PACIFIC JOURNAL OF MATHEMATICS
Vol. 75, No 2, 1978

TRIANGULATIONS WITH THE FREE CELL PROPERTY

W. O. MURRAY AND L. B. TREYBIG

We show that if each of M_1 and M_2 is a connected, orientable, 3-manifold having a triangulation with the free cell property, then the connected sum $M_1 \# M_2$ has a natural triangulation with the free cell property. We also show that if M is a connected, orientable, 3-manifold having a triangulation with the free cell property, and a manifold N is formed from M by adding a handle, then N has a natural triangulation with the free cell property. These theorems are then applied to show that E^3 and various other noncompact 3-manifolds have triangulations with the free cell property.

In this paper all n-manifolds are metrizable and are assumed to be orientable. If T is a triangulation of an n-manifold M with boundary, then a subset S of M is said to be *saturated* (or T-saturated) provided S is the union of simplexes of T. An n-simplex t is said to be *free* in a saturated n-cell S provided $t \cap \mathrm{Bd}(S)$ is an $n-1$-cell or $t = S$. Thus, a triangulation T of an n-manifold with boundary is said to have the *free cell property* provided each nontrivial saturated n-cell contains two free n-simplexes. Closely related to this idea is the concept of shelling. If the n-simplexes of a T-saturated n-cell S can be ordered $t_1, t_2, \cdots, t_m$ such that t_i is free in the n-cell $\bigcup_{i \leq j} t_j$, then we say S can be *shelled* relative to T and that $t_1, t_2, \cdots, t_m$ is a *shelling order* for S. Hence, if a triangulation T has the free cell property, then every T-saturated 3-cell can be shelled relative to T. If v is a vertex of a triangulation T, then $\mathrm{St}(v, T)$ will denote the union of all the simplexes of T which contain v.

We are concerned with the question; which n-manifolds have triangulations with the free cell property? In [11] Sanderson showed that every triangulation of a 2-manifold with boundary has the free cell property. R. H. Bing's "House with two rooms" in [2] and M. E. Rudin's triangulation of a tetrahedron in [10] imply that there are triangulations of 3-manifolds with boundary which do not have the free cell property. Although Sanderson showed in [11] that a triangulation of a 3-cell has a subdivision with two free 3-simplexes, it remains unknown whether a given triangulation has a subdivision such that every nontrivial saturated 3-cell has two free 3-simplexes. However, L. B. Treybig showed in [12] that every compact 3-manifold M with or without boundary has a triangulation T with the free cell property. In [9] W. O.

Murray extended this result by showing that T may be made to agree with a predetermined triangulation of $\mathrm{Bd}(M)$.

For completeness we mention some of the recent results involving shelling. G. Danaraj and V. Klee have shown in [4] that several types of shelling agree in 2-spheres and 3-spheres, and in [5] have given an algorithm for finding a shelling order for a triangulated 2-sphere. Bruggesser and Mani showed in [3] that every triangulation of every "convex" n-cell contains a shellable subdivision. They also showed that if a triangulated 3-sphere is the boundary complex of a polytope in E^4, then it is shellable. Danaraj and Klee in [4] extended this result by showing that the above shelling may be required to satisfy certain conditions on the order of appearance of the simplexes. In [6] Danaraj and Klee are preparing a paper which includes a survey on the current knowledge of shellability. Also, applications of shelling are given by Bing [1], Moise [8], Sanderson [11], and Treybig [13].

