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GEOMETRY OF THE UNIT SPHERE OF A C^*-ALGEBRA AND ITS DUAL

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In this study we exploit as a main tool a polar decomposition for linear functionals on operator algebras, introduced in 1958 by Sakai, to determine various types of extremal behavior in the unit spheres of C^*-algebras and their duals. We discuss exposed points and complex extreme points as well as extreme points.

Throughout this paper M, F, and A will be generic symbols for a von Neumann algebra, its predual, and a C^*-algebra which may not have a unit, respectively. By a C^*-algebra is meant a Banach *-algebra in which $\|x^*x\| = \|x\|^2$ holds for all x. A von Neumann algebra is a C^*-algebra of operators on a Hilbert space which is closed in the weak operator topology and contains the identity operator. Each von Neumann algebra M is equivalent (as a Banach space) to the dual of the Banach space F of ultra-weakly continuous (= normal) linear functionals on M. The space F is called the predual of M (and is unique). References for the preceding facts, as well as any others to follow concerning C^*-algebras and von Neumann algebras, are the two monographs of Dixmier [3], [4], and the lecture notes of Sakai [13]. We will denote by w, s, uw, us the weak operator, strong operator, ultra-weak and ultra-strong topologies of M respectively, and n refers to the norm topology of a Banach space. The subscript 1 denotes solid unit sphere $S_1 = \{x \in S: \|x\| \leqq 1\}$, the subscript h denotes Hermitian part, ext S is the set of extreme points of a set S, $\exp_\tau S$ is the set of exposed points (relative to τ) of S (defined in § 4), and C-ext S is the set of "complex extreme points" of S (defined in § 3). As in [4] $P(A)$ and $E(A)$ denote the sets of pure states and states respectively of A. We denote the normal states of M by $S(M)(S(M) = F \cap E(M))$. A state is faithful if it is nonzero on all nonzero positive elements. If A has a unit it is denoted by 1. If $x \in A$, x' denotes 1-x. $\mathscr{L}(H)$ will denote the algebra of all bounded operators on a Hilbert space H.

Sakai's polar decomposition reads as follows [4, 12. 2. 4]: each f in F can be written $f = u|f|$, i.e., $f(x) = |f|(ux)$, where $|f|$ is a positive element of F with the same norm as f, u is a partial isometry in M with uu^* equal to the support of $|f|$ [3, p. 61]; also $|f| = u^*f$. In connection with this notation, the following remarks are relevant. First, a decomposition $f = |f|'v$, $|f|' = fv^*$ is also valid where $|f|'$ is a positive element of F with the same norm as f, v is a partial isometry in M with v^*v equal to the support of $|f|'$, and $f(x) = |f|'(xv)$. We may refer to this as the "right" polar decomposition of f as opposed to the "left" polar decomposition described above. Second, if $x \in M$ and $f \in F$, then xf and $fx \in F$ where $xf(y) = f(xy)$ and $fx(y) = f(yx)$. Third, a subset I of F is called left invariant if xf belongs to I whenever f belongs to I and x belongs to M. Note that $(xy)f = y(xf)$.

The following lemma, due to Effros, will be used several times [4, 12. 2. 3].

LEMMA 1.1. *If f belongs to F and p is a projection in M with $\|f\| = \|pf\|$, then $f = pf$.*

For each A there is a $*$-isomorphism π (called the universal representation) of A onto a C^*-algebra $\pi(A)$ of operators on a Hilbert space H with the property that A^{**} (the bidual of A) is equivalent (as a Banach space) to the weak closure of $\pi(A)$ [4, 12. 1. 3]. Thus the bidual A^{**} of any A has the same Banach space structure as an M, and therefore any A^* is an F. It follows that the results below concerning F hold automatically for the dual of any C^*-algebra. Only qualitative statements, as opposed to quantitative ones, can be inferred however, since the latter would probably be couched in terms of (elements of) A^{**}. Examples of the former are: C-ext A_1^* is the set of unit vectors in A_1^* (Theorem 3. 1); ext $A_1^* = \exp_n A_1^*$ and is not empty, by Krein-Milman (Proposition 4. 1).

