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ON ALTERNATIVE RINGS AND THEIR ATTACHED JORDAN RINGS

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Let A be an alternative ring and A^q its attached quadratic Jordan ring. We show that if A is finitely generated by n generators then A^q is finitely generated by the monomials in A of degree $\leqq n+1$. It follows that if A is finitely generated then A is nilpotent if and only if A^q is solvable, and for arbitrary A the Levitzki radical of A coincides with the Levitzki radical of A^q. Finally, if A has an involution $*$ and $H(A,*)$ denotes the $*$-symmetric elements of A then several results known for associative rings connecting properties of $H(A,*)$ to those of A apply.

The Levitzki radical $L(R)$ of a ring R (associative, Jordan, alternative) is known to be the maximal locally nilpotent ideal of R and has the properties that $L(R)$ contains all locally nilpotent ideals of R and that $L(R/L(R))=0$. In **[9, 11]** it is shown that if R is an associative or alternative algebra over a commutative ring Φ such that $1/2\in\Phi$ then $L(R)=L(R^+)$ where R^+ denotes the attached linear Jordan algebra. In §1 we extend this by considering an alternative ring A of arbitrary characteristic and its attached quadratic Jordan ring A^q. Recall that A^q is defined to be the additive group of A together with the quadratic operators x^2 and U_x: $a\mapsto xax$ for all x in A. The bilinear operators attached to these are $x\cdot y=xy+yx$ and $U_{x,y}$: $a\mapsto(xa)y+(ya)x=x(ay)+y(ax)$. The key result we prove is that if A is generated by $x_1,x_2,\cdots,x_n$ then $A^{n+2}\subseteq AU_A$ and that A^q is finitely generated by all monomials in A of degree $\leqq n+1$. This enables us to conclude that $L(A)=L(A^q)$ and that if A is finitely generated then A is nilpotent if and only if A^q is solvable.

In §2, we assume that A is a ring with involution $*$ and note that several known results for associative rings in which A inherits properties of $H(A,*)$ apply to alternative rings. In particular, if A is alternative and if the quadratic Jordan ring $H(A,*)$ is nilpotent of index n then A is nil of index $\leqq 2n$. Finally, if A is an algebra over a field with at least n elements and if $H(A,*)$ is nil of bounded index n, then A is nil of bounded index $\leqq 2n$.

1. Throughout we shall make use of the Moufang laws

$$(xax)y=x[a(xy)] \tag{1}$$

(2) $$y(xax) = [(yx)a]x$$

(3) $$(xy)(ax) = x(ya)x$$

It is known that if B, C are ideals of A then BU_C is an ideal of A. For if $b \in B$, $c \in C$, $a \in A$ then

$$(cbc)a = c(b(ca)) + (ca)(bc) - c(ab)c$$

by (1) and (3). But $c(b(ca)) + (ca)(bc) = bU_{c,ca} \in BU_C$ and $c(ab)c = (ab)U_c \in BU_C$. Thus $(BU_C)A \subseteq BU_C$. Similarly $A(BU_C) \subseteq BU_C$. In particular AU_A is an ideal of A.

LEMMA. *If u is a monomial in A of degree $\geqq 2$ in x and $u \neq x^2$ then either $u \equiv 0 \mod AU_A$ or $u \equiv x^2y \mod AU_A$ for some y in A.*

Proof. First note that $x^2y + yx^2 = xU_{x,y} \in AU_A$ so that terms of the form yx^2 are covered by the Lemma. Now in view of the fact that AU_A is an ideal of A and that $(ab)c \equiv -(cb)a \mod AU_A$, it follows that $(x^2a)b \equiv -(ba)x^2 \mod AU_A$ and $(ax^2)b \equiv -(x^2a)b \equiv (ba)x^2 \mod AU_A$. Similarly for their left-right duals: $b(ax^2) \equiv -x^2(ab) \mod AU_A$ and $b(x^2a) \equiv x^2(ab) \mod AU_A$. Thus, if we let $T_a = R_a$ or $T_a = L_a$, an easy induction on s shows that if $u = x^2T_{a_1}T_{a_2}\cdots T_{a_s}$ then $u \equiv x^2y \mod AU_A$ for some $y \in A$. It follows that if a factor of u satisfies the results of the Lemma then so does u itself.

We may assume now that u has a factor u' which takes one of the forms:

(i) $u' = xT_{a_1}T_{a_2}\cdots T_{a_k}T_x$

or

(ii) $u' = (xT_{a_1}T_{a_2}\cdots T_{a_{k_1}})(xT_{b_1}T_{b_2}\cdots T_{b_{k_2}})$

for some $a_i, b_i \in A$.

