

ON THE EXTENSION OF ADDITIVE FUNCTIONALS ON CLASSES OF CONVEX SETS

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Let \mathcal{S} be a class of sets and let $U(\mathcal{S})$ denote the class of all finite unions of sets from \mathcal{S} . This paper concerns itself with the question whether a vector valued function on \mathcal{S} has an additive extension to $U(\mathcal{S})$. Several characterizations of functions with such an extension property are presented. One of these characterizations establishes a relationship between such extensions and integrals of certain types of simple functions. The special cases when \mathcal{S} is the class of all convex polytopes or the class of all compact convex subsets of a euclidean space are investigated in more detail. Some examples are given to show that various extension problems that have been solved previously by methods particularly designed for each individual problem can also be solved by the application of these general results.

1. Introduction. Throughout this paper d is assumed to denote an arbitrary but fixed nonnegative integer. E^d denotes d -dimensional euclidean space. The class of all compact convex sets in E^d will be denoted by \mathcal{K} , and the class of all (compact convex) polytopes in E^d by \mathcal{P} .

Let S be an arbitrary set, and \mathcal{S} a class of subsets of S . We shall say that \mathcal{S} is *intersectional* if $\emptyset \in \mathcal{S}$ and $X \cap Y \in \mathcal{S}$ whenever $X \in \mathcal{S}$, $Y \in \mathcal{S}$. A function λ that maps an intersectional class \mathcal{S} into some vector space is said to be *additive* if $\lambda(X \cup Y) = \lambda(X) + \lambda(Y) - \lambda(X \cap Y)$ for all sets X, Y with $X \in \mathcal{S}$, $Y \in \mathcal{S}$, $X \cup Y \in \mathcal{S}$. Typical examples are the volume or surface area of sets from \mathcal{K} or \mathcal{P} , and these examples show already that it is of interest to investigate whether these functionals can be extended to more general classes of sets. Here we concern ourselves with extensions from \mathcal{S} to the class $U(\mathcal{S})$, i.e. the collection of all unions of finitely many sets from \mathcal{S} .

For the class \mathcal{P} this extension problem has already been considered by Volland [14] (for functions with values in an abelian group) and Hadwiger [7] (for a much more restricted class of real valued functions). Important results regarding the general case are contained in the paper [10] of Perles and Sallee.

In the present paper a rather different approach to this extension problem will be presented. It is closely related to methods that have been used previously (Groemer [2], [3], [4], [5]) for the extension of some

functionals on \mathcal{K} (Euler characteristic, mixed volumes etc.). This method offers a wider range of applications and it also establishes a connection between the extension problem and a seemingly rather different problem in the theory of integration.

To formulate this latter problem we have to introduce the following definitions and notations. If \mathcal{S} is again an intersectional class of subsets of S , and if $X \in \mathcal{S}$ we denote by X^* the characteristic function of X . Hence X^* is defined on S and has the property that $X^*(x) = 1$ if $x \in X$ and $X^*(x) = 0$ if $x \notin X$. The class of all functions X^* with $X \in \mathcal{S}$ will be denoted by \mathcal{S}^* . Any function f of the form

$$(1) \quad f = a_1 X_1^* + a_2 X_2^* + \cdots + a_m X_m^* \quad (X_i \in \mathcal{S}, a_i \text{ real})$$

will be called a *simple function*, or, more precisely, a *simple \mathcal{S} -function*. The simple \mathcal{S} -functions form obviously a vector space (over the reals), and this vector space will be denoted by $V(\mathcal{S})$. If λ is a mapping of \mathcal{S} into some vector space (again over the reals) it appears to be natural to define the integral, more precisely, the λ -integral of a simple function f with the representation (1) by

$$(2) \quad \int f d\lambda = a_1 \lambda(X_1) + a_2 \lambda(X_2) + \cdots + a_m \lambda(X_m).$$