The authors have investigated other types of extremal behavior in this setting but found their study to be less interesting as regards applications than the material reported on here.

2. Extreme points.

For completeness we recall that an extreme point of a set K is a point of K which cannot be the midpoint of a line segment lying in K.

The set ext A_1 was determined by Kadison [5, Th. 1] when A has a unit. The modifications necessary to cover the nonunit case were supplied by Sakai [13, p. 1. 5]. Further discussion appears in Miles [8]. It results that ext A_1 is empty unless A has a unit in which case ext A_1 consists precisely of the partial isometries v in A such that $(1 - vv^*)A(1 - v^*v) = \{0\}$. We will make use of this result in the next section.

We now determine the set ext F_1. The following result is partially known.

PROPOSITION 2.1. *The following are equivalent for an element f in F:*

(1) *f belongs to* ext $F_{h,1}$;

(2) *f or $-f$ belongs to* ext $S(M)$;

(3) *f or $-f$ belongs to* $P(M)$;

(4) *f or $-f$ belongs to $S(M)$ and the support of f is a minimal projection in M.*

Proof. (1) implies (2): Let $f = f_1 - f_2$ where each f_i is positive and normal and $1 = \|f\| = \|f_1\| + \|f_2\|$ [4, 12. 3. 3]. If neither of f_1 or f_2 is 0, then $f = \|f_1\|(f_1/\|f_1\|) + \|f_2\|(-f_2/\|f_2\|)$, which implies that f is 0. Therefore f or $-f$ is positive, and (2) is immediate.

(2) implies (3): Use the fact that $P(M) = \operatorname{ext} E(M)$ and the decomposition of elements of $E(M)$ into normal and singular parts [13, p. 1. 75].

(3) implies (4): The left kernel $I = \{x \in M: f(x^*x) = 0\}$ is a maximal left ideal [4, 2. 9. 5] which is ultra-weakly closed since f is normal. If p is a projection in M with $I = Mp$ [3, p. 45] then p' is minimal and is the support of f. Indeed if $e \leqq p'$ then $e' \geqq p$, $Me' \supset Mp$, so by maximality either $e' = 1$ or $e' = p$, proving the minimality of p'. On the other hand $f(p) = 0$ so that the support of f is dominated by p'. Since p' is minimal, it equals the support of f.

(4) implies (2): If $f = (g + h)/2$ with g, h in $S(M)$ then the support of g is a nonzero projection dominated by the support of f, so equal to the support of f. Denoting by e the support of f, the subalgebra eMe consists of scalar multiples of e which implies that $f = g$.

(2) implies (1): If $f = (g + h)/2$ with g, h in $F_{h,1}$, write $g = g_1 - g_2$, $h = h_1 - h_2$, g_i and h_i positive normal and $1 = \|g\| = \|g_1\| + \|g_2\| = \|h\| = \|h_1\| + \|h_2\|$. The fact that positive functionals assume their norm on the unit and some arithmetic imply that $g_2 = h_2 = 0$ and completes the proof.

We will refer to a state satisfying the conditions of Proposition 2.1 as a normal pure state of M.

THEOREM 2.1 ext F_1 *is the set of functionals f in F such that $|f|$ is a normal pure state.*

Theorem 2.1 is a special case of the following theorem.

THEOREM 2.2. *If I is a norm closed left-invariant subspace of F, then* ext I_1 *is the set of functionals f in I such that $|f|$ is a normal pure state.*

Proof. Let h belong to ext I_1 and let $h = v|h|$ be the polar decomposition of h. If $|h|$ is not a pure state of M, let $|h| = (g_1 + g_2)/2$ for distinct states g_1 and g_2 of M.

Consider the polars $K = I^\circ$ and $J = K^\circ = I^{\circ\circ}$ in M and M^* respectively. Then K is an ultra-weakly closed left ideal in M so has the form $K = Mp$ for some projection p in M [3, p. 45], and J is a left-invariant subspace of M^*.

