

EFFLUENT AND NONEFFLUENT FIXED POINTS ON DENDRITES

HELGA SCHIRMER

This paper gives a partial answer to the problem of establishing conditions for the existence of selfmaps of one-dimensional spaces with prescribed fixed points and fixed point indices. Two types of isolated fixed points on dendrites are defined, and called effluent and noneffluent fixed points. They correspond on polyhedral trees to fixed points of minimal or maximal algebraic index, but are characterized by separation properties. Necessary and sufficient conditions are given for the existence of a selfmap of a dendrite which has a prescribed set of effluent and noneffluent fixed points.

1. Introduction. It is known that dendrites, as well as many polyhedra, have the “complete invariance property” (see e.g. [7], [12]), which means that any arbitrarily given closed and nonempty subset of these spaces can be the fixed point set of a suitable selfmap. For a fairly large class of polyhedra this result has been sharpened considerably: necessary and sufficient conditions have been established for the existence of a selfmap within a given homotopy class for which not only the locations of the (finitely many) fixed points, but also their indices are prescribed [8]. The construction of such a selfmap uses Nielsen’s theory of fixed point classes and the “splitting” and “moving” of fixed points. These methods fail completely in one-dimensional spaces and hence cannot be used for dendrites. Nevertheless the question arises under which conditions selfmaps of dendrites exist for which not only the locations of the fixed points, but also their indices are given.

We give here a partial answer to this question. Dendrites are acyclic continua, and hence our proofs employ methods which belong to continua theory. They consist of an exploitation of the connectedness properties of dendrites which can e.g. be found in [4] and [13], as well as of the partial order structure of dendrites which was developed by L. E. Ward, Jr. [9], [10] and has often been found useful in the investigation of fixed point questions. These methods belong to general topology, and it is therefore necessary to replace the usual definition of the fixed point index by a topological rather than an algebraic one. This is, however, done only for fixed points which correspond to those with a minimal or a maximal index. If c is an isolated fixed point on a polyhedral tree and is of order $o(c)$, then one can easily check that its index $i(c)$ under different

selfmaps can assume values for which $1 - o(c) \leq i(c) \leq 1$. The so-called effluent fixed points defined in §2 correspond to fixed points with index $1 - o(c)$, and the noneffluent ones to those of index one. Inessential fixed points are those which correspond to fixed points of index zero. We do not consider fixed points which correspond to those of an index different from 1, 0, or $1 - o(c)$.

Before we investigate the existence of selfmaps on dendrites with given effluent and noneffluent fixed points, we establish some properties of these types of fixed points. We show that noneffluent fixed points are always essential (Theorem 3.2), and that every selfmap of a dendrite has at least one noneffluent fixed point (Theorem 3.5). An effluent fixed point is essential if and only if it is not an endpoint (Theorem 4.1). A map need not have an effluent fixed point, and the existence of essential effluent fixed points implies the existence of more than one noneffluent one. Two lower bounds for the number of noneffluent fixed points in the presence of effluent fixed points are established in Theorems 4.2 and 4.4.

These properties are used in the proof of the main result (Theorem 5.2), which gives necessary and sufficient conditions for the existence of a selfmap of a dendrite which has a prescribed finite set of effluent and noneffluent fixed points. These conditions are more complicated than in the polyhedral case [8], and the case where all possible values of fixed point indices for selfmaps of dendrites are considered will likely turn out to be quite cumbersome.

A convex metric of the dendrite is used throughout. I do not know how far the results can be extended to trees, i.e., to the nonmetric case. The metric is not used in the definition of effluent and noneffluent fixed points and could be omitted in the definition of inessential fixed points. But it is, in the light of some work on trees by L. E. Ward, Jr. [12], quite possible that at least §5 will not hold for nonmetric trees. An attempt to prove the results of §3 and §4 without the help of a metric seems to have a better chance of success.

I would like to thank the referee for his helpful suggestions.

2. Different types of fixed points on dendrites. Let us first recollect the definition of a dendrite, and those of its properties needed in this paper.

A *dendrite* D is a metric continuum in which every pair of distinct points is separated by a third one. D has a partial order structure which was developed by L. E. Ward, Jr. [9], [10]. It is obtained by selecting an arbitrary point $a \in D$ as root, and defining $x \leq y$ if $x = a$, x separates a

and y , or $x = y$. Then $a \leq x$ for all $x \in D$. The sets

$$L(x) = \{y \in D \mid y \leq x\}$$

and

$$M(x) = \{y \in D \mid x \leq y\}$$

are closed in D , the set $M(x) \setminus \{x\}$ is open, and $L(x)$ is a chain (i.e., is linearly ordered). A point m is called a *maximum* (*minimum*) of a subset A of D if $m \not< x$ ($x \not< m$) for each $x \in A$. It was proved by Ward ([9], Theorem 1; [10], Theorem 1) that every nonempty closed subset of D has a maximum and a minimum. As $L(x)$ is a chain, the maximum and minimum of a closed nonempty subset of $L(x)$ is unique. So is the minimum of a subdendrite ([3], Lemma 2).