For case (i) we induct on k and note that the result is trivial for $k = 1$. Assume then that the result holds for any $w = xT_{d_1}T_{d_2}\cdots T_{d_n}T_x$ with $d_i \in A$ and $n < k$. Now if for some i $T_{a_i} = R_{a_i}$ and $T_{a_{i+1}} = R_{a_{i+1}}$ then

$$\begin{aligned} u' = xT_{a_1}T_{a_2}\cdots T_{a_k}T_x &= (((xT_{a_1}\cdots T_{a_{i-1}})a_i)a_{i+1})T_{a_{i+2}}\cdots T_{a_k}T_x \\ &\equiv -[(a_{i+1}a_i)(xT_{a_1}T_{a_2}\cdots T_{a_{i-1}})]T_{a_{i+2}}\cdots T_{a_k}T_x \mod AU_A \end{aligned}$$

so that $u' = xT_{a_1}\cdots T_{a_{i-1}}L_bT_{a_{i+2}}\cdots T_{a_k}T_x \mod AU_A$ for $b = -a_{i+1}a_i$. By the induction hypothesis on the number of T's we have our result. Similarly if $T_{a_i} = L_{a_i}$ and $T_{a_{i+1}} = L_{a_{i+1}}$ for some i. Thus $T_{a_{2m+1}} = R_{a_{2m+1}}$ and $T_{a_{2m}} = L_{a_{2m}}$ or $T_{a_{2m+1}} = L_{a_{2m+1}}$ and $T_{a_{2m}} = R_{a_{2m}}$ for all m. Therefore, if $k = 2$ we have the cases $((ax)b)x$, $(a(xb))x$, $x((ax)b)$, and $x(a(xb))$. But

$$((ax)b)x \equiv -(xb)(ax) \equiv -x(ba)x \equiv 0 \bmod AU_A \quad \text{by} \quad (3)$$

and

$$(a(xb))x \equiv -(x(xb))a \equiv -(x^2b)a \equiv (ab)x^2 \bmod AU_A$$

and similarly for the last two cases. Thus the result holds for $k = 2$.

Suppose now that $k > 2$ and that $T_{a_{2m+1}} = R_{a_{2m+1}}$ and $T_{a_{2m}} = R_{a_{2m}}$. Then

$$u' = [(a_2(xa_1))a_3]T_{a_4} \cdots T_{a_k}T_x.$$

Since A is alternative we have $a_2(xa_1) = (a_2x)a_1 + (a_2a_1)x - a_2(a_1x)$ so that

$$u' = xL_{a_2}R_{a_1}R_{a_3}T_{a_4} \cdots T_{a_k}T_x + xL_{a_2a_1}R_{a_3}T_{a_4} \cdots T_{a_k}T_x + xL_{a_1}L_{a_2}R_{a_3}T_{a_4} \cdots T_{a_k}T_x.$$

Since the the first term has two consecutive right multiplications, the last term has two consecutive left multiplications, and the middle term fewer than k T's, we have $u' = x^2$, or $u' \equiv 0 \bmod AU_A$, or $u' \equiv x^2y \bmod AU_A$ for some y by the induction hypothesis. If $T_{a_{2m+1}} = L_{a_{2m+1}}$ and $T_{a_{2m}} = L_{a_{2m}}$ we get the same result using the fact that $(a_1x)a_2 = a_1(xa_2) - (xa_1)a_2 + x(a_1a_2)$. Thus we have disposed of case (i).

For case (ii) we induct on $k = \min(k_1, k_2)$ and note that $k = 0$ is case (i). If $k_2 \leqq k_1$, we let $w = xT_{a_1} \cdots T_{a_{k_1}}$, $v = xT_{b_1} \cdots T_{b_{k_2-1}}$ and $c = b_{k_2}$ and we have one of the two cases:

$$(*) \qquad \begin{aligned} &u' = w(vc) \equiv -c(vw) \bmod AU_A \\ \text{or} \quad & \\ &u' = w(cv) \equiv -v(cw) \bmod AU_A. \end{aligned}$$

Now if $k_2 = k = 1$ then vw and $v(cw)$ are of the form of case (i) so that u' satisfies the results of the Lemma. If $k > 1$ then both vw and $v(cw)$ have a lower value of k, so by the induction hypothesis they satisfy the desired conclusion. Hence so does u'. The case $k_1 \leqq k_2$ follows from the left-right dual of (*). Thus, in all cases we get $u \equiv 0 \bmod AU_A$ or $u \equiv x^2y \bmod AU_A$ for some $y \in A$.