Since I is left-invariant, $|h|$ belongs to I, and so $0 = |h|(p) = g_1(p) = g_2(p)$, which implies that g_1 and g_2 belong to J. Also $h = v|h| = (vg_1 + vg_2)/2$, and so either h does not belong to ext J_1 or $vg_1 = vg_2 = h$. We complete this part of the proof by showing that neither alternative holds. Assuming $vg_1 = vg_2 = h$, we get $v^*(vg_1) = v^*(vg_2) = v^*h = |h|$. By Lemma 1.1 $(vv^*)g_1 = g_1$ and $(vv^*)g_2 = g_2$, which implies that $|h| = g_1 = g_2$, a contradiction. Suppose now that h is not in ext J_1. There is a central projection z in M^{**} such that $F = zM^*$ [13, p. 1. 75]. Thus $h = zh = (z(vg_1) + z(vg_2))/2$ so that $\|z(vg_1)\| = \|z(vg_2)\| = 1$. Since z is central,

$$\|vg_1\| = \|z(vg_1)\| + \|z'(vg_1)\| ,$$

so $z'(vg_1) = 0$ and similarly $z'(vg_2) = 0$. Thus vg_1 and vg_2 belong to F and so $vg_1 = vg_2 = h$.

For the converse let $h = v|h|$ be as described in the statement of the theorem. If $h = (h_1 + h_2)/2$, with h_i in I_1, then

$$v^*h = |h| = (v^*h_1 + v^*h_2)/2 .$$

Since $|h|$ is pure, it is easily seen to belong to ext I_1, and so $v^*h_1 = v^*h_2 = |h|$. Thus $v(v^*h_1) = v(v^*h_2) = v|h| = h$. By Lemma 1.1 $h_1 = (v^*v)h_1 = (v^*v)h_2 = h_2$.

We indicate some applications of the preceding results.

In [5], extreme points in factors were grouped into equivalence classes whereby two extreme points (of the unit sphere) are in the same class if there is some linear or conjugate linear isometry of the factor mapping one of the extreme points into the other. Due to the incompleteness of the knowledge of the classes in the II_∞ case, there is possible confusion in distinguishing between factors of type I_∞ and II_∞ by their Banach space structure alone. This situation can be rectified by considering extreme points in the pre-duals of the factors, since minimal projections exist only in factors of type I.

Because, thanks to the Krein-Milman theorem, dual spaces always have extreme points in their unit spheres, one can see immediately that if the bidual of any C^*-algebra is a factor, then it is a factor of type I [4, 12. 5. 5].

In the Σ^*-algebra approach to quantum mechanics [10], a result is that for each nonzero projection e in a Σ^*-algebra there is a σ-state

f such that $f(e) = 1$. Analogously, it is remarked in [10] that a factor of type I has the property: if e is a nonzero projection then there is a *pure* normal state f such that $f(e) = 1$; whereas no factor of type II_1 does have this property. These statements can be sharpened as follows: a von Neumann algebra has the property in question if and only if it is atomic, i.e., a product of factors of type I.

If M is a von Neumann algebra with a faithful semi-finite normal trace φ, with trace ideal m_φ, then the map $t \rightarrow \varphi_t$ is a linear order preserving isometric mapping of m_φ onto a dense subset of F, where $\varphi_t(x) = \varphi(xt)$ [3, p. 105]. Using some spectral theory our result translates into the fact that the set $\operatorname{ext} m_{\varphi,1}$ consists of all partial isometries in M with initial domain a minimal projection in M. This fact was used to help determine the isometries of m_φ in the case of a factor of type I [11], [12].

3. Complex extreme points.

A point t of a subset K of a complex vector space is a complex extreme point of K if

$$\{t + zy: |z| \leqq 1\} \subset K$$

for some vector y implies $y = 0$. Here z is a complex number.