It follows from [13], pp. 88–89, that D is locally connected and that every connected subset is arc-connected. The arc $[x, y]$ between any two points $x, y \in D$ is unique, and consists of all points which separate x and y . Every closed and connected subset of D is a subdendrite ([13], p. 89), and hence $M(x)$ and $(D \setminus M(x)) \cup \{x\}$ are subdendrites. A dendrite has, at each of its points, a neighbourhood basis \mathcal{N} so that every $N \in \mathcal{N}$ is a subdendrite with a finite boundary. (As D is regular ([4], p. 301), there exists for every neighbourhood U of a point $x \in D$ a neighbourhood V of x which has a finite boundary, and we can also ask that its closure $\text{Cl} V$ is contained in U . A neighbourhood $N \in \mathcal{N}$ with $N \subset U$ can then be obtained as the closure of the arc-component of V which contains x .) We call the elements $N \in \mathcal{N}$ the *basic neighbourhoods* of D at x .

The *order* $o(x)$ of a point x of D is defined in [13], p. 48. The point x is called an *endpoint* if its order is one, and a *branchpoint* if its order is greater than two. If either the order of x or the number of components of $D \setminus \{x\}$ is finite, then these two numbers are equal ([13], p. 88).

It is the main purpose of this paper to construct mappings with given effluent and noneffluent fixed points. We now define these, as well as inessential fixed points. All definitions apply only to a fixed point c of $f: D \rightarrow D$ which is *isolated*, i.e., which has a neighbourhood U such that $\text{Cl} U \cap \text{Fix } f = \{c\}$, where $\text{Fix } f$ denotes the fixed point set of f . We give D a convex metric d [1], [6], and denote the boundary of a subset by Bd .

DEFINITION 2.1. Let D be a dendrite and $f: D \rightarrow D$ be a map with fixed point set $\text{Fix } f$. An isolated fixed point $c \in \text{Fix } f$ is called *inessential* if for every neighbourhood U of c with $\text{Cl} U \cap \text{Fix } f = \{c\}$ and every $\epsilon > 0$ there exists a map $g: D \rightarrow D$ such that

- (i) $d(f(x), g(x)) < \epsilon$ for all $x \in D$,
- (ii) $g(x) = f(x)$ for all $x \in D \setminus U$,
- (iii) g is fixed point free on CU .

Otherwise an isolated fixed point is called *essential*.

DEFINITION 2.2. Let D be a dendrite and $f: D \rightarrow D$ be a map with fixed point set $\text{Fix } f$. A fixed point $c \in \text{Fix } f$ is called *effluent* if every neighbourhood of c contains a basic neighbourhood $N(c)$ such that $N(c) \cap \text{Fix } f = \{c\}$ and such that x separates c and $f(x)$ whenever $x \in \text{Bd } N(c)$. A fixed point $c \in \text{Fix } f$ is called *noneffluent* if every neighbourhood of c contains a basic neighbourhood $N(c)$ such that $N(c) \cap \text{Fix } f = \{c\}$ and such that x does not separate c and $f(x)$ whenever $x \in \text{Bd } N(c)$.

We see that a source of a vector field corresponds to an effluent fixed point, and that sinks and circulations correspond to noneffluent fixed points. — It should be noted that noneffluence is a stronger condition than the negation of effluence and that the term “strictly noneffluent” would have been more precise. We have avoided this pedanticism as confusion seems unlikely.

3. Some properties of noneffluent fixed points. We will show in this paragraph that every noneffluent fixed point is essential (Theorem 3.2), prove that there always exists at least one noneffluent fixed point (Theorem 3.5), and finally state a criterion for the existence of exactly one noneffluent fixed point (Theorem 3.6). A frequent tool used in the proofs here and further on is a retraction of the dendrite D onto a given subdendrite D_0 .

LEMMA 3.1. *Let D_0 be a subdendrite of the dendrite D which contains the root of D . Then the function $r: D \rightarrow D_0$ defined by*

$$r(x) = \max(L(x) \cap D_0) \quad \text{for all } x \in D$$

is a retraction, and $r(x) \in \text{Bd } D_0$ if $x \notin D_0$.

Proof. The function r was used by L. Lum in [5], Theorem 2.1, and it is shown there that it is a retraction. From the definition of $r(x)$ as a maximum it is immediate that $r(x) \in \text{Bd } D_0$ if $x \notin D_0$.

With the help of this lemma, the proof of the essential character of every noneffluent fixed point is simple.

THEOREM 3.2. *Every noneffluent fixed point of a dendrite is essential.*

Proof. Let a be a noneffluent fixed point of the selfmap $f: D \rightarrow D$. We choose it as the root of D . As a is noneffluent, we can find a basic neighbourhood $N(a)$ such that $N(a) \cap \text{Fix } f = \{a\}$ and $x \notin (a, f(x))$ for $x \in \text{Bd } N(a)$, where $(a, f(x)) = [a, f(x)] \setminus \{a, f(x)\}$. Let $r: D \rightarrow N$ be the retraction of Lemma 3.1.

