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ON STOPPING RULES AND THE EXPECTED SUPREMUM OF S_n/T_n

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Let S_n and T_n be nth partial sums of two independent sequences of i.i.d. random variables. S_1 and T_1 may have different distributions. Assume $0 \leqq ES_1 < \infty$, $ET_1 < \infty$ and $P[T_1 > 0] = 1$. Let $\mathscr{B}_n$ be the σ-field generated by $S_1, T_1, \cdots, S_n, T_n$, and let R_∞ be the collection of extended-valued stopping rules with respect to $\mathscr{B}_1, \mathscr{B}_2, \cdots$. It is shown that $E \sup_{n \geqq 1} S_n/T_n < \infty$ iff $\sup_{\tau \in R_\infty} ES_\tau/T_\tau < \infty$ iff $ES_1 \log^+ S_1 < \infty$ and $E(T_1^{-1}) < \infty$. The (random) cutoff points characterizing the optimal rules are easily obtained as fixed points of certain contraction mappings. A Markov walk generalization of the Chow and Robbins binomial stopping problem is viewed within the S_n/T_n framework.

1. Introduction. Let $U, U_1, U_2, \cdots$ and $V, V_1, V_2, \cdots$ be independent random variables defined on a common probability space $(\Omega, \mathscr{F}, P)$. Assume the U's are nondegenerate and identically distributed with $0 \leqq EU < \infty$. Assume the V's are identically distributed with $P[V>0]=1$ and $EV<\infty$. Let $S_n = U_1 + \cdots + U_n$ and $T_n = V_1 + \cdots + V_n$. Define the σ-fields $\mathscr{B}_n = \mathscr{B}(U_1, V_1, \cdots, U_n, V_n)$, $\mathscr{B}'_n = \mathscr{B}(U_1, \cdots, U_n)$, $\mathscr{B}''_n = \mathscr{B}(V_1, \cdots, V_n)$, and let R_∞, R'_∞, R''_∞ be the collections of extended-valued stopping rules (Definition 1 [8]) with respect to $\{\mathscr{B}_n\}_{n=1}^\infty$, $\{\mathscr{B}'_n\}_{n=1}^\infty$, $\{\mathscr{B}''_n\}_{n=1}^\infty$, respectively. That is, $\tau \in R_\infty$ (R'_∞, R''_∞) if and only if $[\tau = n] \in \mathscr{B}_n(\mathscr{B}'_n, \mathscr{B}''_n)$ for all $n \geqq 1$ and $P[\tau = \infty] + \sum_{n=1}^\infty P[\tau = n] = 1$. In order that our expected rewards be well defined, we follow the strong law and set S_∞/∞, ∞/T_∞, S_∞/T_∞ equal to EU, $1/EV$, EU/EV, respectively. Unless otherwise mentioned, all suprema and infima are over $\{n: n \geqq 1\}$. We write $E \sup S_n/T_n$ for $E[\sup_{n \geqq 1}(S_n/T_n)]$.

It is well known (Burkholder [1] and McCabe and Shepp [9]) that

$$(1.1) \qquad E \sup S_n/n < \infty \Leftrightarrow EU \log^+ U < \infty \Leftrightarrow \sup_{\tau \in R'_\infty} ES_\tau/\tau < \infty,$$

and in this case an optimal stopping rule exists (Siegmund [10]), i.e., the last supremum in (1.1) is attained by some $\tau \in R'_\infty$.

Operating under successively weaker conditions, Chow and Robbins [2], Teicher and Wolfowitz [11], Dvoretzky [6], Thompson, Basu and Owen [12], Davis [4], and Klass [8] have proved that the (unique) minimal optimal rule is to stop at the first time n such that $S_n \geqq a_n$, where $\{a_n\}_{n=1}^{\infty}$ is the strictly increasing sequence of positive constants satisfying $a_n/n = \sup_{\tau \in R_\infty^+} E[(a_n + S_\tau)/(n+\tau)]$.

One purpose of this paper is to generalize the above results to the reward sequence S_n/T_n. The independence suggests treating S_n and T_n separately, via the elementary inequality

$$(1.2) \qquad \begin{aligned}(E \inf n/T_n)(E \sup S_n/n) &\leqq E \sup S_n/T_n \\ &\leqq (E \sup n/T_n)(E \sup S_n/n).\end{aligned}$$

In light of (1.1) our attentions focus on n/T_n. In §2 is proved a general result (Theorem 1) which implies that $E \sup n/T_n < \infty$ just in case $E(V^{-1}) < \infty$. Section 3 shows that $E \sup S_n/T_n < \infty$ iff $\sup_{\tau \in R_\infty} ES_\tau/T_\tau < \infty$ iff $EU \log^+ U < \infty$ and $E(V^{-1}) < \infty$.