This concept was introduced in [16] where it was shown that the class of complex Banach spaces for which each unit vector is a complex extreme point (of the unit sphere) is precisely the class for which the strong form of the maximum modulus theorem holds. It is also shown in [16] that all L_1 spaces belong to this class.[1]

THEOREM 3.1. *C-$\operatorname{ext} F_1$ is the set of all unit vectors in F.*

Proof. Let f be a unit vector in F and suppose first that f is positive. Then $f(1) = 1$, and if f is not a complex extreme point there is an element $h \neq 0$ in F such that $\| f + zh \| = 1$ for all $|z| = 1$. Let a be a Hermitian element of M such that $h(a) \neq 0$ and let N be the von Neumann algebra generated by a and 1. Since N is commutative it is isomorphic to a space $L_\infty(S)$ with pre-dual $L_1(S)$ [3, p. 117]. Considering the restrictions of f and h to N as elements of $L_1(S)$, the result of Thorp and Whitley implies that h is zero on N, which is a contradiction.

Now let g be an arbitrary unit vector in F with polar decomposition $g = v|g|$. Suppose there is an element $h \neq 0$ in F with $\| g + zh \| = 1$ for all $|z| = 1$. The $\| |g| + zv^*h \| = \| v^*(g + zh) \| \leqq 1$ and so $v^*h = 0$ by the first paragraph. Set $p = v^*v$. Then $ph = 0$ and $pg = v(v^*g) = v|g| = g$. Let u_1 and u_2 be unit vectors in M

[1] See also the dissertation of L. A. Harris, Cornell, 1969.

satisfying $g(u_1) = 1$ and $h(u_2) = \|h\|$. Choose r_1 and r_2 to satisfy $0 < r_1 \leqq r_2 < 1$, $r_1^2 + r_2^2 = 1$ and $r_1 + r_2\|h\| > 1$. Then a simple computation implies that $|(g + h)(r_1pu_1 + r_2p'u_2)| = r_1 + r_2\|h\|$, and the norm of $r_1pu_1 + r_2p'u_2$ is at most 1. Thus $\|g + h\| > 1$, a contradiction.

As remarked in [16] it is sufficient in the definition of complex extreme point to use only the points $z = \pm 1, \pm i$.

THEOREM 3.2 $C\text{-ext}\, A_1 = \text{ext}\, A_1$.

Proof. The proof is a modification of known arguments. Let x be a complex extreme point of A_1. We show that x^*x is a projection. Let B be the C^*-subalgebra of A generated by x^*x. Then B is isomorphic to the algebra of all continuous functions vanishing at infinity on some locally compact Hausdorff space. As in [13, p. 1. 2] choose positive elements y_n in B of unit norm such that

$$\|x^*xy_n - x^*x\| \to 0 .$$

If the function representing x^*x assumes a value in the open unit interval then there is a positive element c in B such that $x^*xc \neq 0$ and the following three vectors are in the unit sphere: $x^*x(y_n + c)^2$, $x^*x(y_n - c)^2$, $x^*x(y_n + c^2)$. It follows that $x \pm xc$ and $x \pm ixc$ all lie in the unit sphere. By definition of complex extreme point $xc = 0$, a contradiction, so x^*x is a projection.

Now let a be any element of $(1 - xx^*)A(1 - x^*x) \cap A_1$, say $a = b - xx^*b - bx^*x + xx^*bx^*x$ (1 is not a unit of A, just a notational convenience). Elementary computations show that $a^*x(a^*x)^* = 0$ and $x^*xa^*a = 0$, so that consequently $a^*x = x^*a = 0$ and $\|x^*x + a^*a\| = \max(\|x^*x\|, \|a^*a\|) = 1$. Using these facts it follows that $x \pm a$ and $x \pm ia$ lie in the unit sphere, so that $a = 0$. Thus x belongs to $\text{ext}\, A_1$ [8, Th. 1].

Our results imply that the strong form of the maximum modulus theorem holds in every F and fails in every A (of dimension larger than 1). That it fails in some A had been remarked in [2].

Our results also show that no F can be (linearly) isometric to any A (of dimension larger than 1). Sakai [14, Corollary 1] had already shown that no infinite dimensional A can be even topologically isomorphic to any F.