Assume now that a is inessential, so that there exists a map $g: D \rightarrow D$ with $g(x) = f(x)$ for all $x \in D \setminus N(a)$ which is fixed point free on $N(a)$. Define $f': N(a) \rightarrow N(a)$ by $f' = r \circ g \upharpoonright N(a)$. If $x \notin \text{Bd } N(a)$ and $g(x) \in N(a)$ then $f'(x) = g(x) \neq x$, if $x \notin \text{Bd } N(a)$ and $g(x) \notin N(a)$ then $f'(x) \in \text{Bd } N(a)$, so in either case $f'(x) \neq x$. If $x \in \text{Bd } N(a)$, then $f'(x) = r \circ f(x)$ and $x \notin (a, f(x))$. Therefore $r \circ f(x) \leq f(x)$ implies $f'(x) \neq x$. Hence f' is a fixed point free selfmap of the dendrite $N(a)$, which contradicts the fact that $N(a)$ has the fixed point property. So a must be essential.

We now want to establish the existence of a noneffluent fixed point. This will involve us in a more complicated proof, and we prepare for it with two lemmas.

LEMMA 3.3. *For every $f: D \rightarrow D$ the set $E = \{x \in D \mid x \leq f(x)\}$ is closed in D .*

Proof. This follows e.g. from [11], Lemma 7.

LEMMA 3.4. *If $x', x'' \in D$ are two points such that $x'' \in (x', f(x''))$ and $x' \in (x'', f(x'))$, then f has a fixed point on (x', x'') .*

Proof. Take $f(x')$ as root. Then by assumption $f(x') < x' < x'' < f(x'')$. The set $E_0 = E \cap [x', x'']$ is closed by Lemma 3.3, is nonempty as $x'' \in E_0$, and is $\neq [x', x'']$ as $x' \notin E_0$. Let $x_0 = \min E_0$. As E_0 is closed, $x_0 \in E_0$, so that $x_0 \leq f(x_0)$. Assume now that $x_0 < f(x_0)$. The partial order of D is order dense [10], therefore we can choose t with $x_0 < t < f(x_0)$, and, as f is continuous, an open set $U(x_0)$ containing x_0 and such that $f(U(x_0)) \subset M(t) \setminus \{t\}$. But then $x \in E_0$ for all $x \in L(x_0) \cap U(x_0)$, contradicting the minimality of x_0 . So we must have $x_0 = f(x_0)$, and x_0 is the desired fixed point.

THEOREM 3.5. *If a selfmap of a dendrite has a finite fixed point set, then it has at least one noneffluent fixed point.*

Proof. Let $f = f_1: D \rightarrow D$ be a selfmap of a dendrite $D = D_1$ with a finite fixed point set $\text{Fix } f$. As $\text{Fix } f \neq \emptyset$, we can choose a point $a_1 \in \text{Fix } f$ as root. If a_1 is not as desired, then there exists a basic neighbourhood

$N_1(a_1)$ with $N_1(a_1) \cap \text{Fix } f = \{a_1\}$, and such that $a_1 < x_1 < f_1(x_1)$ for at least one $x_1 \in \text{Bd } N_1(a_1)$. Now consider the set $E_1 = \{x \in D_1 \mid x \leq f_1(x)\}$. As $x_1 \in E_1$, we have $E_1 \neq \{a_1\}$. An argument similar to the one used in the proof of Lemma 3.4 shows that no x with $x < f_1(x)$ can be a maximal element of E_1 . But E_1 is closed by Lemma 3.3, and hence must contain at least one maximal element, a_2 say. Then $a_2 \in E_1 \cap \text{Fix } f$, and as $x_1 \in E_1$, we have $a_2 \neq a_1$.

Assume now that a_2 is again not as desired. We shall show that in this case there must exist a third fixed point a_3 . For this purpose, select a basic neighbourhood $N_2(a_2)$ with $N_2(a_2) \cap \text{Fix } f = \{a_2\}$, such that $x_2 \in (a_2, f_1(x_2))$ for at least one $x_2 \in \text{Bd } N_2(a_2)$. If $x_2 \in M(a_2) \setminus \{a_2\}$, then $a_2 < x_2 < f_1(x_2)$ implies $x_2 \in E_1$ which contradicts the maximality of a_2 . Hence $x_2 \in \text{Bd } N_2(a_2) \cap (D \setminus M(a_2))$. Let $y_2 = \max(L(a_2) \cap L(x_2))$, so that $a_1 < y_2 < a_2$, and denote by $K(a_2)$ the component of $D_1 \setminus \{y_2\}$ which contains a_2 . As $y_2 \in [a_2, x_2]$, we have $y_2 \in N_2(a_2)$ and therefore $f_1(y_2) \neq y_2$. We also have $f_1(y_2) \notin K(a_2)$, for otherwise the assumptions of Lemma 3.4 would be satisfied for x_2, y_2 (note that $f_1(y_2) \in K(a_2)$ implies $y_2 \neq x_2$) and f_1 would have a fixed point on $(x_2, y_2) \subset N_2(a_2)$, in contradiction to $N_2(a_2) \cap \text{Fix } f = \{a_2\}$. Now let $D_2 = D_1 \setminus K(a_2)$, and define $f_2: D_2 \rightarrow D_2$ by $f_2 = r_2 \circ f_1|_{D_2}$, where $r_2: D_1 \rightarrow D_2$ is the retraction from Lemma 3.1. As a_1 is not a noneffluent fixed point of f_1 , it is not a noneffluent fixed point of f_2 . We can therefore repeat the argument from the beginning of the proof for f_2 and $E_2 = \{x \in D_2 \mid x \leq f_2(x)\}$ instead of f_1 and E_1 , and find a maximal element a_3 of E_2 . Again $E_2 \neq \{a_1\}$: if $y_2 \notin E_2$, then $y_2 \in D_2$ implies $y_2 \not\leq f_1(y_2)$. If also $x_1 \notin E_2$, then $x_1 < f_1(x_1)$ implies $x_1 \notin D_2$ and hence $y_2 < x_1$. But then Lemma 3.4 yields a fixed point of f_1 on (y_2, x_1) which must be different from a_1 , and this cannot happen as now $(y_2, x_1) \subset N_1(a_1)$ and $N_1(a_1) \cap \text{Fix } f = \{a_1\}$. Therefore E_2 must contain at least one of x_1 and y_2 , and we have $a_3 \in \text{Fix } f$, $a_3 \neq a_1$, and also $a_3 \neq a_2$ and $a_2 \notin E_2$.

