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A MEAN VALUE THEOREM FOR BINARY DIGETS

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This paper continues the investigation of the dyadically additive function α defined by $\alpha(n)=$ the number of 1's in the binary expansion of n.

Previously, Bellman and Shapiro (cf. "On a problem in additive number theory." Annals of Mathematics, 49 (1948) 333–340) showed that $\Sigma_{k=1}^{x}\alpha(k)\sim x\log x/2\log 2$. They then considered the iterates of α defined by $\alpha_q=\alpha_{q-1}\circ\alpha$ and observed that $A_r(x)=\Sigma_{k=1}^{x}\alpha_r(k)$ is not asymptotic to any elementary function for $r\geqq 2$.

In this paper the function $A_2(x)$ will be examined more closely. Defining $c(x)$ by $A_2(x)=c(x)x\log\log x/2\log 2$, we will prove the following theorems.

THEOREM 1. *As x ranges over the positive integers, $c(x)$ ranges densely over $[1/2,3/2]$. Furthermore, given any $c\in[1/2,3/2]$, there is an explicit way to construct a sequence of integers x for which $c(x)\to c$ as $x\to\infty$.*

THEOREM 2.

$$1/2+O(\log\log\log x/\log\log x)\leqq c(x)\leqq 3/2+O(\log\log\log x/\log\log x). \tag{1.1}$$

THEOREM 3.

$$\liminf c(x)=1/2,\qquad \limsup c(x)=3/2. \tag{1.2}$$

Note. Theorem 3 is an immediate consequence of Theorems 1 and 2.

2. The proof of Theorem 1 is obtained by considering a special set of integers.

Let $\mathcal{M}=\{x: x=2^M-1,\ M \text{ even},\ M/2=\Sigma_{i=1}^{r}2^{a_i}-1,\ a_1>a_2>\cdots>a_r \text{ positive integers, and } a_r/a_1\geqq 1/2+\log\log\log x/\log\log x\}$.

LEMMA 1. *If $x\in\mathcal{M}$, then*

$$A_2(x) = \left(r + \frac{a_r}{2} + O(1)\right) x. \tag{2.1}$$

Proof. If $x \in \mathcal{M}$, then (cf. [1])

$$A_2(x) = \sum_{k<2^M} \alpha(\alpha(k)) = \sum_{n \leqq M} \binom{M}{n} \alpha(n). \tag{2.2}$$

We can then write

$$A_2(x) = \sum_1 \binom{M}{n} \alpha(n) + \sum_2 \binom{M}{n} \alpha(n) \tag{2.3}$$

where Σ_1 is the sum over $\{n: |M/2 - n| < 2a_r\}$ and Σ_2 is the sum over $\{n: |M/2 - n| \geqq 2a_r\}$.

Chebyshev's inequality yields

$$\begin{aligned} \sum_2 \binom{M}{n} \alpha(n) &\ll 2^M M \cdot 2^{-2a_r} \log M \\ &\ll 2^M \cdot M \cdot 2^{-a_1(1+2\log\log\log x/\log\log x)} \log M \end{aligned} \tag{2.4}$$

which implies

$$\sum_2 \binom{M}{n} \alpha(n) = O(x). \tag{2.5}$$

(Here and further on, inequalities such as $M = O(\log x)$, $a_1 = O(\log M)$, $\alpha(n) = O(\log n)$ and $r = O(\log M)$ will be used without comment).

We will use the symmetry of the binomial coefficients to estimate Σ_1.

$$\begin{aligned} \sum_1 \binom{M}{n} \alpha(n) = \frac{1}{2} \sum_{0 \neq |t| < 2^{a_r}} \binom{M}{M/2+t} \left\{\alpha\left(\frac{M}{2} - t\right) + \alpha\left(\frac{M}{2} + t\right)\right\} \\ + \binom{M}{M/2} \alpha(M/2). \end{aligned} \tag{2.6}$$

Writing $t = \sum_{j=1}^{w} 2^{b_j}$, we obtain

$$\alpha\left(\frac{M}{2} + t\right) = \alpha\left(\sum_{i=1}^{r} 2^{a_i} + \sum_{j=1}^{w} 2^{b_j} - 1\right) = r + w - 1 + b_w \tag{2.7}$$

and

$$\alpha\left(\frac{M}{2}-t\right)=\alpha\left(\sum_{i=1}^{r} 2^{a_i}-1-\sum_{j=1}^{w} 2^{b_j}\right)=r-1+a_r-w \tag{2.8}$$

so that

$$\alpha\left(\frac{M}{2}+t\right)+\alpha\left(\frac{M}{2}-t\right)=2r-2+a_r+b_w. \tag{2.9}$$