If F is the pre-dual of a factor of type I, i.e., if F is the trace class operators on some Hilbert space H, a proof of Theorem 3.1 can be given which does not use the commutative case, i.e., the Thorp-Whitley result. It is based (as is the proof of Thorp and Whitley) on when equality holds in Minkowski's inequality (cf. [7, Th. 2. 4]).

4. Exposed points. An exposed point of a convex set K in a topological linear space X is a point p of K such that K is supported at p by a closed hyperplane which intersects K only at p [6]. If τ is a locally convex topology on a complex Banach space X (whose continuous linear functionals are bounded) then a vector x of norm 1 belongs to $\exp_\tau X_1$ if there is a τ-continuous linear functional f on X such that $1 = \|f\| = f(x) > \operatorname{Re} f(y)$ for all $y \neq x$ in the unit sphere of X.

For M a von Neumann algebra we have the following relations: $\operatorname{ext} M_1 \supset \exp_n M_1 \supset \exp_{uw} M_1 = \exp_{us} M_1 \supset \exp_w M_1 = \exp_s M_1$; the equalities obtaining since the appropriate topologies have the same continuous linear functionals [3, p. 40].

THEOREM 4.1. *$\operatorname{ext} M_1 = \exp_{uw} M_1$ if and only if M has a faithful normal state; otherwise $\exp_{uw} M_1$ is empty.*

Note that a von Neumann algebra has a faithful normal state if and only if it is countably decomposable [3, p. 61].

LEMMA 4.1. *If t belongs to $\exp_{uw} M_1$ and is semi-unitary, i.e., at least one of t^*t or tt^* is 1, then 1 belongs to $\exp_{uw} M_1$.*

Proof. Since the involution is uw-continuous it is enough to assume that $t^*t = 1$. Let f be an element of F of norm 1 which assumes its maximum on M_1 only at t, and write $f = v|f|$. Then $f(v^*) = |f|(vv^*) = 1$, so that $t = v^*$. Since $t^*t = 1$, it ensues that $|f|$ assumes its maximum on M_1 only at 1.

Proof of Theorem 4.1. Let t be an element of $\exp_{uw} M_1$. There is a central projection p in M such that tp and $t(1-p)$ are semi-unitary in M_p and M_{1-p} respectfully [8, Th. 2]. By Lemma 4.1, p and $1-p$ are uw-exposed in the unit spheres of M_p and M_{1-p} respectively, which implies by averaging two functionals that 1 belongs to $\exp_{uw} M_1$. If f is the element of F which assumes its maximum on M_1 only at 1 then f is a faithful normal state; indeed $0 \leqq x \leqq 1$ and $f(x) = 0$ implies $f(1-x) = 1$, so $1 - x = 1$.

Suppose that M has a faithful normal state, say f, where $f(x) = \sum_{i=1}^{\infty}(x\xi_i, \xi_i)$ with $\|f\| = \Sigma\|\xi_i\|^2 = 1$ [3, p. 54]. Let t belong to $\operatorname{ext} M_1$ and suppose for the moment that $t^*t = 1$. Then the functional $g = t^*f$ assumes its norm only at t. Indeed, $g(t) = 1 = \|g\|$ and if $g(s) = 1$ for some s in the unit sphere, then

$$1 = g(s) = \sum_{i=1}^{\infty}(s\xi_i, t\xi_i) \leqq \Sigma\|s\xi_i\|\,\|t\xi_i\| \leqq \Sigma\|\xi_i\|^2 = 1\ ,$$

so that $(s-t)\xi_i = 0$ for all i. It follows that $|s-t|\xi_i = 0$, $f(|s-t|) =$

0 and since f is faithful, $|s - t| = 0$, $s - t = 0$. The remainder of the proof follows as in the preceding paragraph by patching.

Using [3, p. 40] it is easily seen that $\exp_w \mathscr{L}(H)_1$ is empty if H is infinite dimensional. This fact and Theorem 4.1 yield a quick proof that the weak and ultra-weak topologies are distinct on $\mathscr{L}(H)$ for H separable and infinite dimensional (cf. [3, p. 283]). The same is true of the strong and ultra-strong topologies.