For future reference and some immediate methodology we recall here that

$$(1.3) \qquad \{S_n/n\}_{n=\infty}^{1} \text{ is a reversed martingale,}$$

so that the conditional Jensen's inequality and independence imply

$$(1.4) \qquad \{n/T_n\}_{n=\infty}^{1} \text{ and } \{S_n/T_n\}_{n=\infty}^{1} \text{ are reversed submartingales.}$$

Application of a well known submartingale inequality (Doob **[5]**, p. 317) to (1.3) yields the sufficiency of $EU \log^+ U < \infty$ in (1.1). A possible approach to characterizing $E \sup S_n/T_n < \infty$ (or $E \sup n/T_n < \infty$) might then be to apply the same inequality to obtain the sufficient condition $E(U/V)\log^+(U/V) < \infty$ ($EV^{-1}\log^+(V^{-1}) < \infty$). As our results show, these conditions are not "sufficiently" weak. After all, $EV^{-1}\log^+(V^{-1}) < \infty$ precisely when $E \sup n^{-1}\Sigma_{i=1}^n V_i^{-1} < \infty$, and $n^{-1}\Sigma_{i=1}^n V_i^{-1}$ almost surely dominates n/T_n, by the inequality of the arithmetic and harmonic means. The underlying idea in the proof of Theorem 1 is the classical inequality relating the arithmetic and geometric means.

In §4 we employ contractions to obtain the cutoff points which characterize the optimal rules. The situation is somewhat novel in that the optimal stopping times depend on the intrinsic times k only through the values of T_k at those times, and the cutoff points are themselves random, owing to dependence on the T_k. This section relies heavily on §§1 and 2 of Klass **[8]**.

In §5 we indicate how a Markov chain generalization of the Chow and Robbins **[2]** example may be viewed as an S_n/T_n problem.

2. Expected suprema of inverse generalized means. For simplicity we now assume (w.l.o.g.) that $V_k(\omega) > 0$ for all $k \geqq 1$ and all $\omega \in \Omega$. Let

$$M_n(t,\omega) = \left(n^{-1}\sum_{k=1}^{n} (V_k(\omega))^t\right)^{1/t} \quad \text{for} \quad t \neq 0;$$

$$M_n(0,\omega) = \lim_{t\to 0} M_n(t,\omega) = \left(\prod_{k=1}^{n} V_k(\omega)\right)^{1/n}.$$

For n and ω fixed, $M_n(t,\omega)$ is an increasing function of t (Chapter 2 of Hardy, Littlewood and Polya [7]).

For $r > 0$ let $\|X\|_r = [E(|X|^r)]^{1/r}$ if the expectation is finite; otherwise let $\|X\|_r = \infty$.

THEOREM 1. *For all $t \geqq 0$ and $N \geqq 1$*

$$E\left(\sup_{n\geqq N} [M_n(t,\omega)]^{-1}\right) \leqq \|V^{-1}\|_{1/N}(2^N + N\log 2 + 1). \tag{2.1}$$

Consequently, for all $t \geqq 0$

$$E(V^{-1}) \leqq E\left(\sup_{n\geqq 1} [M_n(t,\omega)]^{-1}\right) \leqq (3 + \log 2)E(V^{-1}), \tag{2.2}$$

whence $E(\sup_{n\geqq 1}[M_n(t,\omega)]^{-1}) < \infty$ for (all) $t \geqq 0$ if and only if $E(V^{-1}) < \infty$. More generally,

$$E\left(\sup_{n\geqq N} [M_n(t,\omega)]^{-1}\right) < \infty \text{ for (all) } t > 0 \quad \text{if and only if } E\min_{1\leqq j\leqq N}(V_j^{-1}) < \infty, \tag{2.3}$$

whereas

$$E\left(\sup_{n\geqq N} [M_n(0,\omega)]^{-1}\right) < \infty \text{ if and only if } E(V^{-1/N}) < \infty. \tag{2.4}$$

Proof. First we establish (2.1). Since for n and ω fixed and $t \geqq 0$ the $[M_n(t,\omega)]^{-1}$ are all majorized by the inverse geometric mean $[M_n(0,\omega)]^{-1} = (\Pi_{k=1}^{n} V_k^{-1})^{1/n}$, it suffices to prove (2.1) for $t = 0$.