We can now rewrite (2.6), obtaining

$$\sum_1 \binom{M}{n}\alpha(n)=\sum_{0<|t|<2^{a_r}}\binom{M}{M/2+1}\left(r-1+\frac{a_r}{2}+\frac{b_w}{2}\right)+\binom{M}{M/2}a_r. \tag{2.10}$$

Chebyshev's inequality implies that

$$\sum_{|t|\geqq 2^{a_r}}\binom{M}{M/2+t}\left(r+\frac{a_r}{2}\right)=O(x)$$

as in the analysis of Σ_2. Since

$$\binom{M}{M/2}(r+a_r/2)=O(x\log\log x/\sqrt{\log x})=O(x),$$

we obtain

$$\sum_{|t|<2^{a_r}}\binom{M}{M/2+t}\left(r+\frac{a_r}{2}\right)=2^M\left(r+\frac{a_r}{2}\right)+O(x)=\left(r+\frac{a_r}{2}\right)+O(x). \tag{2.11}$$

Thus it remains only to show that each remaining term is $O(x)$. We have already seen that

$$\binom{M}{M/2}a_r/2=O(x) \tag{2.12}$$

and easily obtain

$$\sum_{|t|<2^{a_r}}\binom{M}{M/2+t}(-1)=O(2^M)=O(x). \tag{2.13}$$

We estimate the remaining term by observing that $b_w=b_w(t)$ is the largest exponent such that $2^{b_w}\mid t$. Thus we can write

$$\sum_{0<|t|<2^{a_r}} \binom{M}{M/2+t} b_w \leqq \sum_{\substack{\iota,t \\ 2^\iota | t}} \binom{M}{M/2+t} \leqq \sum_{\iota \geqq 0} \sum_{q>0} \binom{M}{M/2+2^\iota q}$$

(2.14)

$$\leqq \sum_{\iota \geqq 0} \frac{1}{2^\iota} \sum_q \binom{M}{q} = O(x).$$

This completes the proof of Lemma 1.

Lemma 1 implies that

$$c(x) = \frac{r + a_r/2}{\log\log x/2\log 2} + o(1). \tag{2.15}$$

Since $a_1 = \log\log x/2\log 2 + O(1)$, we obtain

$$c(x) = \frac{r + a_r/2}{a_1} + o(1). \tag{2.16}$$

We now complete the proof of Theorem 1 by showing that if $\epsilon > 0$ then there exist arbitrarily large q such that if $(1/2+\epsilon)q < z < (3/2-\epsilon)q$ is an integer, then there exists $x \in \mathcal{M}$ such that $a_1 = q$ and $r + a_r/2 = z$.

Suppose we choose $(1/2+\epsilon)q - 4 \leqq s < (1/2+\epsilon)q - 2$, s even. As t takes on all possible integer values between 2 and $q - s$, $t + s/2$ certainly takes on all integer values between $(1/2+\epsilon)q$ and $(3/2-\epsilon)q$.

If q is large enough, it is certainly possible to find $x \in \mathcal{M}$ such that $a_1 = q$, $r = t$ and $a_r = s$, completing the proof of Theorem 1.

3. We carry out the proof of Theorem 2 in a series of steps.

Let $\mathcal{M}^1 = \{x : x = 2^M - 1,\ M \text{ even},\ M/2 = \sum_{i=1}^{r} 2^{a_i} - 1,\ a_1 > a_2 > \cdots > a_r \text{ integers and } a_r/a_1 \geqq (1/2)\log\log\log x/(\log\log x + \log\log 2)\}$.

We begin by proving the conclusion of Theorem 2 holds for element of $\mathcal{M}^1$.

LEMMA 2. *If $x \in \mathcal{M}^1$ then*

$$\frac{1}{2} + O\left(\frac{\log\log\log x}{\log\log x}\right) < c(x) < \frac{3}{2} + O\left(\frac{\log\log\log x}{\log\log x}\right). \tag{3.1}$$

Proof. We begin as in Lemma 1, writing

$$A_2(x) = \sum_1 \binom{M}{n} \alpha(n) + \sum_2 \binom{M}{n} \alpha(n) \tag{3.2}$$

except where Σ_1 is the sum over $\{n: |M/2 - n| < 2^{(\frac{1}{2}+\epsilon)a_1}\}$ and Σ_2 is the sum over $\{n: |M/2 - n| \geqq 2^{(1/2+\epsilon)a_1}\}$, where $\epsilon = \epsilon(x) = \log\log\log x/(\log\log x + \log\log 2)$.