2. The theorems.

Both the connected sum operation and the operation of adding a handle, as we use them here, require a slight subdivision of the original triangulation before the operation is performed. The necessary subdividing takes place in only one 3-simplex of the original triangulation and is described as follows. Let $t = abcd$ denote a 3-simplex in a triangulation T of a 3-manifold. Pick a_1 in $\mathrm{Int}(t)$ and subdivide t radially from a_1. Denote by $R_1(t)$ the subdivision of t containing these four 3-simplexes. Now, pick b_1 in $\mathrm{Int}(a_1bcd)$ and subdivide a_1bcd radially from b_1. Let $R_2(t)$ denote the subdivision of t defined by $(R_1(t) - a_1bcd) \cup R_1(a_1bcd)$. Let $c_1 \in \mathrm{Int}(a_1b_1cd)$ and subdivide radially from c_1. Denote by $R_3(t)$ the subdivision of t defined by $(R_2(t) - a_1b_1cd) \cup R_1(a_1b_1cd)$. Finally, pick d_1 in $\mathrm{Int}(a_1b_1c_1d)$ and subdivide $a_1b_1c_1d$ radially from d_1. Denote by $R(t)$ the subdivision of t defined by $(R_3(t) - a_1b_1c_1d) \cup R_1(a_1b_1c_1d)$. We note that $R(t)$ consists of thirteen 3-simplexes and one of these, $a_1b_1c_1d_1$, lies in $\mathrm{Int}(t)$. Also, since no vertices are added to $\mathrm{Bd}(t)$, $(T - t) \cup R(t)$ is a subdivision of T. The following theorem shows that this subdivision has the free cell property if T does.

THEOREM 2.1. *Suppose M is a 3-manifold with boundary and T is a triangulation of M with the free cell property. Denote by S the triangulation of M defined by $(T - t) \cup R_1(t)$, where t is a 3-simplex of T. Then S has the free cell property.*

Proof. Suppose S does not have the free cell property and B is a nontrivial, S-saturated 3-cell which has a minimal number of 3-simplexes, while containing at most one free 3-simplex.

If there is a 2-simplex xyz in B such that $xyz \cap \mathrm{Bd}(B) = \mathrm{Bd}(xyz)$, then $B = C_1 \cup C_2$, where each C_i is a 3-cell and $C_1 \cap C_2 = xyz$. Since C_1, C_2 have fewer 3-simplexes than B, there are 3-simplexes g_1 in C_1 and g_2 in C_2 which are free in B, a contradiction. This implies there are no 3-simplexes in B with three faces in $\mathrm{Bd}(B)$ and each 3-simplex with two faces in $\mathrm{Bd}(B)$ is free in B.

Let $t = abcd$ and suppose $R_1(t)$ is a radial subdivision of t from a_1 in $\mathrm{Int}(t)$. If $a_1 \notin B$, then B is a T-saturated 3-cell and has two free 3-simplexes in T. These 3-simplexes are also simplexes of S, a contradiction. Thus, $a_1 \in B$ and we now consider the number of 3-simplexes from $\mathrm{St}(a_1, S)$ in B.

Case 1. Suppose there is precisely one 3-simplex of $\mathrm{St}(a_1, S)$ in B. This 3-simplex has three faces in $\mathrm{Bd}(B)$, a contradiction.

Case 2. Suppose there are precisely two 3-simplexes of $\mathrm{St}(a_1, S)$ in B. Each of these 3-simplexes would have two faces in $\mathrm{Bd}(B)$ and would thus be free in B, a contradiction.

Case 3. Suppose there are precisely three 3-simplexes of $\mathrm{St}(a_1, S)$ in B, say a_1abc, a_1abd, and a_1acd. If a_1bcd intersects B in four faces then $M - a_1bcd \subset B$ and a_1abc, a_1acd are free in B. Thus, a_1bcd intersects B in only three faces. Since a_1bcd intersects B in three faces of a_1bcd, $B \cup a_1bcd$ is an S-saturated 3-cell. Consider a T-saturated 3-cell B^* which is obtained from $B \cup a_1bcd$ by replacing $\mathrm{St}(a_1, S)$ with $t = abcd$. Since T has the free cell property, there are two 3-simplexes g_1 and g_2 which are free in B^*. If $g_1 \neq abcd \neq g_2$, then a_1 is not in g_1 nor g_2, and g_1, g_2 are free in B. If $g_1 = abcd$ and g_1 has two faces in $\mathrm{Bd}(B^*)$, then one of a_1abc, a_1acd or a_1abd has two faces in $\mathrm{Bd}(B)$ and is thus free in B. If g_1 has only one face in $\mathrm{Bd}(B^*)$ then $g_1 \cap \mathrm{Bd}(B^*) = bcd$. Since $a \in \mathrm{Int}(B^*)$, $a \in \mathrm{Int}(B)$ and all of a_1abc, a_1acd, a_1abd are free in B. In any event, there are two 3-simplexes, g_2 and one other, which are free in B, a contradiction.