If a_3 is again not as desired, we repeat the last step of the proof, and obtain a fixed point $a_4 \in D_3 = D_2 \setminus K(a_3)$, where $K(a_3)$ is defined in D_2 analogously to $K(a_2)$ in D_1 , and such that $a_4 \in \text{Fix } f \setminus \{a_1, a_2, a_3\}$. This process, if continued, must eventually lead to a noneffluent fixed point as the fixed point set $\text{Fix } f$ is finite, and therefore Theorem 3.5 must be true.

We finally state a result which shows that usually more than one noneffluent fixed point can be found.

THEOREM 3.6. *Let f be a selfmap of a dendrite with a finite fixed point set. Then f has exactly one noneffluent fixed point if and only if all but one of its fixed points are inessential.*

The proof of Theorem 3.6 is lengthy and uses similar methods to

those employed in the proofs of Theorems 3.2 and 3.5. It is omitted as the theorem is not used in the rest of this paper.

4. Some properties of effluent fixed points. We now turn our attention to effluent fixed points. An effluent fixed point need not be essential, but we can show that it is inessential if and only if it is an endpoint (Theorem 4.1). There is no counterpart to Theorem 3.5, as effluent fixed points need not exist. But if they do, then the number of noneffluent fixed points increases. We establish in Theorems 4.2 and 4.4 two lower bounds for the number of noneffluent fixed points for maps which have effluent fixed points.

THEOREM 4.1. *An effluent fixed point of a selfmap of a dendrite is inessential if and only if it is an endpoint.*

Proof. (i) Let b be an effluent and inessential fixed point of the selfmap f of the dendrite D , and $\text{Fix } f$ be its (not necessarily finite) fixed point set. We use b as root. Assume by way of contradiction that b is not an endpoint, so that $D \setminus \{b\}$ has at least two components K_1 and K_2 . As b is effluent we can find a basic neighbourhood $N(b)$ with the following properties: $N(b) \not\supseteq K_i$ for $i = 1, 2$, $N(b) \cap \text{Fix } f = \{b\}$, and $x < f(x)$ for all $x \in \text{Bd } N(b)$. If $y_i \in K_i \setminus N(b)$, then the arc $[b, y_i]$ is contained in the arc-connected set $K_i \cup \{b\}$. From $b \in N(b)$ and $y_i \notin N(b)$ it follows that there exists $x_i \in (b, y_i) \cap \text{Bd } N(b)$ for $i = 1, 2$. As b is inessential, we can find a selfmap g of D such that $g(x) = f(x)$ for all $x \in D \setminus N(b)$ and such that g is fixed point free on $N(b)$. But then $g(x_i) = f(x_i)$ and therefore $x_i < g(x_i)$ for $i = 1, 2$, so that Lemma 3.4 asserts that g has a fixed point on $(x_1, x_2) \subset N(b)$. Hence we arrive at a contradiction.

(ii) Now consider an effluent fixed point $b \in D$ which is an endpoint, and select it as root. Choose any $\epsilon > 0$, and any neighbourhood $U(b)$ with $\text{Cl } U(b) \cap \text{Fix } f = \{b\}$. Take $\delta > 0$ such that $\text{diam } P < \delta$ implies $\text{diam } f(P) < \epsilon$ for all $P \subset D$. As b is effluent there exists a basic neighbourhood $N(b) \subset U(b)$ such that $x < f(x)$ for all $x \in \text{Bd } N(b)$; we can also require that $\text{diam } N(b) < \delta$ and $N(b) \neq D$. As b is an endpoint, we can choose an open neighbourhood $V(b)$ with $\text{Cl } V(b) \subset N(b)$ and such that $\text{Bd } V(b)$ consists of exactly one point x_1 . Let $K(b)$ be the component of $V(b)$ which contains b . Then $K(b)$ is open in $V(b)$, $\text{Bd } K(b) = \{x_1\}$, and it is easy to see that $K(b) \cup M(x_1) = D$. As $K(b) \cup M(x_1) = D$ and $K(b) \subset N(b) \neq D$, there exists a point $x' \in M(x_1) \setminus N(b)$, and therefore a point $x_2 \in [x_1, x'] \cap \text{Bd } N(b)$. Then $x_2 < f(x_2)$, and f is fixed point free on $(x_1, x_2) \subset N(b)$, so that Lemma 3.4 implies $x_1 < f(x_1)$. We see that $f(x_1) \notin K(b)$, for otherwise $x_1 \in$

$[b, f(x_1)] \subset K(b)$. But x_1 is the boundary of the open set $K(b)$, so $x_1 \notin K(b)$.