In [6, p. 96] it was shown that the dual of a separable Banach space has the property: "every w^*-compact convex set is the w^*-closed convex hull of its w^*-exposed points". On the other hand, if X is a Banach space considered as a subspace of its second dual X^{**}, then it is plain that $(\exp_{w^*} X_1^{**}) \cap X_1$ is a subset of $\exp_n X_1$. It follows that if A is a C^*-algebra with unit, identified with its universal representation, then $\exp_{uw} M_1 \cap A_1 \subset \exp_n A_1$, where M is the weak closure of A. Since it is easy to verify that $\operatorname{ext} M_1 \cap A_1 = \operatorname{ext} A_1$ we see that if a C^*-algebra has a separable dual then $\operatorname{ext} A_1 = \exp_n A_1$.

On the other hand if a C^*-algebra has a separable dual, then a faithful state can be easily constructed from a dense sequence in the state space. Thus the result of the preceding paragraph is also a consequence of the following theorem.

THEOREM 4.2. $\operatorname{ext} A_1 = \exp_n A_1$ *if and only if* A *has a faithful state; otherwise* $\exp_n A_1$ *is empty.*

In the case of a commutative A this has been proved by Phelps [9].

Proof. Let v be an exposed point, say $p = v^*v$, $q = vv^*$, and let h be a linear functional which assumes its norm 1 only at v. We may replace A by its universal representation. Let $h = u|h|$ be the enveloping polar decomposition of h [4, 12. 2. 8]. Then we may replace u by v^* in the sense that $h = v^*|h|$. Indeed $|h|(x) = (x\xi, \xi)$ for some unit vector ξ, so that $h(x) = (ux\xi, \xi)$. Thus

$$1 = h(v) = (uv\xi, \xi) = (v\xi, u^*\xi) ,$$

which implies that $v\xi = u^*\xi$ and therefore that $h = v^*|h|$.

We now assert that $|h|$ is faithful on p in the sense that $pxp = 0$ whenever $|h|((pxp)^*(pxp)) = 0$ and x is in A. Indeed, if $b = px^*pxp \leqq 1$ and $|h|(b) = 0$, then $h(v - vb) = 1$ and $\|v(1 - b)\| \leqq 1$, hence $vb = 0$, so that $b = pb = v^*vb = 0$.

Using the "right" polar decomposition shows that there is a faithful state on q hence on p' since $p + q - pq = 1$ implies $p' \leqq q$. It follows that there is a faithful state on A.

Suppose now that A has a faithful state, and that v belongs to $\text{ext}\, A_1$, say with $p = vv^*$, $q = v^*v$. Let f be a faithful state on p and g be a faithful state on q, and set $h = (fv^* + v^*g)/2$. Then $h(v) = 1$, and we will show that if $h(a) = 1$ for a unit vector a, then $a = v$. Assuming $h(a) = 1$ we have $f(av^*) = g(v^*a) = 1$. Thus, since $f = pfp$ and $g = qgq$, $f(av^*p) = 1$, so that $av^*p = p$. To see this let $b = (p - av^*p)^*(p - av^*p)$ and observe that $f(b) = 0$. Indeed by [4, 2. 1. 5] $0 \leqq f(b) = -1 + f(pva^*av^*p) \leqq -1 + \| va^*av \| \leqq 0$ so that $b = pbp = 0$. Also $pav^* = p$, so that $av^* = p + p'av^*p'$. Similarly $v^*a = q + q'v^*aq'$. Now $v^*p' = 0 = q'v^*$, so $av^* = p$ and $v^*a = q$. Therefore $a(v^*v) = aq = (av^*)v = pv = v$, and $pa = v$ similarly. Since v is extreme, $0 = p'aq' = aq' - paq' = aq' - vq' = aq'$. Thus $a = a(q + q') = aq + 0 = v$. The proof is complete.

COROLLARY. $\exp_n M_1 = \exp_{uw} M_1$.

Proof. Using Takesaki's criterion for normality [15, Th. 1], one sees that M has a faithful state if and only if it has a faithful normal state.