Case 4. Suppose $\mathrm{St}(a_1, S) \subset B$. Replace $\mathrm{St}(a_1, S)$ by $abcd$ in B to obtain a T-saturated 3-cell B^*. As before there are two 3-simplexes g_1, g_2 free in B^*. If $g_1 \neq abcd \neq g_2$, then g_1, g_2 are free in B. If $g_1 = abcd$, then there is a 3-simplex of $\mathrm{St}(a_1, S)$ which is free in B. In either case, there are two 3-simplexes which are free in B. The proof is now complete.

COROLLARY 2.2. *If T is a triangulation of M with the free cell property then $(T - t) \cup R_2(t)$, $(T - t) \cup R_3(t)$ and $(T - t) \cup R(t)$ all have the free cell property.*

Proof. By successive applications of Theorem 2.1 to $(T - t) \cup R_1(t)$ we see that $(T - t) \cup R_2(t)$, $(T - t) \cup R_3(t)$ and $(T - t) \cup R(t)$ have the free cell property, which completes the proof.

In [7], Milnor defined the *connected sum* $M_1 \# M_2$ of two connected, orientable 3-manifolds M_1 and M_2 by removing the interior of a 3-cell in each of M_1, M_2, and then matching the resulting boundaries using an orientation reversing homeomorphism. We modify Milnor's definition slightly, by requiring that the 3-cells be 3-simplexes in triangulations T_1, T_2 of M_1, M_2, respectively, and that the boundaries are identified by means of an affine, orientation reversing homeomorphism. When $M_1 \# M_2$ is defined in this manner, there results a natural triangulation T of $M_1 \# M_2$. Although $M_1 \# M_2$ is well defined up to homeomorphism, the resulting triangulation T depends on the 3-simplexes which are removed and the identification map on the boundaries of the 3-simplexes. Thus, we denote by $T_1 \# T_2$ the class of all such triangulations T of $M_1 \# M_2$. We are now prepared to show that $M_1 \# M_2$ has a triangulation T in $S_1 \# S_2$ with the free cell property, where S_1 and S_2 are certain triangulations of M_1 and M_2, respectively, with the free cell property.

THEOREM 2.3. *Suppose M_1 and M_2 are connected 3-manifolds with triangulations T_1 and T_2, respectively, which have the free cell property. Let $S_1 = (T_1 - t_1) \cup R(t_1)$ and $S_2 = (T_2 - t_2) \cup R(t_2)$ be subdivisions of, respectively, T_1 and T_2, as defined previously. Denote by $a_1b_1c_1d_1$ and $a_2b_2c_2d_2$ the 3-simplexes of $R(t_1)$, $R(t_2)$ which lie in $\mathrm{Int}(t_1)$, $\mathrm{Int}(t_2)$, respectively. Let T denote a triangulation of $M_1 \# M_2$, where $T \in S_1 \# S_2$, and the identification is made along $a_1b_1c_1d_1$ and $a_2b_2c_2d_2$. Then, T has the free cell property.*

Proof. Suppose T does not have the free cell property, and B is a nontrivial, T-saturated 3-cell in $M_1 \# M_2$ which has a minimal number of 3-simplexes, while having at most one free 3-simplex. If B lies entirely in M_1 or in M_2, then since S_1 and S_2 have the free cell property by Corollary 2.2, we are done. Thus, suppose B contains 3-simplexes in both M_1 and M_2. Let $B_i = \mathrm{Cl}[\mathrm{Int}(B \cap M_i)]$, $i = 1$ and 2. We consider the following preliminary case.

Case 1. Suppose there is a 2-simplex xyz in B such that $xyz \cap \mathrm{Bd}(B) = \mathrm{Bd}(xyz)$. Then B is the union of two 3-cells C_1 and C_2 such that $C_1 \cap C_2 = xyz$. Since each of these contains fewer 3-simplexes than B, there are 3-simplexes g_1 in C_1 and g_2 in C_2 such that g_1 and g_2 are free in B, a contradiction.