Fix $N \geqq 1$. We may assume $\|V^{-1}\|_{1/N} < \infty$. Let $C = E(V^{-1/N})$ and $B = [2E(V^{-1/N})]^N$. Then

$$
\begin{aligned}
E \sup_{n \geqq N} \left(\prod_{i=1}^{n} V_i^{-1/n} \right) &= \int_0^\infty P\left[\sup_{n \geqq N} \prod_{i=1}^{n} V_i^{-1/n} \geqq y \right] dy \\
&\leqq B + [E(V^{-1/N})]^N + \sum_{n=N+1}^{\infty} \int_B^\infty P\left[\prod_{i=1}^{n} V_i^{-1/N} \geqq y^{n/N} \right] dy \\
&\leqq B(1 + 2^{-N}) + \sum_{n=N+1}^{\infty} C^n \int_B^\infty y^{-n/N} dy \\
&= \| V^{-1} \|_{1/N} (2^N + N \log 2 + 1).
\end{aligned}
$$

This proves (2.1), from which (2.2) and (2.4) follow readily. To prove (2.3) note that for $t > 0$

$$
N^{-1/t} \max_{1 \leqq j \leqq N} V_j \leqq M_N(t, \omega) \leqq \max_{1 \leqq j \leqq N} V_j.
$$

Hence for $t > 0$

$$
\begin{aligned}
E \min_{1 \leqq j \leqq N} (V_j^{-1}) &\leqq E\left(\sup_{n \geqq N} [M_n(t, \omega)]^{-1} \right) \\
&\leqq N^{1/t} E \min_{1 \leqq j \leqq N} (V_j^{-1}) + E\left(\sup_{n > N} [M_n(t, \omega)]^{-1} \right).
\end{aligned}
$$

We may assume $E \min_{1 \leqq j \leqq N} (V_j^{-1}) < \infty$, in which case $\lim_{y \to \infty} y(P(V^{-1} > y))^N = \lim_{y \to \infty} yP[\min_{1 \leqq j \leqq N} V_j^{-1} > y] = 0$. We may conclude that $E(V^{-1/\alpha}) < \infty$ for any $\alpha > N$. Take $\alpha = N + 1$ and use (2.1) to complete the proof.

Taking $t = 1$ in (2.2) yields

COROLLARY 1. $E \sup n/T_n < \infty \Leftrightarrow E(V^{-1}) < \infty$.

REMARK 1. To illustrate the (qualitative) sharpness of (2.1) for $t = 1$, fix $N \geqq 2$ and let V be a gamma random variable with mean and variance both equal to $1/(N - 1)$. Then $En/T_n = \infty$ for $1 \leqq n < N$, while by (2.1) $E \sup_{n \geqq N} n/T_n < \infty$.

To underline the distinction between (2.3) and (2.4), take $N \geqq 2$ and $P[V^{-1} > y] = (y^{1/N} \log(ey))^{-1}$ for $y \geqq 1$. Then

$$
E(V^{-1/N}) = 1 + \int_1^\infty (yN \log(e^{1/N} y))^{-1} dy = \infty,
$$

while

$$
E \min_{1 \leqq j \leqq N} (V_j^{-1}) = 1 + \int_1^\infty (P(V^{-1} > y))^N dy < \infty.
$$

Whenever $E(V^{-1}) = \infty > E(V^{-1/N})$, Theorem 1 yields that $E \sup_{n \geqq 1} n/T_n = \infty > E \sup_{n \geqq N} n/T_n$, so that the infinite expected supremum owes exclusively to the behavior of the first few terms. Our next result sheds additional light on this.

THEOREM 2. *Let* $V, V_1, V_2, \cdots$ *be* i.i.d. *nonnegative random variables with* $P[V > 0] > 0$. *Then*

$$E \sup n/(b + T_n) < \infty \text{ for each } b > 0. \tag{2.5}$$

Proof. We use ladder variables to transform the given reward sequence to an S_n/n reward sequence.