The corollary should be compared with Theorem 4.3 where separability is required and in turn suggests the question: if X is a separable Banach space then does $\exp_n X_1^* = \exp_{w^*} X_1^*$?[2]

PROPOSITION 4.1. $\text{ext}\, F_1 = \exp_n F_1$.

Proof. Let h be an extreme point and write $h = v|h|$, with $p = vv^*$ the support of h. Then $h(v^*) = 1$. Suppose that g belongs to F_1 and that $g(v^*) = 1$. Then $v^*g(1) = 1 = \| v^*g \|$, so v^*g is a state [4, 2.1.9] with support dominated by p. Since p is minimal, $v^*g = |h|$. Therefore, $h = v|h| = v(v^*g) = (v^*v)g = g$ by Lemma 1.1.

If A is separable, then by the result of Klee quoted above and the converse to the Krein-Milman Theorem one knows that each f in $\text{ext}\, A_1^*$ is the w^*-limit of a sequence in $\exp_{w^*} A_1^*$. With some effort this can be improved.

THEOREM 4.3. $\exp_{w^*} A_1^* = \text{ext}\, A_1^*$ *if A is separable and has a unit.*

Proof. Replace A by its universal representation. Let $h \in \text{ext}\, A_1^*$ with enveloping polar decomposition $h = vf$ so that $f(x) = (x\xi, \xi)$ for some unit vector ξ (see the proof of Theorem 4.2). Let $\{b_n\}$ be a

[2] No, according to a communication of R. R. Phelps.

sequence dense in the unit sphere of $N_f = \{x \in A: f(x^*x) = 0\}$ and set $b = 1 - \sum_1^\infty 2^{-n} b_n^* b_n$ so that $f(b) = 1$, $\| b \| = 1$, and $b\xi = \xi$. Since f is pure and $\| v^*\xi \| = \| \xi \|$ there is a unitary element u in A such that $u\xi = v^*\xi$. For this it suffices to observe that since v^* is in the strong closure of A $v^*\xi$ is a limit of elements $a\xi$ for $a \in A$ and to apply [4, 2. 8. 3] to the restriction of the identity representation of A to $A\xi$.

Now $h(ub) = f(vub) = (vub\xi, \xi) = (ub\xi, u\xi) = (b\xi, \xi) = f(b) = 1$. Suppose that $g \in A_1^*$ and $g(ub) = 1$. Let $g = wp$ be its enveloping polar decomposition. Then

$$1 = g(ub)^2 = p(wub)^2 = p((wub^{1/2})b^{1/2})^2 \leqq p(wubu^*w^*)p(b) \leqq 1$$

so that $p(b) = 1$. It follows that $N_f \subset N_p$ and since f is pure that $f = p$. Indeed if $x \in N_f$ then some subsequence $b_{n_j} \rightarrow x/\| x \|$, so $b_{n_j}^* b_{n_j} \rightarrow x^*x/\| x \|^2$ and therefore $p(x^*x) = \lim_j p(b_{n_j}^* b_{n_j}) = 0$ since $p(b) = p(1) = 1$. Now $1 = g(ub) = p(wub) = f(wub) = (wub\xi, \xi) = (u\xi, w^*\xi)$ and thus $u\xi = w^*\xi$. Finally $g(x) = p(wx) = f(wx) = (wx\xi, \xi) = (x\xi, u\xi) = (x\xi, v^*\xi) = (vx\xi, \xi) = vf(x) = h(x)$.

COROLLARY. *If A is separable and has a unit then $P(A) = \exp_{w^*} E(A)$.*

For A nonseparable, Theorem 4.3 is false since $P(A)$ may contain $\exp_{w^*} E(A)$ properly (take $A = C(X)$ where X has no G_δ points e.g., $X = \beta N - N$). On the other hand the following result (whose proof is essentially that of [1, Th. 1. 1]) is of greater interest: if $f \in P(A) \cap \exp_{w^*} A_1^*$ then there is a maximal abelian C^*-subalgebra B of A such that f is a pure state on B and has a unique pure state extension to A.

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