For the remainder of the proof we assume Case 1 does not hold. This implies that no 3-simplex in B has three faces in $\mathrm{Bd}(B)$, and if a 3-simplex in B has two faces in $\mathrm{Bd}(B)$, then it is free in B. The proof is now finished by considering the possible ways in which B_i may intersect $a_ib_ic_id_i$, $i=1$ and 2. Note that Case 1 implies that $B_i \cap a_ib_ic_id_i$ contains at least two faces of $a_ib_ic_id_i$, $i=1$ and 2.

Case 2. Suppose $B_i \cap a_ib_ic_id_i = \mathrm{Bd}(a_ib_ic_id_i)$, $i=1$ or 2. This implies that one of $M_1 - a_1b_1c_1d_1$ or $M_2 - a_2b_2c_2d_2$ lies in B, say $M_2 - a_2b_2c_2d_2$. Then, $B_1 \cup a_1b_1c_1d_1$ is an S_1-saturated 3-cell and thus, has two free 3-simplexes g_1 and g_2. If $g_1 \neq a_1b_1c_1d_1 \neq g_2$, then g_1 and g_2 are free in B. If $g_1 = a_1b_1c_1d_1$, then we may choose a 3-simplex g_1^* in M_2 which has a face in $\mathrm{Bd}(B) \cap a_2b_2c_2d_2 = \mathrm{Bd}(B_1 \cup a_1b_1c_1d_1) \cap g_1$. Then, g_1^* and g_2 are free in B, a contradiction.

Case 3. Suppose Case 2 does not hold and, in $M_1 \# M_2$, $B_1 \cap \mathrm{Bd}(a_1b_1c_1d_1) = B_2 \cap \mathrm{Bd}(a_2b_2c_2d_2) =$ union of three faces of $a_1b_1c_1d_1$ (or $a_2b_2c_2d_2$). Then $B_1^* = B_1 \cup a_1b_1c_1d_1$ and $B_2^* = B_2 \cup a_2b_2c_2d_2$ are saturated 3-cells in M_1 and M_2, respectively.

If $a_1b_1c_1d_1$, $a_2b_2c_2d_2$ are free in B_1^* and B_2^*, then there are 3-simplexes g_1 in B_1^* and g_2 in B_2^* which are free in B_1^* and B_2^*, respectively, and $g_1 \neq a_1b_1c_1d_1$, $g_2 \neq a_2b_2c_2d_2$. Since $a_1b_1c_1d_1$ and $a_2b_2c_2d_2$ are free in B_1^* and B_2^*, respectively, g_1 and g_2 are free in B.

If $a_1b_1c_1d_1$ is not free in B_1^*, then there are two 3-simplexes $g_1 \neq a_1b_1c_1d_1 \neq g_2$ in B_1^* which are free in B_1^*. Since $a_1b_1c_1d_1$ is not free in B_1^*, and B is a 3-cell, $a_2b_2c_2d_2$ must be free in B_2^*. This implies that $\mathrm{Bd}(B) \cap B_1^* = \mathrm{Cl}[\mathrm{Bd}(B_1^*) - a_1b_1c_1d_1]$. Thus, g_1 and g_2 are free in B, a contradiction.

For the remainder of the proof we assume that Cases 2 and 3 do not hold. Each of B_1 and B_2 may now intersect $\mathrm{Bd}(a_1b_1c_1d_1) = \mathrm{Bd}(a_2b_2c_2d_2)$ in only one of three ways. We consider each of these cases for B_1 and show there is a 3-simplex g_1 in B_1 which is free in B. Since B_2 must also intersect $\mathrm{Bd}(a_2b_2c_2d_2)$ in one of these ways, we obtain g_2 in B_2 which is free in B, and the proof will be completed.