Clearly, this definition is only meaningful if $\int f d\lambda$ does not depend on the special representation (1) of f . If this is the case for every simple \mathcal{S} -function we say that \mathcal{S} permits a λ -integral. Typical examples are provided by taking for \mathcal{S} the class of intervals or the class of Lebesgue measurable subsets of E^1 and for λ the length of an interval or the Lebesgue measure; the integral defined by (2) is then the Riemann integral for step functions or the Lebesgue integral for (Lebesgue) simple functions. It is worth mentioning that, in general, the additivity of λ does not imply that \mathcal{S} permits a λ -integral. For example, if one assumes that $S = \{1, 2, 3\}$, $\mathcal{S} = \{\emptyset, \{1\}, \{2\}, \{3\}, \{1, 2, 3\}\}$ and $\lambda(\emptyset) = 0$, $\lambda(X) = 1$ for $X \neq \emptyset$ then \mathcal{S} is intersectional, λ is additive and $\{1\}^* + \{2\}^* + \{3\}^* - \{1, 2, 3\}^* = \emptyset^*$; but $\lambda(\{1\}) + \lambda(\{2\}) + \lambda(\{3\}) - \lambda(\{1, 2, 3\}) = 2$ and $\lambda(\emptyset) = 0$.

Any function λ on \mathcal{S} can also be viewed as a function on \mathcal{S}^* (simply by setting $\lambda(X^*) = \lambda(X)$). Then, instead of asking whether λ has an additive extension to $U(\mathcal{S})$ one may ask whether λ has a linear extension to $V(\mathcal{S})$.

The following section is devoted to a discussion of the relationship between this problem, the existence of λ -integrals, and the existence of additive extensions of λ to $U(\mathcal{S})$. It also deals with a concept that will

be referred to as general additivity. A vector valued function λ on an intersectional class \mathcal{S} is said to be *generally additive* if

$$(3) \quad \begin{aligned} &\lambda(X_1 \cup X_2 \cup \cdots \cup X_m) \\ &= \sum_i \lambda(X_i) - \sum_{i < j} \lambda(X_i \cap X_j) + \sum_{i < j < k} \lambda(X_i \cap X_j \cap X_k) - + \cdots \end{aligned}$$

whenever $X_i \in \mathcal{S}$ and $X_1 \cup X_2 \cup \cdots \cup X_m \in \mathcal{S}$.

Section 3 concerns itself with the special cases that result when \mathcal{S} is taken to be the class of convex polytopes or the class of all convex bodies of E^d . In §4 several examples are presented, and in §5 the case of spherically convex sets is briefly discussed.

2. A general extension theorem. Part of the following theorem, namely the equivalence (iii) \Rightarrow (iv), has already been formulated and proved by Perles and Sallee [10]. Our proof of this result is based on the relationship between extensions and integrals.

THEOREM 1. *Let \mathcal{S} be an intersectional class of sets, and let λ be a function mapping \mathcal{S} (or \mathcal{S}^*) into a vector space (over the reals) so that $\lambda(\emptyset) = 0$. Then, the following statements are equivalent:*

- (i) \mathcal{S} permits a λ -integral,
- (ii) λ has a unique linear extension from \mathcal{S}^* to $V(\mathcal{S})$,
- (iii) λ has a unique additive extension from \mathcal{S} to $U(\mathcal{S})$,
- (iv) λ is generally additive.

Proof. First we note that (i) is equivalent to the following statement:

$$(i') \quad \text{If } b_1 Y_1^* + b_2 Y_2^* + \cdots + b_k Y_k^* = 0 \quad (Y_i \in \mathcal{S}, \quad b_i \text{ real}) \quad \text{then} \\ b_1 \lambda(Y_1) + b_2 \lambda(Y_2) + \cdots + b_k \lambda(Y_k) = 0.$$

Indeed, if \mathcal{S} permits a λ -integral then (i') is certainly satisfied since the second sum in (i') is the integral of the first sum and therefore equal to $\int 0 d\lambda = 0$; and if \mathcal{S} does not permit a λ -integral there are two representations of the form (1) that yield different values (2), but if (i') were true it could be applied to the difference of these two representations of f and one would immediately arrive at a contradiction.

As another preliminary remark we note that the characteristic function on an intersectional class \mathcal{S} is generally additive. In other words, if $X_i \in \mathcal{S}$, $X_1 \cup X_2 \cup \cdots \cup X_m \in \mathcal{S}$ then

$$(4) \quad \begin{aligned} & (X_1 \cup X_2 \cup \cdots \cup X_m)^* \\ &= \sum_i X_i^* - \sum_{i < j} (X_i \cap X_j)^* + \sum_{i < j < k} (X_i \cap X_j \cap X_k)^* - + \cdots. \end{aligned}$$

This is a simple consequence of the binomial theorem.

Our proof of the theorem proceeds now by showing that (i') \Rightarrow (ii) \Rightarrow (iii) \Rightarrow (iv) \Rightarrow (i').