There exists $c > 0$ such that $P[V \geqq c] \geqq c$. Let $\tau(0) = 0$. Having defined $\tau(0), \cdots, \tau(k)$, let $\tau(k+1) = 1\text{st } n$ s.t. $V_1 + \cdots + V_n \geqq c + V_1 + \cdots + V_{\tau(k)}$. Then $T_{\tau(k)} \geqq kc$. The random variables $\tau(k)$ (for $k \geqq 1$) are sums of k i.i.d. ladder variables $q_1, \cdots, q_k$. Note that $P[q_1 > n] = P[\tau(1) > n] \leqq P[\bigcap_{j=1}^{n} \{V_j < c\}] = [P(V < c)]^n \leqq (1-c)^n$, so that all moments of q_1 are finite. Further,

$$\begin{aligned} E \sup n/(b + T_n) &= E \sup_{k \geqq 0} \sup_{\tau(k) < n \leqq \tau(k+1)} n/(b + T_n) \\ &\leqq E \sup_{k \geqq 0} \tau(k+1)/(b + kc) \\ &\leqq (1/b)E\tau(1) + (2/c)E \sup_{k \geqq 1} \tau(k)/k, \end{aligned}$$

which is finite by (1.1).

REMARK 2. We conclude this section by mentioning another condition equivalent to $E \sup n/T_n < \infty$. One can show that

$$[r/(r+1)]EY_1 \sup Y_n^{-(r+1)} \leqq E \sup Y_n^{-r} \leqq EY_1 \sup Y_n^{-(r+1)} \tag{2.6}$$

for any $r > 0$ and any positive reversed martingale $\cdots Y_2, Y_1$ (the upper bound is trivial; the lower bound follows from an integration by parts, inequality (3.4″) of Doob [**5**, p. 314], and Fubini's theorem). It follows from (2.6) and (1.3) that

$$\sup n/T_n \in L_1(P) \quad \text{iff} \quad T_1^{1/2} \sup n/T_n \in L_2(P). \tag{2.7}$$

3. $E \sup S_n/T_n < \infty \Leftrightarrow EU \log^+ U < \infty$ **and** $E(V^{-1}) < \infty$. The following lemma is a consequence of the strong law. The corollary follows from the lemma and (1.2).

LEMMA 1. *$P[\inf n/T_n = 0] = 0$ and $0 < E \inf n/T_n < \infty$.*

COROLLARY 2. *$E \sup S_n/T_n = \infty$ whenever $E \sup S_n/n = \infty$.*

THEOREM 3. *The following are equivalent.*
(i) $\sup_{\tau\in R_\infty} ES_\tau/T_\tau < \infty$
(ii) $E \sup S_n/T_n < \infty$
(iii) $E \sup S_n/n < \infty$ *and* $E \sup n/T_n < \infty$
(iv) $EU \log^+ U < \infty$ *and* $E(V^{-1}) < \infty$.

Proof. (iii) and (iv) are equivalent by (1.1) and Corollary 1. (iii) implies (ii) by (1.2). (ii) implies (i) since $E \sup Y_n \geqq \sup_{\tau\in R_\infty} EY_\tau$ for any reward sequence $\{Y_n\}_{n=1}^\infty$. The chain will be completed by showing the inverse of [(iv) ⇒ (i)].

Suppose first that $E(V^{-1}) = \infty$. Define $\tau \in R'_\infty$ by $\tau = 1$ if $U_1 > 0$, $\tau = \infty$ otherwise. Then $ES_\tau/T_\tau = \infty$ since $P[U_1 > 0] > 0$.

Now suppose $EU \log^+ U = \infty$. Then $\sup_{t\in R'_\infty} ES_t/t = \infty$ **[9]**. It follows that for every $m \geqq 1$ there exists $\tau_m \in R'_\infty$ such that $ES_{\tau_m}/\tau_m > m/E \inf(n/T_n)$; Lemma 1 has been invoked here $(0 < E \inf n/T_n < \infty)$. Because each τ_m is independent of the V_i, and $\mathscr{B}(S_1, T_1, \cdots, S_m, T_m) \supseteq \mathscr{B}(S_1, \cdots, S_m)$ for every m, we have

$$\sup_{\tau\in R_\infty} ES_\tau/T_\tau \geqq \sup_{\tau\in R'_\infty} ES_\tau/T_\tau \geqq \sup_{m\geqq 1} E\left[(S_{\tau_m}/\tau_m) \inf_{n\geqq 1} n/T_n\right]$$
$$= \sup_{m\geqq 1}\left[E(S_{\tau_m}/\tau_m)E \inf_{n\geqq 1} n/T_n\right] = \sup_{m\geqq 1} m = \infty.$$

This completes the proof.