Case 4. Suppose B_1 intersects $\mathrm{Bd}(a_1b_1c_1d_1)$ in three faces of $a_1b_1c_1d_1$, with precisely one of these faces in $\mathrm{Bd}(B)$. Also, suppose no 3-simplex in B_1 is free in B. Let t_1 be denoted by $abcd$.

First assume $B_1 \cap \mathrm{Bd}(a_1b_1c_1d_1) = a_1b_1d_1 \cup a_1c_1d_1 \cup b_1c_1d_1$, and one of these faces, say $a_1b_1d_1$, is in $\mathrm{Bd}(B)$. Since $a_1b_1c_1d_1$ intersects B_1 in exactly three faces, $B_1 \cup a_1b_1c_1d_1$ is a 3-cell which we denote by B_1^*. Note that the addition of $a_1b_1c_1d_1$ affects the freeness of only those 3-simplexes in B_1 containing d_1. Now consider $a_1b_1c_1d$ as a single 3-simplex in B_1^*. Since the triangulation $(T_1 - t_1) \cup R_3(t_1)$ of M_1 has the free cell property, there is a 3-simplex g_1 free in B_1^* such that $g_1 \neq a_1b_1c_1d$. Since $d_1 \notin g_1$, g_1 is free in B.

Now suppose $B_1 \cap \mathrm{Bd}(a_1b_1c_1d_1)$ consists of $a_1b_1c_1$ and two other 2-simplexes containing d_1. If one of the above 2-simplexes containing d_1 lies in $\mathrm{Bd}(B)$, then the 3-simplex in B_1 containing it has two faces in $\mathrm{Bd}(B)$, and is thus free in B. Hence we assume $a_1b_1c_1$ lies in $\mathrm{Bd}(B)$. Since $a_1b_1c_1c$ does not have two faces in $\mathrm{Bd}(B)$, both b_1c_1dc and a_1c_1cd are in B. If $b_1c_1d_1d$ is not in B, then $a_1c_1d_1d$ and $a_1b_1d_1d$ are in B. Since $\mathrm{Bd}(a_1b_1c_1d_1 \cup b_1c_1d_1d)$ lies in B, $M_1 - (a_1b_1c_1d_1 \cup b_1c_1d_1d)$ is in B and b_1c_1cd is free in B. Thus, $b_1c_1d_1d$ is in B and, likewise, $a_1c_1d_1d$ is in B. Since $a_1b_1c_1d_1$ intersects B_1 in exactly three faces, $B_1 \cup a_1b_1c_1d_1$ is a 3-cell in M_1. Note that in adding $a_1b_1c_1d_1$ to B_1, only the freeness of those 3-simplexes in B_1 containing c_1 is affected. If $a_1b_1d_1d$ intersects $B_1 \cup a_1b_1c_1d_1$ in four faces, then $M_1 - (a_1b_1c_1d_1 \cup a_1b_1d_1d)$ lies in B and $a_1c_1d_1d$ is free in B. Thus, $a_1b_1d_1d$ intersects the 3-cell $B_1 \cup a_1b_1c_1d_1$ in exactly three faces, and $B_1^* = B_1 \cup a_1b_1c_1d_1 \cup a_1b_1d_1d$ is a 3-cell. Now consider a_1b_1cd as a single 3-simplex in B_1^*. Since the triangulation $(T_1 - t_1) \cup R_2(t_1)$ of M_1 has the free cell property, and B_1^* is a saturated 3-cell under this triangulation, there is a 3-simplex g_1 free in B_1^* such that $g_1 \neq a_1b_1cd$. Since the addition of $a_1b_1c_1d_1$ and $a_1b_1d_1d$ to B_1 affected the freeness of only those 3-simplexes containing c_1 or d_1 and $c_1, d_1 \notin g_1$, g_1 is free in B.

Case 5. Suppose B_1 intersects $\mathrm{Bd}(a_1b_1c_1d_1)$ in exactly two faces of $a_1b_1c_1d_1$, say $a_1b_1c_1 \cup b_1c_1d_1$. Since Case 1 does not hold, $b_1c_1 \not\subset \mathrm{Bd}(B)$. Now $B_1^* = B_1 \cup a_1b_1c_1d_1$ is a 3-cell in M_1 and so has two free 3-simplexes one of which, say g_1, is not $a_1b_1c_1d_1$. Since $b_1c_1 \not\subset \mathrm{Bd}(B)$, g_1 is free in B.

