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TAUTNESS FOR ALEXANDER–SPANIER COHOMOLOGY

E. H. SPANIER

The purpose of this note is to give a straightforward unified proof of the tautness of Alexander–Spanier cohomology in the cases where it is known to be valid and to give a necessary condition that every closed (arbitrary) subspace be taut with respect to zero dimensional cohomology.

Let F denote a contravariant functor from the category of topological spaces to the category of abelian groups. A subspace A of a topological space X is said to be *taut with respect to* F if the canonical map $\varinjlim\{F(U)\} \to F(A)$ is an isomorphism (the direct limit is taken over the family of all neighborhoods of A in X, the family being directed downward by inclusion). The subspace A is *taut* in X if it is taut with respect to the Alexander–Spanier cohomology theory $\bar{H}$ for every dimension and every coefficient group (for notation and terminology dealing with $\bar{H}$ see [6]).

This concept of tautness has proved to be important. In [6] and [7] it is shown that a closed subspace of a paracompact Hausdorff space is taut, and this is used to deduce a strong excision property for $\bar{H}$. This tautness property is also used in [6] to derive the continuity property for $\bar{H}$. In [4] it is shown that an arbitrary subspace of a metric space is taut with respect to Čech cohomology, and this is used to obtain a general duality in spheres. Since the Čech cohomology is isomorphic to $\bar{H}$ [3], every subspace of a metric space is taut. In [2] it is shown that every neighborhood retract of X is taut in X, and this is used to prove a generalized homotopy property for compact spaces. In [1] tautness is considered for sheaf cohomology and used in proving the Vietoris–Begle mapping theorem.

We shall prove a simple lemma which gives a sufficient condition for tautness. This sufficient condition is enough to establish tautness in all the various cases where it is known.

Let $\mathcal{U}$ be a collection of subsets of X and A a subset of X. The *star of* A *with respect to* $\mathcal{U}$, denoted by $\mathrm{st}(A, \mathcal{U})$, is defined to be the union of those elements of $\mathcal{U}$ whose intersection with A is nonempty. An *open covering of* A *in* X is a collection $\mathcal{U}$ of open sets of X such that $A \subset \mathrm{st}(A, \mathcal{U})$.

The following seems to be the main fact underlying tautness (see [2] and [6]).

LEMMA. *Let A be a subspace of X and suppose that for every open covering $\mathcal{U}$ of A in X there are an open covering $\mathcal{V}$ of A in X and a function (not necessarily continuous) f: $\operatorname{st}(A, \mathcal{V}) \to A$ such that:*

(1) *$f(a) = a$ for all $a \in A$.*

(2) *For each $V \in \mathcal{V}$ with $V \cap A \neq \varnothing$ there is $U \in \mathcal{U}$ such that $V \cup f(V) \subset U$.*

Then A is taut in X.

Proof. (Recall the notation is as in **[6]**.) An arbitrary q-dimensional cohomology class of A is represented by a q-cochain $\varphi \in C^q(A)$ such that $\delta\varphi = 0$ on $\mathcal{U}^{q+2} \cap A^{q+2}$ where $\mathcal{U}$ is an open covering of A in X. Choose $\mathcal{V}$ and f with respect to this $\mathcal{U}$ to satisfy (1) and (2). Then $f^*\varphi \in C^q(\operatorname{st}(A, \mathcal{V}))$ is a q-cochain such that $\delta f^*\varphi = f^*\delta\varphi$, and, by (2), the latter vanishes on $\{V \in \mathcal{V} \mid V \cap A \neq \varnothing\}^{q+2}$. Thus, $f^*\varphi$ represents an element of $\bar{H}^q(\operatorname{st}(A, \mathcal{V}))$, and, by (1), its restriction to A is the element of $\bar{H}^q(A)$ represented by φ. Therefore, the canonical map $\varinjlim\{\bar{H}^q(U)\} \to \bar{H}^q(A)$ is an epimorphism.