THEOREM 1. *If A is generated by n elements then $A^{n+2} \subseteq AU_A$.*

Proof. Let $u \in A^{n+2}$. Then since A has n generators it follows that either there is at least one generator, say x, such that the degree of u in x is $\geqq 3$ or there are at least two generators, say w and z, such that the degree of u in w is $\geqq 2$ and the degree of u in z is $\geqq 2$. If the latter holds then by the lemma if $u \not\equiv 0 \mod AU_A$ we have $u \equiv z^2y \mod AU_A$. Since y is of degree at least two in w we get $y = w^2$ or $y \equiv w^2a \mod AU_A$ for some $a \in A$. Thus, either $u \equiv z^2w^2 \mod AU_A$ or $u \equiv z^2(w^2a) \mod AU_A$. But $z^2w^2 \equiv -wz^2w \equiv 0 \mod AU_A$ and $z^2(w^2a) \equiv -a(w^2z^2) \equiv 0 \mod AU_A$. Thus in this case $u \equiv 0 \mod AU_A$.

If the former holds then $u \equiv x^2y \mod AU_A$ where y contains a factor x. Thus $u \equiv x^2(xT_{a_1}T_{a_2}\cdots T_{a_k}) \mod AU_A$ for some $a_i \in A$. Thus $u \equiv 0 \mod AU_A$ by induction on k. For if $k = 1$ then we get $u \equiv x^3a_1 \equiv 0 \mod AU_A$ or $u \equiv x^2(ax) \equiv 0 \mod AU_A$. As in the lemma we may assume that no two consecutive T's represent R or L so that the case $k = 2$ reduces to $x^2(a_2(xa_1))$ or $x^2((a_1x)a_2)$. But $x^2(a_2(xa_1)) = x[x(a_2(xa_1))] = x[(xa_2x)a_1] \equiv 0 \mod AU_A$ and $x^2((a_1x)a_2) \equiv -a_2((a_1x)x^2) \equiv 0 \mod AU_A$. The inductive step is obtained precisely as in case (i) of the lemma. Thus $u \in AU_A$ and the theorem is proven.

REMARK. The advance in Theorem 1 is not the fact that a power of A is contained in AU_A but rather in the precise value $n + 2$. For, as noted by Professor McCrimmon in a private communication, if A is finitely generated then $\bar{A} = A/AU_A$ is finitely generated and nil satisfying the polynomial identity $x^3 = 0$. This, by an earlier result of his [**6**, Theorem 3] implies that A is nilpotent so there is an integer k such that $A^k \subseteq AU_A$.

THEOREM 2. *If A is generated by $x_1, x_2, \cdots, x_n$ then the Jordan ring A^q is finitely generated by all monomials of degree $< n + 2$.*

Proof. Let F be the free alternative ring generated by $x_1, x_2, \cdots, x_n$. Then if u is an element of minimal degree in A^q not generated by the monomials of degree $\leqq n + 1$ then $\deg u \geqq n + 2$ so that $u \in F^{n+2} \subseteq FU_F$. Thus, $u = \Sigma_i a_iU_{b_i} + \Sigma_i p_iU_{q_i,r_i}$ for monomials a_i, b_i, p_i, q_i, r_i in F. Therefore a_i, b_i, p_i, q_i, r_i have lower degree than u and are generated in F^q by the monomials of degree $< n + 2$. Thus u is generated by these monomials also and we have the result for F. Now $A^q \cong F^q/K$ for some ideal K of A^q. Therefore A^q is also generated by the monomials of degree $< n + 2$.

Recall that if J is a Jordan algebra then $D(J) = JU_J$ is a quadratic ideal of J, and the derived series of J is given by

$$J = D^0(J) \supset D(J) \supset D^2(J) \supset \cdots \supset D^n(J) \supset \cdots$$

where $D^{i+1}(J) = D(D^i(J))$. J is solvable if $D^n(J) = 0$ for some n. The degree of an element is defined by $\deg(aU_b) = 2\deg b + \deg a$, $\deg(aU_{b,c}) = \deg a + \deg b + \deg c$, $\deg a^2 = 2\deg a$, and $\deg a \cdot b = \deg a + \deg b$. J is nilpotent if there is an n such that all monomials of degree $\geqq n$ are zero. McCrimmon has shown that if J is finitely generated then J is solvable iff J is nilpotent [4]. In our situation we write $D^t(A)$ to denote $D^t(A^q)$.