The second term can be estimated as the corresponding term was in Lemma 1, yielding

$$\sum_2 \binom{M}{n} \alpha(n) = O(x). \tag{3.3}$$

We estimate the first sum by considering two cases.

Case 1. $a_r \geqq (1/2 + \epsilon)a_1$. We can treat this case as we treated Lemma 1, obtaining $\Sigma_1 \binom{M}{n} \alpha(n) = (r + a_r/2)x + O(x)$ and hence

$$A_2(x) = (r + a_r/2)x + O(x). \tag{3.4}$$

Since $0 \leqq r \leqq a_1 - a_r + 1$, we obtain $a_r/2 \leqq r + a_r/2 \leqq a_1 - a_r/2 + 1$. Since $(1/2 + \epsilon)a_1 \leqq a_r \leqq a_1$, we obtain

$$\left(\frac{1}{4} + \frac{\epsilon}{2}\right) a_1 \leqq r + a_r/2 \leqq \left(\frac{3}{4} - \frac{\epsilon}{2}\right) a_1 + 1. \tag{3.5}$$

But $a_1 = (\log\log x/\log 2) + O(1)$, so

$$\left(\frac{1}{4} + \frac{\epsilon}{2}\right) \frac{\log\log x}{\log 2} + O(1) \leqq r + a_r/2 \leqq \left(\frac{3}{4} - \frac{\epsilon}{2}\right) \frac{\log\log x}{\log 2} + O(1) \tag{3.6}$$

which implies

(3.7)

$$\frac{1}{4}\frac{\log\log x}{\log 2} + O(\log\log\log x) \leqq r + a_r/2 \leqq \frac{3}{4}\frac{\log\log x}{\log 2} + O(\log\log\log x).$$

Thus

$$\begin{aligned} &\left(\frac{1}{2} + O\left(\frac{\log\log\log x}{\log\log x}\right)\right)\left(\frac{x \log\log x}{2\log 2}\right) \leqq A_2(x) \\ &\leqq \left(\frac{3}{2} + O\left(\frac{\log\log\log x}{\log\log x}\right)\right)\left(\frac{x \log\log x}{2\log 2}\right) \end{aligned} \tag{3.8}$$

which proves Lemma 2 for this case.

Case 2. $(1/2-\epsilon)a_1 < a_r < (1/2+\epsilon)a_1$.

As in Lemma 1, we write

$$\sum_1 \binom{M}{n}\alpha(n) = \frac{1}{2} \sum_{0\neq t<2^{(1/2+\epsilon)a_1}} \binom{M}{M/2+t}\left\{\alpha\left(\frac{M}{2}-t\right)+\alpha\left(\frac{M}{2}+t\right)\right\} \tag{3.9}$$
$$+\binom{M}{M/2}\alpha\left(\frac{M}{2}\right).$$

Here, however, an overlap of the nonzero digits in the binary representations of t and $M/2+1$ forces us to use the subadditive properties of α. Writing $M/2$ and t as before, we obtain

$$\frac{M}{2}+t = 2^{a_1}+\cdots+2^{a_r}+2^{b_1}+\cdots+2^{b_w}-1 \tag{3.10}$$
$$\frac{M}{2}-t = 2^{a_1}+\cdots+2^{a_r}-2^{b_1}-\cdots-2^{b_w}-1.$$

The subadditivity of α implies $\alpha(M/2+t) \leqq \alpha(M/2+1) + \alpha(t-1)$ so that

$$\alpha\left(\frac{M}{2}+t\right) \leqq r+w+b_w. \tag{3.11}$$

Also, $\alpha(M/2+t)$ is at least $\alpha(t)$ minus the overlap between the binary expansions of $M/2$ and t, so that

$$\alpha\left(\frac{M}{2}+t\right) \geqq w-2\epsilon a_1. \tag{3.12}$$

Since $\alpha(M/2-t)$ is no greater than the number of places available, less $\alpha(t)$, plus the overlap, we obtain

$$\alpha\left(\frac{M}{2}-t\right) \leqq a_1+1-w+2\epsilon a_1. \tag{3.13}$$

Also, $\alpha(M/2-t)$ must be at least the number of 1's that $M/2$ ends with less $\alpha(t)$, so that

$$\alpha\left(\frac{M}{2}-t\right) \geqq a_r-w. \tag{3.14}$$

Combining (3.11)–(3.14) we obtain

$$(3.15)\quad a_r - 2\epsilon a_1 \leqq \alpha\left(\frac{M}{2}+t\right) + \alpha\left(\frac{M}{2}-t\right) \leqq a_1 + r + b_w + 2\epsilon a_1 + 1.$$

