let $A = \prod_{i=0}^{\infty} D(i)$. With the usual product topology A is a compact Hausdorff space. For every $f \in A$ and $g \in A$, define $(f + g)(i) = f(i) + g(i)$ and $(fg)(i) = \sum_{k+m=i} f(k)\sigma^k[g(m)]$. With the above operations A is a compact integral domain with an identity and has characteristic p. A is not commutative if σ is not the identity automorphism. The above construction is called a Hilbert construction [1, pp. 43-44].

It is interesting to note that although commutativity is an algebraic property, it may depend upon a topological property as is seen in the main result.

THEOREM 6. *If A is a compact integral domain, then A is commutative if and only if its center is open.*

Proof. Assume that Z is the center of A and that it is open. The center of R, $Z(R)$, contains $Z \cap R$. Because Z is open and $0 \in Z$, for every $x \in R$, there exists an $n(x) \geqq 1$ such that $x^{n(x)} \in Z$, and hence $x^{n(x)} \in Z(R)$. Since R itself is an integral domain, R is commutative [2, Corollary, p. 219]. Futhermore, $R \subset Z$ since for every $x \in R$ and $a \in A$, $axx = xax$ which implies $ax = xa$.

If A has no identity, then $A = R$, and A is commutative.

If A has an identity, then A/R is a finite field containing p^q elements for some prime p, and for every $x \in A$, $x^{p^q} - x \in R$. Thus for every $x \in A$, $x^{p^q} - x \in Z$ which implies that A is commutative by a result of Herstein [2, Th. 2, p. 221].

The other implication is trivial.

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