Case 6. Suppose B_1 intersects $\mathrm{Bd}(a_1b_1c_1d_1)$ in two faces of $a_1b_1c_1d_1$ and a 1-simplex xy belonging to neither of these faces. We now consider the various possibilities for xy on $a_1b_1c_1d_1$.

If xy contains d_1, say $x = d_1$, then there is a 3-simplex in B_1 which contains d_1y. But, each 3-simplex in M_1 which contains d_1y shares a face with $a_1b_1c_1d_1$, which contradicts the manner in which B_1 intersects $a_1b_1c_1d_1$.

If xy contains c_1, say $x = c_1$, then there are at most four 3-simplexes in M_1 which contain c_1y. One of these 3-simplexes is $a_1b_1c_1d_1$, which is not in B_1. Since the faces adjacent to c_1y on $a_1b_1c_1d_1$ are not in B_1, there are two other 3-simplexes containing c_1y which are not in B_1. This implies that only one 3-simplex g_1 containing c_1y is in B_1. Thus, g_1 has two faces in $\mathrm{Bd}(B)$ and is free in B.

Now suppose $B_1 \cap a_1b_1c_1d_1 = a_1b_1 \cup b_1c_1d_1 \cup a_1c_1d_1$. There are exactly five 3-simplexes in M_1 which contain a_1b_1 and, as above, three of these are not in B_1. The two remaining 3-simplexes are a_1b_1bc and a_1b_1bd. If only one of a_1b_1bc, a_1b_1bd is in B, it would have two faces in $\mathrm{Bd}(B)$ and would be free in B. Thus, suppose both a_1b_1bc and a_1b_1bd are in B. Since each of these has one face in $\mathrm{Bd}(B)$, b_1bcd, a_1abc and a_1abd are in B, for otherwise a_1b_1bc or a_1b_1bd would have two faces in $\mathrm{Bd}(B)$. We wish to show that a_1acd is in B. Suppose a_1acd is not in B. If a_1c_1cd is in B then, since $a_1b_1c_1c$ is not in B, a_1c_1cd has two faces, a_1cd and a_1c_1c, in $\mathrm{Bd}(B)$, and is thus free in B. If a_1c_1cd is not in B then, since $a_1b_1d_1d$ is not in B, $a_1c_1d_1d$ has two faces $a_1c_1d_1$ and a_1d_1d in $\mathrm{Bd}(B)$, and is thus free in B. We have produced a free cell in either case, which implies a_1acd is in B.

We now have the 2-sphere $\mathrm{Bd}(abcd)$ in B, which implies that $M_2 - \mathrm{Int}(abcd)$ is in B. It follows that $b \in \mathrm{Int}(B)$ and a_1b_1bc is free in B. This completes Case 6 and the proof of Theorem 2.3.

An important aspect of Theorem 2.3 is that the triangulation T of $M_1 \# M_2$ agrees with $T_1 \cup T_2$ outside of two 3-simplexes, one in each of T_1 and T_2. That is, the triangulation T agrees with $T - t_1$ on M_1 and $T_2 - t_2$ on M_2. As will be seen in later examples, this fact allows us to construct triangulated 3-manifolds with the free cell property which are not compact.

In [7], Milnor defines adding a handle to a connected, orientable 3-manifold M by choosing two disjoint 3-cells in M, removing their interiors, and matching the resulting boundaries under an orientation reversing homeomorphism. As in the case of connected sums, this operation is well defined up to homeomorphism. For example, if a handle is added to the 3-sphere S^3, the result is isomorphic to $S^1 \times S^2$. For our purpose, we modify this definition slightly by requiring that the disjoint 3-cells be 3-simplexes with disjoint stars in a triangulation T of M, and that the boundaries be identified under an affine, orientation reversing homeomorphism. If $H(M)$ denotes the resulting 3-manifold, then there is a natural triangulation S of $H(M)$. As before, the triangulation S depends on which 3-simplexes are removed, and the identification map on the boundaries of the 3-simplexes. Thus, we denote by $H(T)$ the class of all such triangulations S of the 3-manifold $H(M)$.