4. The form of the optimal rule.

We assume throughout this section that $E(V^{-1})$ and $EU \log^+ U$ are both finite. Our return sequences $Y_n(a, b)$ are defined by $Y_n(a, b) = (a + S_n)/(b + T_n)$, a real, $b \geqq 0$. Since $Y_n(a, b) \xrightarrow{\text{a.s.}} EU/EV$ and $T_n \uparrow \infty$ a.s., we set $Y_\infty(a, b) = EU/EV$ and $T_\infty = \infty$. By the results of §3, $E \sup Y_n(a, b) < \infty$. We thus see that assumptions A_1, A_2, A_3 of Klass **[8]** hold for our $Y_n(a, b)$, so that the entirety of §1 there is applicable. In particular

$$M_b(a) = \sup_{\tau\in R_\infty} E(a + S_\tau)/(b + T_\tau) \tag{4.1}$$

is well-defined, finite, and attained by some $\tau \in R_\infty$ (Klass **[8]**, Theorem 1).

We omit the proof of the following lemma ($EV < \infty$ is used).

LEMMA 2. *For each $b \geqq 0$ there exists $\epsilon(b) > 0$ such that for any $\tau \in R_\infty$*

$$0 \leqq E[1/(b + T_\tau)] \leqq E[1/(b + T_1)] = 1/(b + \epsilon(b)).$$

If $P[\tau < \infty] > 0$ the leftmost inequality is strict.

REMARK 3. In the S_τ/τ problem ($\tau \in R'_\infty$), the form of the minimal strictly semi-optimal rule (Definitions 4 and 5 of Klass [8]) is dictated by the fact that for each $n \geqq 0$ there is a unique a_n such that $M_n(a_n) = a_n/n$. Our result, in addition to being more general, is obtained with a considerable economy of effort over earlier ones through the observation that the maps $a \to bM_b(a)$ contract the reals.

THEOREM 4. *Fix $b \geqq 0$. $M_b(a) > EU/EV \geqq 0$ for each a. $M_b(a)$ is a continuous strictly increasing function of a. bM_b is a contraction of the reals, and so has a unique fixed point a_b ($bM_b(a_b) = a_b$).*

Proof. The theorem is proved with the appropriate modifications of the proof of Lemma 8, page 729 of Klass [8]. Fix $b \geqq 0$.

For the first assertion, it suffices to show that $P[\sup(a + S_n)/(b + T_n) > EU/EV] = 1$ for any a. But $(a + S_n)/(b + T_n) > EU/EV$ if and only if $\Sigma_{i=1}^n (U_i - (EU/EV)V_i) > b(EU/EV) - a$. Since a nondegenerate mean zero random walk almost surely exceeds any real number infinitely often, the first assertion is proved.

Again fix $b \geqq 0$, let $a_1 < a_2$, and let τ_i attain $M_b(a_i)$, $i = 1, 2$. Then $P[\tau_i < \infty] > 0$ since $M_b(a_i) > EU/EV$, $i = 1, 2$, and two applications of Lemma 2 yield

$$\begin{aligned} 0 &< (a_2 - a_1)E[1/(b + T_{\tau_1})] \leqq M_b(a_2) - M_b(a_1) \\ &\leqq (a_2 - a_1)E[1/(b + T_{\tau_2})] \leqq (a_2 - a_1)/[b + \epsilon(b)]. \end{aligned}$$

The continuity of M_b follows, as does the last assertion of the theorem:

$$|bM_b(a_2) - bM_b(a_1)| \leqq \frac{b}{b + \epsilon(b)} |a_2 - a_1|.$$

Lemmas 6 and 7 and Remark 2 of Klass [8] carry over in straightforward fashion to our case, culminating in

LEMMA 3. *For* $b \geqq 0$:
(i) $a < a_b \Rightarrow bM_b(a) > a$
(ii) $a > a_b \Rightarrow bM_b(a) < a$
(iii) $\epsilon > 0 \Rightarrow a_{b+\epsilon} > a_b$.