Since $a_r > (1/2-\epsilon)a_1$ and $r \leqq a_1 - a_r + 1 < a_1 - (1/2-\epsilon)a_1 + 1 = (1/2+\epsilon)a_1 + 1$ we obtain

$$(3.16)\quad \left(\frac{1}{2}-3\epsilon\right) a_1 \leqq \alpha\left(\frac{M}{2}+t\right) + \alpha\left(\frac{M}{2}-t\right) \leqq \left(\frac{3}{2}+3\epsilon\right) a_1 + b_w + 1.$$

Plugging the first inequality of (3.16) into (3.9) yields

$$\sum_1 \binom{M}{n} \alpha(n) \geqq \frac{1}{2}\left(\frac{1}{2}-3\epsilon\right) a_1 \sum_{0\neq t<2^{(1/2+\epsilon)a_1}} \binom{M}{M/2+t} + \binom{M}{M/2} \alpha\left(\frac{M}{2}\right).$$

Chebyshev's inequality yields

$$\sum_{0\neq t<2^{(1/2+\epsilon)a_1}} \binom{M}{M/2+t} = x + O\left(\frac{x}{2^{2\epsilon a_1}}\right)$$

which implies

$$(3.17)\qquad \sum_1 \binom{M}{n} \alpha(n) \geqq \frac{1}{4} a_1 x - \frac{3}{2} \epsilon a_1 x + O(x).$$

Recalling $a_1 = (\log\log x)/\log 2) + O(1)$ and combining (3.17) with (3.3) yields

$$(3.18)\qquad A_2(x) \geqq \frac{1}{2}\frac{x \log\log x}{2\log 2} + O(x \log\log\log x)$$

which implies

$$(3.19)\qquad c(x) \geqq \frac{1}{2} + O\left(\frac{\log\log\log x}{\log\log x}\right).$$

Plugging the second inequality of (3.16) into (3.9) yields

$$\sum_1 \binom{M}{n} \alpha(n) \leqq \frac{1}{2}\left(\frac{3}{2}+3\epsilon\right) a_1 \sum_{0\neq t<2^{(1/2+\epsilon)a_1}} \binom{M}{M/2+t}$$

$$(3.20)\qquad +\frac{1}{2} \sum_{0<|t|<2^{(1/2+\epsilon)a_1}} \binom{M}{M/2+t} (b_w+1)$$

$$+ \binom{M}{M/2} \alpha\left(\frac{M}{2}\right).$$

As in (2.14) we see that

$$\sum_{0<|t|<2^{(1/2+\epsilon)a_1}} \binom{M}{M/2+t}(b_w+1) = O(x)$$

to obtain

$$\sum_1 \binom{M}{n}\alpha(n) \leqq \frac{3}{4}a_1x + \frac{3}{2}\epsilon a_1 x + O(x). \tag{3.21}$$

Repeating the reasoning of (3.17)–(3.19) yields

$$c(x) \leqq \frac{3}{2} + O\left(\frac{\log\log\log x}{\log\log x}\right). \tag{3.22}$$

Combining (3.19) with (3.22) completes the proof of Lemma 2.

We now consider a lemma which will enable us to extend the conclusion of Theorem 2 to all integers of the form $2^n - 1$.

LEMMA 3. *If $x = 2^N - 1$, then there exists an even integer $M \geqq N$ such that $M - N \leqq \sqrt{N}/\log N$, $M/2 = \Sigma_{i=1}^r 2^{a_i} - 1$ with $a_r/a_1 \geqq 1/2 - \epsilon$ and*

$$A_2(x) = \frac{A_2(2^M - 1)}{2^{M-N}} + O(x), \tag{3.23}$$

where

$$\epsilon = \epsilon(x) = \frac{\log\log\log x}{\log\log x + \log\log 2}.$$

Proof. Let $N = \Sigma_{i=1}^l 2^{a_i}$, $a_1 > a_2 > \cdots > a_l$. Define n by $n + 1 = \Sigma_j 2^{c_j}$, where $\{c_j\}$ runs over all integer values in the interval $[1, (1/2)a_1(1-2\epsilon)+1]$ not equal to any of the a_i's. If no such c's exist, let $n = 0$ if N is even, $n = 1$ if N is odd. Let $M = N + n$. Clearly $n = M - N \ll 2^{(1/2)a_1(1-2\epsilon)} \ll N^{1/2-\epsilon} \ll \sqrt{N}/\log N$ and only (3.23) requires further analysis.