We define $g: D \rightarrow D$ by

$$f(x) = \begin{cases} f(x_1) & \text{if } x \in \text{Cl } K(b), \\ f(x) & \text{if } x \in D \setminus K(b). \end{cases}$$

Then g is continuous, $d(f(x), g(x)) < \epsilon$, $g(x) = f(x)$ for $x \in D \setminus U(b)$ and g is fixed point free on $\text{Cl } U(b)$. Hence b is inessential.

It was not necessary in Theorem 4.1 to assume that f has a finite fixed point set, but in the following theorem the assumption is needed.

THEOREM 4.2. *If b is an effluent fixed point of a selfmap of a dendrite D with a finite fixed point set, then every component of $D \setminus \{b\}$ contains at least one noneffluent fixed point.*

Proof. Take the effluent fixed point b of the map $f: D \rightarrow D$ with fixed point set $\text{Fix } f$ as root, and let K be a component of $D \setminus \{b\}$. As $D_0 = K \cup \{b\}$ is a subdendrite, a retraction $r: D \rightarrow D_0$ can be defined, as in Lemma 3.1, by $r(x) = x$ if $x \in D_0$ and $r(x) = b$ if $x \notin D_0$. It follows from Theorem 3.5 that the map $g: D_0 \rightarrow D_0$ given by $g(x) = r \circ f(x)$ for all $x \in D_0$ has a noneffluent fixed point a . If $a \neq b$, then the theorem is proved, as a noneffluent fixed point of g is clearly a noneffluent fixed point of f if it is contained in the open subset K of D . It remains to rule out the possibility that $a = b$.

If $a = b$ and b is effluent for f , then we can select a neighbourhood $U(b)$ in D with $U(b) \cap \text{Fix } f = \{b\}$ which contains a basic neighbourhood $N(b)$ with $x < f(x)$ for $x \in \text{Bd } N(b)$, and such that $K \not\subset N(b)$. Then $K \cap \text{Bd } N(b) \neq \emptyset$, so that there exists an $x_1 \in K \cap \text{Bd } N(b)$ with $x_1 < f(x_1)$. Let $U'(b)$ be the neighbourhood of b in D_0 which consists of all points x with $d(b, x) < \frac{1}{2}d(b, x_1)$. As a is noneffluent for g , $U'(b)$ contains a basic neighbourhood $N'(b)$ of b in D_0 with $x \not< g(x)$ for $x \in \text{Bd}_{D_0} N'(b)$, where Bd_{D_0} denotes the boundary in D_0 . The fact that $x_1 \notin N'(b)$ allows us to choose $x_2 \in \text{Bd}_{D_0} N'(b) \cap (b, x_1)$, and $x_2 \not< g(x_2)$ implies $x_2 \not< f(x_2)$. Therefore Lemma 3.4 yields a fixed point of f on $(x_1, x_2) \subset N(b) \subset U(b)$, in contradiction to $U(b) \cap \text{Fix } f = \{b\}$. So $a = b$ is impossible.

COROLLARY 4.3. *If b is an effluent fixed point of a selfmap of a dendrite with a finite fixed point set, then the order of b is finite.*

Proof. If b is not of finite order, then $D \setminus \{b\}$ has infinitely many components. But Theorem 4.2 asserts the existence of a noneffluent

fixed point in each of them, which cannot happen if the fixed point set of f is finite.

Note that the counterpart of Corollary 4.3 for noneffluent fixed points is not true. If the order of the point $a \in D$ is not finite, then the map $f: D \rightarrow D$ given by $f(x) = a$ for all $x \in D$ has a as its only fixed point, and a is noneffluent. Less trivial counterexamples can also be easily constructed.

Theorem 4.2 provides information about the location as well as the number of noneffluent fixed points, and the lower bound for the number of noneffluent fixed points obtained in it is the best possible one if only one effluent fixed point is present. But we shall now sharpen this bound considerably for the case where more than one effluent fixed point exists. We denote by $\# A$ the number of elements of the finite set A .

THEOREM 4.4. *Let a selfmap of a dendrite have a finite fixed point set, and let A be the set of its noneffluent fixed points and B be the set of its effluent fixed points. Then*

$$\# A \geq 1 + \sum_{b \in B} (o(b) - 1).$$

Proof. We know from Theorem 3.5 that $A \neq \emptyset$, so we can select a noneffluent fixed point a_0 as root. Let now $b_i \in B$ be an effluent fixed point of the selfmap f of order $o(b_i)$. It follows from Corollary 4.3 that $o(b_i)$ is finite, so that $D \setminus \{b_i\}$ has $o(b_i)$ components. Let K_{ik} ($k = 1, 2, \dots, o(b_i) - 1$) be the components of $D \setminus \{b_i\}$ which do not contain a_0 .