COROLLARY. *If A is finitely generated then for each t there is a k such that $A^k \subseteq D^t(A)$. Also $D^t(A)$ is finitely generated for every t.*

Proof. The second statement follows immediately from Theorem 2, since it is known that if a Jordan algebra J is finitely generated then so is $D^t(J)$ for all t [4]. Thus, by Theorem 2, $D^t(A)$ is finitely generated as a Jordan ring and hence, as an alternative ring. The first statement is arrived at by induction on t. The case $t = 1$ is the statement of Theorem 1. Assume true for t. Since $D^t(A)$ is a finitely generated alternative ring then by Theorem 1 there is an integer m such that $(D^t(A))^m \subseteq D(D^t(A)) = D^{t+1}(A)$. Thus $(A^k)^m \subseteq (D^t(A))^m \subseteq D^{t+1}(A)$. By a result of Zwier [12] there is an integer r such that $A^r \subseteq (A^k)^m$. Thus $A^r \subseteq D^{t+1}(A)$.

The following theorem extends a result of Shirshov for alternative algebras over a field of characteristic $\neq 2$.

THEOREM 3. *If A is a finitely generated alternative ring then A is nilpotent iff A^q is solvable iff A^q is nilpotent.*

Proof. Clearly, A nilpotent implies A^q solvable. The equivalence of A^q solvable and A^q nilpotent is the result of McCrimmon mentioned earlier. Since to each t there is a k such that $A^k \subseteq D^t(A)$ we conclude that A^q solvable implies A nilpotent.

THEOREM 4. *If A is an alternative ring then $L(A) = L(A^q)$.*

Proof. Clearly $L(A)$ is an ideal of A^q and since it is locally nilpotent in A, it is also locally nilpotent in A^q. Thus $L(A) \subseteq L(A^q)$.

For the converse it is sufficient to prove that $L(A) = 0$ implies that $L(A^q) = 0$. For under this assumption if $L(A) \neq 0$ then, since $L(A/L(A)) = 0$, we get $L(A^q/L(A)) = 0$. Since the homomorphic image of a locally nilpotent ideal is locally nilpotent we get $L(A^q)/L(A) \subseteq L(A^q/L(A)) = 0$. Thus $L(A^q) \subseteq L(A)$.

Recall that if B is an ideal of A^q then $\operatorname{Ker} B = \{b \in B \mid bA + Ab \subseteq B\}$ is an ideal of A. It is shown in [5] that $AU_B \subseteq \operatorname{Ker} B$ and that $L(A) = 0$ implies that A is strongly semiprime in the sense that $AU_a = 0$ implies that $a = 0$. Assume now that $L(A) = 0$ and that $L(A^q) \neq 0$. If $\operatorname{Ker} L(A^q) = 0$ then $AU_{L(A^q)} = 0$ contradicting the fact that A is strongly semiprime. Thus $L(A^q)$ contains a nonzero alternative ideal $K = \operatorname{Ker} L(A^q)$. We show that $K \subseteq L(A)$ to obtain a contradiction. For if

R is a finitely generated alternative subring of K then by Theorem 2 R^q is a finitely generated quadratic Jordan algebra. Since $R^q \subseteq L(A^q)$ it follows that R^q is nilpotent. Then, by Theorem 3, R is a nilpotent ring. Thus K is a locally nilpotent ideal of A and $K \subseteq L(A)$ for the desired contradiction. It follows that $L(A) = 0$ implies that $L(A^q) = 0$ and the proof is complete.

REMARK. Note that the proof of Theorem 4 can be used equally well to show that the locally finite dimensional radical of A coincides with the locally finite dimensional radical of A^q.

2. In the following let A be an alternative ring with involution $*$ and let $H(A,*)$ denote the Jordan ring of $*$-symmetric elements of A. In **[3]** McCrimmon asked the question: If B is an associative algebra with involution $*$ such that all $*$-symmetric elements are nilpotent, does it follow that B is itself necessarily nil? Osborn **[8]** answered the question in the affirmative if B is an algebra over an uncountable field Φ. In an analogous result Montgomery has shown that if B is an associative algebra with involution over an uncountable field and if the symmetric elements of B are algebraic then B is algebraic **[7]**. We note that both of these results apply to an alternative algebra A with involution. For if $a \in A$ then by Artin's theorem $A_0 = \Phi[a, a^*]$ is an associative algebra. Since the symmetric elements of A_0 are nil (algebraic) it follows that A_0 is nil (algebraic). Thus the elements of A are nilpotent (algebraic).