As before, $A_2(2^M - 1) = \Sigma_{s \leqq M}\binom{M}{s}\alpha(s)$.

We rewrite this as

$$A_2(2^M - 1) = s_1 + s_2 \tag{3.24}$$

where

$$s_1 = 2^n \sum_s \binom{N}{s} \alpha(s) = 2^{M-N} A_2(2^N - 1) \tag{3.25}$$

$$s_2 = \sum_s \left\{ \binom{M}{s} - 2^n \binom{N}{2} \right\} \alpha(s). \tag{3.26}$$

We bound s_2 from above by writing

$$s_2 \ll \log M \sum_s \left| \binom{M}{s} - 2^N \binom{N}{2} \right|. \tag{3.27}$$

But

$$\begin{aligned}
\sum_s \left| \binom{M}{s} - 2^N \binom{N}{s} \right| &= \sum_s \left| \sum_q \binom{N}{s-q} \binom{n}{q} - \binom{N}{s} \binom{n}{q} \right| \\
&\leqq \sum_q \binom{n}{q} \sum_s \left| \binom{N}{s-q} - \binom{N}{s} \right| \\
&\leqq \sum_q \binom{n}{q} \cdot 2q \max \binom{N}{s} \ll n \cdot 2^n \cdot 2^N / \sqrt{N} \\
&\ll \frac{\sqrt{N}}{\log N} \cdot 2^n \cdot 2^N / \sqrt{N} = \frac{2^{N+n}}{\log N}
\end{aligned}$$

so $s_2 \ll 2^{N+n} \ll 2^n x = 2^{M-N} x$ and

$$A_2(2^M - 1) = 2^{M-N} \{A_2(x) + O(x)\}, \tag{3.28}$$

proving the lemma.

COROLLARY 1. *If* $x = 2^N - 1$, *then*

$$\frac{1}{2} + O\left(\frac{\log\log\log x}{\log\log x}\right) \leqq c(x) \leqq \frac{3}{2} + O\left(\frac{\log\log\log x}{\log\log x}\right).$$

Proof. Find an M as in Lemma 3 so that

$$A_2(x) = \frac{A_2(2^M - 1)}{2^{M-N}} + O(x).$$

Applying Lemma 2 to $2^M - 1$ immediately yields this result.

LEMMA 4. *Let* $x = \sum_{i=1}^r 2^{s_i}$, $s_1 > s_2 > \cdots > s_r$. *Then*

$$A_2(x) = \sum_{i=1}^{r} A_2(2^{s_i} - 1) + O(x \log\log x / \sqrt{\log x}). \tag{3.29}$$

Proof.

$$A_2(x) = \sum_{i=1}^{r} \sum_{n<2^{s_i}} \alpha_2 \left(\sum_{j=1}^{i-1} 2^{s_j} + n \right).$$

Since $\alpha(\sum_{j=1}^{i-1} 2^{s_j} + n) = \alpha(n) + i - 1$ we obtain

$$A_2(x) = \sum_{i-1}^{r} \sum_{n<2^{s_i}} \alpha(\alpha(n) + i - 1).$$

Letting $E_i = \sum_{n<2^{s_i}} \{\alpha(\alpha(n) + i - 1) - \alpha(\alpha(n))\}$, we obtain

$$A_2(x) = \sum_{i=1}^{r} A_2(2^{s_i} - 1) + \sum_{i=2}^{r} E_i. \tag{3.30}$$

We now must merely show that $\sum_{i=2}^{r} E_i = O(x \log\log x / \sqrt{\log x})$. Rewrite

$$E_i = \sum_{l \leq s_i} \sum_{\substack{n<2^{s_i}-1 \\ \alpha(n)=l}} \{\alpha(l + i - 1) - \alpha(l)\}$$

$$= \sum_{l} \binom{s_i}{l} \{\alpha(l + i - 1) - \alpha(l)\}.$$

Summing by parts,

$$E_i = \sum_{l} \alpha(l) \left\{ \binom{s_i}{l - i + 1} - \binom{s_i}{l} \right\}.$$