THEOREM 2.4. *Suppose M is a connected 3-manifold and T is a triangulation of M with the free cell property. Suppose further that t_1 and t_2 are disjoint 3-simplexes in T, and $a_1b_1c_1d_1$ and $a_2b_2c_2d_2$ are the 3-simplexes of $R(t_1)$ and $R(t_2)$ in $\mathrm{Int}(t_1)$ and $\mathrm{Int}(t_2)$, respectively. Let S denote the triangulation of M defined by $(T - t_1 - t_2) \cup R(t_1) \cup R(t_2)$. Let K be a triangulation from $H(S)$, where the identification is defined as* *above on* $\mathrm{Bd}(a_1b_1c_1d_1)$ *and* $\mathrm{Bd}(a_2b_2c_2d_2)$. *Then, K has the free cell property.*

Proof. Note first that since t_1 and t_2 are disjoint, the stars of $a_1b_1c_1d_1$ and $a_2b_2c_2d_2$ are disjoint, and indeed, K is a triangulation. To see that K has the free cell property, consider 3-manifolds $M_1 = M_2 = M$ with triangulations $S_1 = S_2 = S$. Let K^* denote the triangulation of $M_1 \# M_2$ which is defined by the same identification along $\mathrm{Bd}(a_1b_1c_1d_1)$ and $\mathrm{Bd}(a_2b_2c_2d_2)$ as in K. By Theorem 2.3, K^* has the free cell property. Since any K saturated 3-cell B has an isomorphic copy B^* in K^*, K has the free cell property.

3. Applications.

EXAMPLE 3.1. We now give a general method of constructing a noncompact triangulated 3-manifold with the free cell property from a sequence of compact 3-manifolds. Let $M_1, M_2, M_3, \cdots$ be compact 3-manifolds with triangulations $T_1, T_2, T_3, \cdots$, respectively, where each T_i has the free cell property. This is feasible since we could assume each T_i has a minimal number of 3-simplexes, and by **[12]**, each T_i would have the free cell property. Let t_{1b} be a 3-simplex in T_1 and let t_{ia}, t_{ib} be disjoint 3-simplexes in T_i, for $i = 2, 3, 4, \cdots$. We may assume such t_{ia} and t_{ib} exist since, by Theorem 2.1, we could subdivide each T_i sufficiently to produce disjoint 3-simplexes, while preserving the free cell property. Now, let $S_1 = (T_1 - t_{1b}) \cup R(t_{1b})$ and let $S_i = (T_i - t_{ia} - t_{ib}) \cup R(t_{ia}) \cup R(t_{ib})$, for $i = 2, 3, 4, \cdots$. Denote by r_{1b}, r_{ia}, r_{ib} the 3-simplexes of $R(t_{1b})$, $R(t_{ia})$, $R(t_{ib})$ which lie in the interiors of t_{1b}, t_{ia}, t_{ib}, respectively, for $i = 2, 3, 4, \cdots$. Remove the interiors of r_{1b}, r_{ia}, r_{ib} $(i = 2, 3, 4, \cdots)$ and then match the resulting boundary of t_{jb} with that of $t_{j+1,a}$ $(j = 1, 2, 3, \cdots)$ using, as in Theorem 2.3, an affine, orientation reversing homeomorphism. Note that, since r_{ia} and r_{ib} are disjoint, the identification map is well defined. The resulting 3-manifold M may be thought of as $M_1 \# M_2 \# M_3 \# \cdots$ with a resulting triangulation S in $S_1 \# S_2 \# S_3 \# \cdots$. Since S contains an infinite number of 3-simplexes, M is not compact.