The key result needed by Osborn is the result of Amitsur that if A is an associative algebra over a field Φ such that the cardinality of Φ exceeds the dimension of A over Φ then the Jacobson radical of A is nil ideal. We note that the proof of Amitsur's theorem as presented in **[2**, pp. 19–20] carries over verbatim to the alternative case once the following two observations are made. (1): the proof in **[2]** that the elements in the radical are either nilpotent or transcendental uses associativity but can be easily adjusted. For if $a \in \operatorname{Rad} A$ is algebraic then $\Phi[a]$ is finite dimensional. From the power-associativity of A we know that $\Phi[a]$ is nil or contains an idempotent e **[10**, p. 32]. The latter implies that $e \in \operatorname{Rad} A$ which is impossible. Thus a is nilpotent. (2): the proof of Proposition 2 in **[2]** requires the fact that $(ab)b^{-1} = a$ for all $a, b \in A$. This is also true in alternative rings **[9**, p. 38].

Some other results which relate nilpotency in $H(R,*)$ with nilpotency in R for an associative ring R are given in **[9]** under the assumption that $2x = a$ is solvable for all a in R. We note that these results also apply to an alternative ring A with involution and do not require any characteristic assumptions. For the key result needed is that if $\widehat{\alpha\beta}(0,0) = 1$ and $\widehat{\alpha\beta}(n,k)$ denotes the sum of all monomials of degree n

in α and degree k in β, then for any $x \in R$ we get

$$(4) \qquad x^{2n} = \left[\sum_{k=0}^{n-1} \widehat{\alpha\beta}(2n-2k-1, k)\right] x + \left[\sum_{k=0}^{n-1} \widehat{\alpha\beta}(2k, n-k-1)\right] \beta$$

for $\alpha = x + x^*$ and $\beta = -x^*x$. Since all of the computations take place in the subring generated by x and x^*, by Artin's theorem this identity holds for an alternative ring A. Thus we get:

THEOREM. *If A is an alternative ring with involution $*$ and if the quadratic Jordan ring $H(A, *)$ is nilpotent of index n, then A is nil of index $\leqq 2n$.*

Proof. As in **[8]**, if $x \in A$ let $\alpha = x + x^*$, $\beta = -x^*x$. Then if K_x denotes the quadratic Jordan subring of $H(A, *)$ generated by α and β then K_x is nilpotent of index $\leqq n$. If K_x^t denotes the set of all sums of monomials in K_x of degree $\geqq t$ then the proof of [**9**, Lemma 6] shows (without any characteristic assumptions) that $\widehat{\alpha\beta}(m, t) \in K^{m+t}$ for all m, t such that $m + t \geqq 1$. Thus, by (4) $x^{2n} = 0$.

COROLLARY. *If $H(A, *)$ is solvable then A is a nil ring.*

Proof. The proof of the previous theorem shows that if $x \in A$ and K_x is nilpotent of index n then $x^{2n} = 0$. Now since $H(A, *)$ is solvable it follows that K_x is solvable. Since K_x is finitely generated it is nilpotent of index t for some t. Therefore $x^{2t} = 0$.

With our previous remarks the following theorem of **[9]** carries over to the alternative case with no changes.

THEOREM. *Let A be an alternative algebra with involution $*$ over a field Φ with at least n elements. Then if $H(A, *)$ is nil with bounded nilindex n, A is nil with bounded nilindex $\leqq 2n$.*

REMARK. In [**9**, theorem 3] it is shown that if A is an associative algebra over a field F of characteristic $\neq 2$ with involution then $L(H(A, *)) = H(A, *) \cap L(A)$. We note that the same result holds for the locally finite dimensional radical $\mathscr{L}$. For, as in **[9]**, the proof reduces to showing that if U is a nonzero ideal of A and $U \cap H(A, *) \subseteq \mathscr{L}(H(A, *))$ then $U \subseteq \mathscr{L}(A)$. Assume then that B is a finitely generated subalgebra of U. Then by the result of Osborn mentioned in **[9]**, $H(B, *)$ is finitely generated and thus finite dimensional of dimension n for some n. But then $H(B, *)$ is algebraic and satisfies a polynomial identity. Then, by a result of Baxter and Martindale **[1]**, B is finite dimensional. Thus, U is a locally finite ideal of A so that $U \subseteq \mathscr{L}(A)$.

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