Since $\alpha(l) = O(\log(s_i + i))$ and

$$\sum_{l} \left| \binom{s_i}{l - i + 1} - \binom{s_i}{l} \right| = O\left(i \binom{s_i}{[s_i]/2} \right) = O\left(i \frac{2^{s_i}}{\sqrt{s_i}} \right)$$

we obtain

$$E_i \ll i \cdot \log(s_i + i) 2^{s_i} / \sqrt{s_i}. \tag{3.31}$$

Thus

$$\sum_{i=2}^{r} E_i \ll \sum_{i=2}^{r} i \log(s_i + i) 2^{s_i} / \sqrt{s_i}.$$

Since $s_i \leqq s_1 - i + 1$ and $s_i + i \ll \log x$, and writing $s = s_1$, we obtain

$$\sum_{i=2}^{r} E_i \ll \log\log x \sum_{i=1}^{r} i 2^{s-i} / \sqrt{s-i}. \tag{3.32}$$

Now

$$\sum_{i \leqq s/2} i 2^{s-i} / \sqrt{s-i} \ll \frac{2^s}{\sqrt{s}} \sum \frac{i}{2^i} \ll 2^s / \sqrt{s}$$

while

$$\sum_{i > s/2} i 2^{s-i} / \sqrt{s-i} \ll \sum_{i > s/2} i 2^{s-i} \ll s \cdot 2^{s/2} \ll \frac{2^s}{\sqrt{s}}.$$

Since $2^s = O(x)$ and $s = \log x / \log 2 + O(1)$, we obtain

$$\sum_{i=2}^{r} E_i = O(x \log\log x / \sqrt{\log x}), \tag{3.33}$$

completing the proof of Lemma 4.

We can now easily prove Theorem 2.

Proof of Theorem 2. Let $x = \sum_{i=1}^{r} 2^{s_i}$. By Lemma 4,

$$A_2(x) = \sum_{i=1}^{r} A_2(2^{s_i} - 1) + O(x). \tag{3.34}$$

Corollary 1 implies that

$$A_2(2^{s_i} - 1) \leqq \frac{3}{2} \frac{2^{s_i} \log\log x}{2 \log 2} + O(2^{s_i} \log\log\log x),$$

so that

$$A_2(x) \leqq \frac{3}{2} \sum_{i=1}^{r} \left(\frac{2^{s_i} \log\log x}{2 \log 2} + O(2^{s_i} \log\log\log x) \right) + O(x)$$

and hence

$$A_2(x) \leqq \frac{3}{2} \frac{x \log\log x}{2 \log 2} + O(x \log\log\log x) \tag{3.35}$$

which is equivalent to

$$c(x) \leqq \frac{3}{2} + O\left(\frac{\log\log\log x}{\log\log x}\right). \tag{3.36}$$

We now obtain a lower bound. Again using Corollary 1, we obtain

$$A_2(x) \geqq \frac{1}{2} \sum_{i=1}^{r} \left(\frac{2^{s_i} \log\log 2^{s_i}}{2 \log 2} + O(2^{s_i} \log\log\log x)\right) + O(x)$$

and hence

$$A_2(x) \geqq \frac{1}{2} \sum_{i=1}^{r} \frac{2^{s_i} \log\log 2^{s_i}}{2 \log 2} + O(x \log\log\log x). \tag{3.37}$$

Since $\log\log 2^{s_i} = \log\log x + O(1)$ if $s_i \geqq s_1/2$, we obtain

$$A_2(x) \geqq \frac{1}{2} \sum_{s_i \geqq s_1/2} \frac{2^{s_i} \log\log x}{2 \log 2} + O(x \log\log\log x). \tag{3.38}$$

But $\Sigma_{s_i \geqq s_1/2} 2^{s_i} = x + O(2^{s_1/2}) = x + O(\sqrt{x})$ yielding

$$A_2(x) \geqq \frac{1}{2} \frac{x \log\log x}{2 \log 2} + O(x \log\log\log x) \tag{3.39}$$

which implies

$$c(x) \geqq \frac{1}{2} + O\left(\frac{\log\log\log x}{\log\log x}\right), \tag{3.40}$$

completing the proof of Theorem 2.

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