For each K_{ik} the set

$$D_{ik} = [K_{ik} \cup \{b_i\}] \setminus [\cup (M(b_i) \setminus \{b_i\}) \mid b_i \in K_{ik}]$$

(where the set in the second square bracket can be empty) is a subdendrite of D , and the function $r_{ik}: D \rightarrow D_{ik}$ given by

$$r_{ik}(x) = \begin{cases} x & \text{if } x \in D_{ik}, \\ b_i & \text{if } x \in D \setminus K_{ik}, \\ b_i & \text{if } x \in M(b_i) \text{ and } b_i \in K_{ik} \end{cases}$$

is a retraction. Define

$$g_{ik}: D_{ik} \rightarrow D_{ik} \quad \text{by} \quad g_{ik}(x) = r_{ik} \circ f(x)$$

for all $x \in D_{ik}$. It follows from Theorem 3.5 that g_{ik} has at least one

noneffluent fixed point, say a_{ik} , and an argument similar to the one used in the proof of Theorem 4.2 shows that $a_{ik} \neq b_i$ and $a_{ik} \neq b_j$ for all $b_j \in K_{ik}$. Therefore a_{ik} is also a noneffluent fixed point of f . In this way, we associate with each effluent fixed point b_i a set of $o(b_i) - 1$ noneffluent fixed points a_{ik} , which are different from the noneffluent fixed point a_0 .

It remains to show that all a_{ik} are different. As $b_i < a_{ik}$, the fact that $a_{ik} = a_{il}$ would imply $b_i < b_l$ or $b_j < b_i$, say $b_i < b_j$. But then $b_i < b_j < a_{ik}$, which is not possible according to the construction of the a_{ik} in D_{ik} . So a_0 and the points a_{ik} form a set of $1 + \sum_{b \in B} (o(b) - 1)$ noneffluent fixed points of f .

REMARK. In general, f has in fact more noneffluent fixed points than the minimum number required by Theorem 4.4. As an example, consider the cross in the xy -plane given by

$$D = \{(x, y) \in \mathbb{R}^2 \mid |x| \leq 1 \text{ and } y = 0, \text{ or } x = 0 \text{ and } |y| \leq 1\}$$

and define $f: D \rightarrow D$ by

$$\begin{aligned} f(x, 0) &= \left(\frac{3}{2}x - \frac{1}{2}|x|, 0\right) & \text{for all } (x, 0) \in D, \\ f(0, y) &= \left(0, \frac{1}{2}y + \frac{1}{2}|y|\right) & \text{for all } (0, y) \in D \end{aligned}$$

(i.e. f moves the points on the x -axis away from the origin, and the points on the y -axis towards the origin).

The selfmap f has five fixed points at $(0, 0)$, $(\pm 1, 0)$, and $(0, \pm 1)$. The points $(\pm 1, 0)$ are noneffluent and the points $(0, \pm 1)$ are effluent and all are of order one. The point $(0, 0)$ is neither effluent nor noneffluent. Hence

$$\# A = 2 > 1 + \sum_{b \in B} (o(b) - 1) = 1.$$

But we will show in the next paragraph, in Theorem 5.2, that f has precisely $1 + \sum_{b \in B} (o(b) - 1)$ noneffluent fixed points if f has a finite fixed point set and all fixed points of f are either effluent or noneffluent.

5. Mappings of dendrites with prescribed effluent and noneffluent fixed points. The last paragraph of this paper contains its main result, the construction of a selfmap of a dendrite with a prescribed set of effluent and noneffluent fixed points. Necessary and sufficient conditions for the existence of such maps will be given in Theorem 5.2. They are consequences of Corollary 4.3 and Theorem 4.4,

supplemented by one further condition which is motivated by Lemma 5.1. The proof of Theorem 5.2 depends heavily on the fact that D has a convex metric.

LEMMA 5.1. *Let $A \neq \emptyset$ and B be two finite subsets of a dendrite, and let the order $o(b)$ be finite for each $b \in B$. If*

$$(a', a'') \cap B \neq \emptyset \quad \text{for all distinct } a', a'' \in A,$$

then

$$(i) \quad \# A \leq 1 + \sum_{b \in B} (o(b) - 1).$$

Proof. Select an element $a_1 \in A$ as root of the dendrite D , and index the elements of A so that $L(a_i)$ does not contain more elements of B than $L(a_j)$ whenever $i < j$. This obviously implies $a_i \not\prec a_j$. Let, for all $k = 2, 3, \dots, \# A$, the finite subdendrite D_k of D be given by $D_k = \cup \{[a_1, a_i] \mid i = 2, 3, \dots, k\}$, and denote by $o_k(b_j)$ the order of b_j in D_k . We shall proceed by induction on k , and associate with each a_k an element $b_k \in B \cap D_k$ of order $o_k(b_k) \geq 2$, so that either $b_k \neq b_j$ for all $j < k$, or $o_k(b_j) = o_{k-1}(b_j) + 1$ for some $j < k$. If the points a_1, a_2, \dots, a_{k-1} and b_2, b_3, \dots, b_{k-1} satisfy

$$k - 1 \leq 1 + \sum_{j \leq k-1} (o_{k-1}(b_j) - 1),$$

then the points a_1, a_2, \dots, a_k and b_2, b_3, \dots, b_k satisfy

$$(ii) \quad k \leq 1 + \sum_{j \leq k} (o_k(b_j) - 1).$$

As $o_k(b) \leq o(b)$ for all $b \in B$ and as $\{b_j \mid 2 \leq j \leq \# A\} \subseteq B$, this gives the desired result (i).