We now show that S has the free cell property. Let B be a saturated 3-cell in M. Since B contains at most a finite number of 3-simplexes, there exists an integer N such that B lies in $M_1 \# M_2 \# M_3 \# \cdots \# M_N$ and is saturated under a triangulation K such that K agrees with S on $M_1 \# M_2 \# M_3 \# \cdots \# M_N$. From the construction of S we see that K satisfies the hypothesis of Theorem 2.3, and thus K has the free cell property. Therefore, there are two 3-simplexes g_1 and g_2 from

K which are free in B. But these 3-simplexes may be considered as 3-simplexes from S, and so S has the free cell property.

EXAMPLE 3.2. Following the previous example, we now construct a triangulation of E^3 which has the free cell property. First, consider a triangulation of the 3-sphere S^3 described as follows. We view S^3 as consisting of two 3-simplexes t_1 and t_2 such that $t_1 \cap t_2 = \mathrm{Bd}(t_1) = \mathrm{Bd}(t_2)$. Let T denote the triangulation $t_1 \cap R_1(t_2)$ of S^3. Since T consists of only five 3-simplexes, it is easily verified that T has the free cell property. By repeated applications of Theorem 2.1 to T, the triangulation $K = t_1 \cup R(t_2)$ has the free cell property. Let s_1 denote the 3-simplex of K which lies in $\mathrm{Int}(t_2)$.

Now let $M_i = S^3$, $T_i = K$, $t_{ia} = t_1$ and $t_{ib} = s_1$, for each $i = 1, 2, 3, \cdots$. It follows from Example 3.1 that the resulting triangulation S in $S_1 \# S_2 \# S_3 \# \cdots$ of $E^3 = M_1 \# M_2 \# M_3 \# \cdots$ has the free cell property. As a note, the above triangulation S could also be realized as the "limit" of a sequence of radial subdivisions of K. It would then follow from Theorem 2.1 that S has the free cell property.

By similar methods, it can be shown that if S is a triangulation of a 3-manifold M with the free cell property, then there is a triangulation S' of $M - \{x\}$ with the free cell property, where x is a point of M. Moreover, if X is a countable set of points in M with no limit points, then $M - X$ also has a triangulation with the free cell property.

REFERENCES

1. R. H. Bing, *An alternative proof that 3-manifolds can be triangulated*, Ann. of Math., (1) **59** (1959), 37–65.

2. ———, *Topology of 3-manifolds related to the Poincaré conjecture*, Lectures on Modern Mathematics, (T. L. Saaty, ed.), New York, Wiley, 1964, 93–128.

3. H. Bruggesser and P. Mani, *Shellable decompositions of cells and spheres*, Math. Scand., **29** (1972), 197–205.

4. G. Danaraj and V. Klee, *Shellings of spheres and polytopes*, Duke Math. J., **41** (1974), 443–451.

5. ———, *Shelling algorithms*, Report RC 5301, IBM Watson Research Center, 1975.

6. ———, *Which spheres are shellable?*, (in preparation).

7. J. Milnor, *A unique decomposition theorem for 3-manifolds*, Amer. J. Math., **84** (1962), 1–7.

8. E. E. Moise, *Affine structures in 3-manifolds. II. Positional properties of 2-spheres*, Ann. of Math., (2) **55** (1952), 172–176.

9. W. O. Murray, *Shelling in low dimensional manifolds*, Masters thesis, Texas A & M University, 1974.

10. M. E. Rudin, *An unshellable triangulation of a tetrahedron*, Bull. Amer. Math. Soc., **64** (1958), 90–91.

11. D. E. Sanderson, *Isotopy in 3-manifolds. I. Isotopic deformations of 2-cells and 3-cells*, Proc. Amer. Math. Soc., **8** (1957), 912–922.

12. L. B. Treybig, *Shelling 3-cells in compact triangulated 3-manifolds*, Proc. Amer. Math. Soc., **33** (1972), 171–174.
13. ———, *Bounds in piecewise linear topology*, Trans. Amer. Math. Soc., **201** (1975), 383–405.

Received October 29, 1976 and in revised form June 8, 1977. Some of the results herein are modifications of a portion of the first author's dissertation.

TEXAS A & M UNIVERSITY
COLLEGE STATION, TX 77843

Current address W. O. Murray:
Hendrix College
Conway, AR 72032