Let U be a neighborhood of A. An element of $\bar{H}^q(U)$ whose restriction to A is 0 is represented by a q-cochain $\varphi \in C^q(U)$ such that $\delta\varphi = 0$ on $\mathcal{U}_1^{q+2}$ where $\mathcal{U}_1$ is an open covering of U and such that there is a $(q-1)$-cochain $\varphi' \in C^{q-1}(A)$ with $\varphi \mid A = \delta\varphi'$ on $\mathcal{U}_2^{q+1} \cap A^{q+1}$ where $\mathcal{U}_2$ is an open covering of A in X. Let $\mathcal{U} = \{U_1 \cap U_2 \mid U_1 \in \mathcal{U}_1$ and $U_2 \in \mathcal{U}_2\}$. Then $\mathcal{U}$ is an open covering of A in X such that $\delta\varphi = 0$ on $\mathcal{U}^{q+2}$ and $\varphi \mid A = \delta\varphi'$ on $\mathcal{U}^{q+1} \cap A^{q+1}$. Let $\mathcal{V}$ and f satisfy (1) and (2) with respect to this $\mathcal{U}$. It follows from (1) and (2) using the Fundamental Lemma 9.1 of **[5]** that $\varphi \mid \operatorname{st}(A, \mathcal{V})$ and $f^*(\varphi \mid A)$ represent the same element of $\bar{H}^q(\operatorname{st}(A, \mathcal{V}))$. Since $f^*(\varphi \mid A) = f^*\delta\varphi' = \delta f^*\varphi'$ on $\{V \in \mathcal{V} \mid V \cap A \neq \varnothing\}^{q+1}$, we see that $f^*(\varphi \mid A)$ represents 0 in $\bar{H}^q(\operatorname{st}(A, \mathcal{V}))$. Therefore, $\varphi \mid \operatorname{st}(A, \mathcal{V})$ represents 0 in $\bar{H}^q(\operatorname{st}(A, \mathcal{V}))$, and the canonical map $\varinjlim\{\bar{H}^q(U)\} \to \bar{H}^q(A)$ is a monomorphism.

THEOREM 1. *In each of the following cases A is taut in X.*

(1) *A is compact and X is Hausdorff.*

(2) *A is closed and X is paracompact Hausdorff.*

(3) *A is arbitrary and every open subset of X is paracompact Hausdorff.*

(4) *A is a neighborhood retract of X.*

Proof. In each of the first three cases it is easy to verify that if $\mathcal{U}$ is any open covering of A in X there is an open covering $\mathcal{V}$ of A in X such that the collection $\{\operatorname{st}(V, \mathcal{V}) \mid V \in \mathcal{V}$ and $V \cap A \neq \varnothing\}$ is a refinement of $\mathcal{U}$. If f: $\operatorname{st}(A, \mathcal{V}) \to A$ is defined so that $f(a) = a$ for $a \in A$ and so that for every $x \in \operatorname{st}(A, \mathcal{V})$ there is $V' \in \mathcal{V}$ with x and $f(x)$ both in V', then $\mathcal{V}$

and f satisfy (1) and (2) of the Lemma with respect to $\mathcal{U}$ (see Lemma 1 on p. 316 of [6]). Therefore, A is taut in X.

In the fourth case let $r: N \to A$ be a retraction of an open neighborhood N of A to A. If $\mathcal{U}$ is an open covering of A in X let $\mathcal{V} = \{U \cap r^{-1}(U \cap A) \mid U \in \mathcal{U}\}$. Then $\mathcal{V}$ is an open covering of A in X. Define $f: \operatorname{st}(A, \mathcal{V}) \to A$ by $f = r \mid \operatorname{st}(A, \mathcal{V})$. Then $\mathcal{V}$ and f satisfy (1) and (2) of the Lemma with respect to $\mathcal{U}$ and so A is taut in X.

The following result is a necessary condition for tautness of every closed (arbitrary) subspace with respect to $\bar{H}^0$. It can be used to provide examples where tautness fails to hold.

THEOREM 2. *If X is a space such that every closed (arbitrary) subspace is taut with respect to $\bar{H}^0$, then X is normal (completely normal).*

Proof. We present the proof in the completely normal case, the normal case being analogous. To show X is completely normal it suffices to show that if E and F are subsets of X such that $\bar{E} \cap F = \varnothing = E \cap \bar{F}$ then E and F can be separated by open sets in X. Given such E and F let $A = E \cup F$. Then A is a subspace of X and E and F are both open and closed in A. Let φ be the 0-cocycle on A which is 0 on E and 1 on F. Assuming A is taut in X, there is an open neighborhood W of A in X and a 0-cocycle ψ on W such that $\psi \mid A = \varphi$. Since a 0-cocycle is a locally constant function, $U = \{x \in W \mid \psi(x) = 0\}$ and $V = \{x \in W \mid \psi(x) = 1\}$ are disjoint open sets in W, hence in X, which separate E and F.

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UNIVERSITY OF CALIFORNIA
BERKELEY, CA 94720