If $k = 2$, then $(a_1, a_2) \cap B \neq \emptyset$ allows us to choose $b_2 \in (a_1, a_2)$, and as $o_2(b_2) = 2$, we see that (ii) holds for $k = 2$.

Now assume that (ii) holds for the $k - 1$ points a_1, a_2, \dots, a_{k-1} in D_{k-1} . Let $m_k = \max(L(a_k) \cap D_{k-1}) \in D_{k-1}$. If $m_k \in B \cap D_{k-1}$, then we choose $b_k = m_k$. In this case we either have $b_k = b_j$ for some $j < k$, and hence $o_k(b_k) = o_{k-1}(b_k) + 1$, or $b_k \neq b_j$ for all $j < k$. If $m_k \in A \cap D_{k-1}$, then $(m_k, a_k) \cap B \neq \emptyset$, and we can select $b_k \in (m_k, a_k)$. If finally $m_k \notin (A \cup B) \cap D_{k-1}$, then there exists an $a_i \in D_{k-1}$ with $m_k < a_i$. It cannot happen that $(m_k, a_k) \cap B \neq \emptyset$, for $(a_i, a_k) = (a_i, m_k] \cup [m_k, a_k)$ must

intersect B , and $(m_k, a_i) \cap B \neq \emptyset$ with $(m_k, a_k) \cap B \neq \emptyset$ and $i < k$ contradicts the way in which A was indexed. So we can again choose $b_k \in (m_k, a_k)$. We see that in each of these possible cases (ii) is satisfied for a_1, a_2, \dots, a_k .

We now proceed to establish the criteria for maps with prescribed sets of effluent and noneffluent fixed points.

THEOREM 5.2. *Let A and B be two finite subsets of a dendrite. Then there exists a selfmap with fixed point set $A \cup B$ for which all points in A are noneffluent and all points in B are effluent, if and only if the following three conditions hold:*

- (i) *The order $o(b)$ is finite for each $b \in B$,*
- (ii) *$(a', a'') \cap B \neq \emptyset$ for all distinct $a', a'' \in A$,*
- (iii) *$\# A \geq 1 + \sum_{b \in B} (o(b) - 1)$.*

Proof. **A. Necessity.** Condition (i) follows from Corollary 4.3 and (iii) follows from Theorem 4.4, so we only have to deal with (ii). Assume therefore that a selfmap $f: D \rightarrow D$ of a dendrite D exists with the prescribed fixed point set, and that there are two distinct points $a', a'' \in A$ for which $(a', a'') \cap B = \emptyset$. We can choose a root of D such that $a' < a''$, and can clearly also assume that $(a', a'') \cap A = \emptyset$. As a' and a'' are noneffluent we can with the help of suitable neighbourhoods find points $x', x'' \in (a', a'')$ with $a' < x' < x'' < a''$ and so that $x' \notin (a', f(x'))$ and $x'' \notin (a'', f(x''))$. But then $x'' \in (x', f(x''))$ and $x' \in (x'', f(x'))$, so Lemma 3.4 asserts that f has a fixed point on (x', x'') . This cannot happen if $A \cup B$ is the fixed point set of f , and (ii) must hold.

B. Sufficiency. Assume now that $\text{Fix } f = A \cup B$, is a finite subset of D which satisfies conditions (i), (ii), and (iii). As $\# A \geq 1$, we can index the points in A as in the proof of Lemma 5.1, and see from Lemma 5.1 (i) and Theorem 5.2 (iii) that the points b_k obtained in this proof are all the points of B apart from possibly some points of order one, i.e., we have $B = B' \cup B''$, where $B' = \{b_j \mid 2 \leq j \leq \# A\}$ with $o(b_j) \geq 2$ and not necessarily all b_j distinct, and where $o(b) = 1$ for all $b \in B''$. We also have, in the notation of the proof of Lemma 5.1, $o(b) = o_{*A}(b)$ for all $b \in B'$. Let D_k , for $2 \leq k \leq \# A$, be the subdendrite from the proof of Lemma 5.1, and $D_1 = \{a_1\}$. We first define maps $f_k: D_k \cup A \cup B' \rightarrow D_k \cup A \cup B'$ with fixed point set $A \cup B'$ inductively for $k = 1, 2, \dots, \# A$.