Rather than introduce randomization (which is "unnecessary"; see Theorem 5.3, p. 111 of Chow, Robbins and Siegmund [3]) and determine up to equivalence the collection of all τ which attain $M_b(a)$, we content ourselves with exhibiting one such τ. The situation is somewhat novel in that the optimal stopping time depends on intrinsic time k only through the values of the T_k at those times, and the cutoff points a_{T_k} are themselves random. Then a_b in Theorem 5 are in accordance with those of Theorem 4.

THEOREM 5. *Given a real,* $b \geqq 0$, *define* $\tau \in R_\infty$ *by*

$$\tau = \min\{k : a + S_k > a_{b+T_k}\}$$
$$= \infty \quad \text{if} \quad a + S_k \leqq a_{b+T_k} \text{ for all } k.$$

Then $E(a + S_\tau)/(b + T_\tau) = M_b(a)$.

Proof. Clearly $\tau \in R_\infty$. To show that τ is optimal for the reward sequence $Y_n(a, b)$, it suffices to show that τ is minimal strictly semi-optimal (Definitions 4 and 5 and Theorem 6 of Klass [8]).

Suppose $S_n = s_n$, $T_n = t_n$ and τ instructs us to stop at time n for the reward $(a + s_n)/(b + t_n) > a_{b+t_n}/(b + t_n)$. By continuing we would expect to get at most $M_{b+t_n}(a + s_n)$, which is strictly less than $(a + s_n)/(b + t_n)$, by (ii) of Lemma 3. Hence τ is strictly semi-optimal.

The proof that τ is minimal (strictly semi-optimal) is as in the proof of Theorem 7 of Klass [8, p. 734], with a_{n+k} replaced by a_{b+T_k}.

5. A Markov walk example. The following example generalizes the fair coin tossing problem treated in Chow and Robbins [2]. Let $\{X_k\}_{k=1}^\infty$ be a $\{0, 1\}$-valued stationary Markov chain with $P[X_{k+1} = 1 \mid X_k = 0] = p = 1 - q$ and $P[X_{k+1} = 0 \mid X_k = 1] = p' = 1 - q'$. In order that the chain have stationary initial distribution we must have $P[X_1 = 1] = a = p/(p + p')$. We consider the optimal stopping problem with reward sequence $S_n^*/n = (X_1 + \cdots + X_n)/n$. Let $v = \sup_{\tau \in R_\infty^*} ES_\tau^*/\tau$, where R_∞^* is the collection of stopping rules w.r.t. $\{\mathcal{B}(X_1, \cdots, X_n)\}_{n=1}^\infty$.

Clearly any optimal rule has $\tau = 1$ if $X_1 = 1$ (otherwise τ is not regular; see Definition 2 and Theorem 2 of Klass [8]).

Now suppose $X_1 = 0$. We thrust independence into the picture as follows. Suppose the statistician gets to see the data, not a digit (0 or 1) at a time, but in blocks (more formally, instead of observing the original X_i, he views the sojourn times $V_1, U_1, V_2, U_2, \cdots$, where $V_i(U_i)$ is the time spent in the ith visit to $\{0\}$ ($\{1\}$)). The idea here is that, in the context of the original "game", it is clearly more profitable to stop at the end of some string of 1's as opposed to stopping in the middle of a 1-block or somewhere in a 0-block.

So let $U, U_1, U_2, \cdots$ be i.i.d. geometric r.v.'s with $P[U=k] = (q')^{k-1}p'$, $k \geqq 1$, and let $V, V_1, V_2, \cdots$ be i.i.d. geometric r.v.'s with $P[V=k] = q^{k-1}p$. Then the foregoing heuristics show that

$$\begin{aligned} v &\leqq a + (1-a)E\sup[S_n/(S_n+T_n)] \\ &\leqq a + (1-a)(E\sup S_n/T_n)/(E\sup S_n/T_n + 1). \end{aligned} \tag{5.1}$$

Here we have used Jensen's inequality and the fact that $f(x) = x/(x+1)$ is concave increasing for $x > 0$. In this way an upper bound on $E\sup S_n/T_n$ may be employed in majorizing v.

For example, one may use Theorem 3.4, p. 317 of Doob **[5]**, together with the fact that $\{S_n/T_n\}_{n=\infty}^{1}$ is a reversed submartingale, to obtain

$$E\sup S_n/T_n \leqq [e/(e-1)][1 + E(U/V)\log^+(U/V)].$$

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