If $k = 1$, then $D_1 \cup A \cup B' = A \cup B'$, therefore we put $f_1(x) = x$ for all $x \in D_1 \cup A \cup B'$. Now assume that $f_{k-1}: D_{k-1} \rightarrow D_{k-1}$ has been defined. Let again $m_k = \max(L(a_k) \cap D_{k-1})$, so that $D_k = D_{k-1} \cup [m_k, a_k]$, where $[m_k, a_k] = [m_k, b_k] \cup [b_k, a_k]$, with $m_k = b_k$ or

$m_k \neq b_k$, but always $b_k \neq a_k$. In the definition of f_k we shall use parameters λ and μ , with $0 \leq \lambda \leq 1$ and $\mu = \lambda + \frac{1}{2}\lambda(1 - \lambda)$. Note that then $0 \leq \mu \leq 1$, and $\mu \geq \lambda$ with $\mu = \lambda$ if and only if $\lambda = 0$ or $\lambda = 1$. We put

$$f_k(x) = \begin{cases} f_{k-1}(x) & \text{if } x \in D_{k-1} \cup A \cup B', \\ \mu a_k + (1 - \mu)b_k & \text{if } x = \lambda a_k + (1 - \lambda)b_k, \\ \mu f_{k-1}(m_k) + (1 - \mu)b_k & \text{if } x = \lambda m_k + (1 - \lambda)b_k. \end{cases}$$

We also write $D' = D_{*A} = D_{*A} \cup A \cup B'$ and $f': D' \rightarrow D'$ for the case $k = \# A$. It is easy to check that f' is continuous, that its fixed point set is $\text{Fix } f = A \cup B'$, and that the map $f': D' \rightarrow D'$ has the points of A as noneffluent fixed points and the points of B' as effluent fixed points.

If $B'' = \emptyset$, let $D'' = D'$ and $f'' = f'$. If $B'' \neq \emptyset$, i.e., if a map with inessential effluent fixed points is to be constructed, then let $D'' = D' \cup (U([a_i, b] \mid b \in B''))$, and extend $f': D' \rightarrow D'$ to $f'': D'' \rightarrow D''$ as follows: if $x \in D'$, let $f''(x) = f'(x)$. If $x \in D'' \setminus D'$, then $x = \lambda m + (1 - \lambda)b$, for some $b \in B''$, with $m = \max(L(b) \cap D')$ and $0 \leq \lambda < 1$. Let $f''(x) = \mu f'(m) + (1 - \mu)b$, where again $\mu = \lambda + \frac{1}{2}\lambda(1 - \lambda)$. Then f'' has the fixed point set $A \cup B$, with the points of A as noneffluent and the points of B as effluent fixed points.

It remains to extend $f'': D'' \rightarrow D''$ to $f: D \rightarrow D$. For this purpose, let $r: D \rightarrow D''$ be the retraction of Lemma 3.1, $i: D'' \rightarrow D$ be the injection, and define f by $f = i \circ f'' \circ r$. Clearly f has the fixed point set $\text{Fix } f = A \cup B$. The points of A are noneffluent, as $x \notin (a, f(x))$ for $x \in \text{Bd } N(a)$, where $N(a)$ is any subdendrite of D with $N(a) \cap \text{Fix } f = \{a\}$. In order to see that the points of B are effluent take any $b \in B$, and any neighbourhood $U(b)$. Choose a neighbourhood $V(b) \subset U(b)$ with $\text{Cl } V(b) \cap \text{Fix } f = \{b\}$, and so that $\text{Cl } V(b) \setminus \{b\}$ contains no branchpoints of D'' . Let $W(b)$ be an open neighbourhood $W(b) \subset V(b)$ and so that $\text{Bd } W(b)$ consists of at most $o(b)$ points, and let $K(b)$ be the component of $W(b)$ which contains b . Finally put $N(b) = \text{Cl } K(b)$. Then $N(b)$ is a neighbourhood of b and is a subdendrite. As $N(b) \cap D'' \subset \text{Cl } V(b) \cap D''$ and $o(b) = o_{D''}(b)$, we have $N(b) \cap D'' = \cup([b, x_i] \mid i = 1, 2, \dots, o(b))$, where $[b, x_i] \cap [b, x_j] = \{b\}$ for $i \neq j$. Hence $\{x_i \mid i = 1, 2, \dots, o(b)\}$ is the boundary of $N(b) \cap D''$ in D'' . Now $\text{Bd}_{D''}(N(b) \cap D'') \subseteq \text{Bd } W(b)$ and $\text{Bd } W(b)$ consists of at most $o(b)$ points. So we see that $\text{Bd } N(b) = \{x_i \mid i = 1, 2, \dots, o(b)\}$. By construction of f we have $x_i \in (b, f(x_i))$ for all i , and therefore $N(b)$ satisfies the conditions in the definition of an effluent fixed point.

REMARK. In consequence of Lemma 5.1, we can replace the condition (iii) in Theorem 5.2 by

$$(iii)' \quad \# A = 1 + \sum_{b \in B} (o(b) - 1).$$

But condition (ii) cannot be omitted even in the presence of condition (iii)'. Consider, e.g., the dendrite D which consists of the points $0 \leq x \leq 4$ on the real line, let $A = \{0, 1, 4\}$ and $B = \{2, 3\}$. Then A and B satisfy (i) and (iii)', but there is no selfmap of D which has the points of A as its noneffluent fixed points and the points of B as its effluent fixed points.

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CARLETON UNIVERSITY, OTTAWA, CANADA