Proof of (i') \Rightarrow (ii): If (i') is satisfied this implies, as already remarked, that \mathcal{S} permits a λ -integral $\int f d\lambda$ for every $f \in V(\mathcal{S})$, and it is obvious that this integral is a linear functional on $V(\mathcal{S})$. Also, if $X \in \mathcal{S}$ then $\int X^* d\lambda = \lambda(X) = \lambda(X^*)$, which shows that the integral is an extension of λ as a function on \mathcal{S}^* . Finally, if λ' is any linear extension of λ from \mathcal{S}^* to $V(\mathcal{S})$ then it follows from (1) that

$$\begin{aligned} \lambda'(f) &= a_1 \lambda'(X_1) + \cdots + a_m \lambda'(X_m) \\ &= a_1 \lambda(X_1) + \cdots + a_m \lambda(X_m) = \int f d\lambda. \end{aligned}$$

Hence, a linear extension of λ from \mathcal{S} to $V(\mathcal{S})$ is unique.

Proof of (ii) \Rightarrow (iii): If $T \in U(\mathcal{S})$ then $T = X_1 \cup X_2 \cup \cdots \cup X_m$ with $X_i \in \mathcal{S}$, and it follows from (4) that $T^* \in V(\mathcal{S})$. Hence, if λ_v is the extension of λ on \mathcal{S} to $V(\mathcal{S})$ then, λ_u defined by $\lambda_u(T) = \lambda_v(T^*)$ is an extension of λ from \mathcal{S} to $U(\mathcal{S})$. Moreover, if $A \in U(\mathcal{S})$, $B \in U(\mathcal{S})$ then

$$\begin{aligned} \lambda_u(A \cup B) &= \lambda_v((A \cup B)^*) \\ &= \lambda_v(A^* + B^* - (A \cap B)^*) \\ &= \lambda_v(A^*) + \lambda_v(B^*) - \lambda_v((A \cap B)^*) \\ &= \lambda_u(A) + \lambda_u(B) - \lambda_u(A \cap B), \end{aligned}$$

which shows that λ_u is additive. The uniqueness of λ_u can be derived inductively from the fact that for $X_i \in \mathcal{S}$ and any additive function μ on $U(\mathcal{S})$

$$(5) \quad \begin{aligned} & \mu(X_1 \cup X_2 \cup \cdots \cup X_m) = \mu(X_1) + \mu(X_2 \cup \cdots \cup X_m) \\ & \quad - \mu((X_1 \cap X_2) \cup (X_1 \cap X_3) \cup \cdots \cup (X_1 \cap X_m)) \end{aligned}$$

since this relation shows that the value of μ for a union of m members from \mathcal{S} is uniquely determined by the value of μ for unions of less than m terms.

Proof of (iii) \Rightarrow (iv). The general additivity of any additive function on $U(\mathcal{S})$ can be deduced from (5) by an obvious induction argument. Hence, if λ has an additive extension λ_u to $U(\mathcal{S})$ then λ_u is generally additive on $U(\mathcal{S})$, and λ (since it is the restriction of λ_u to \mathcal{S}) is a generally additive function on \mathcal{S} .

Proof of (iv) \Rightarrow (i'). Let us assume that (iv) is true but that there exist sets Y_1, Y_2, \dots, Y_k in \mathcal{S} so that (i') is not satisfied. It will be shown that these assumptions lead to a contradiction. We define sets D_1, D_2, \dots, D_p (where $p = 2^k - 1$) by the following rules: $D_1 = Y_1, D_2 = Y_2, \dots, D_k = Y_k$; $D_{k+1}, D_{k+2}, \dots, D_{k_1}$ are the intersections $Y_i \cap Y_j$, with $i < j$ (in some order); $D_{k_1+1}, D_{k_1+2}, \dots, D_{k_2}$ are the intersections $Y_i \cap Y_j \cap Y_{k_1}$ with $i < j < 1$, and so on until one meets $D_p = Y_1 \cap Y_2 \cap \dots \cap Y_k$. Then, every intersection of the form $D_g \cap D_h$ is again some D_s , with $s \cong \max\{g, h\}$. Because of $D_1 = Y_1, \dots, D_k = Y_k$, and since (i') is not satisfied it follows that there are real numbers c_i , so that

$$(6) \quad \sum_{i \cong r} c_i D_i^* = 0, \quad c_r \neq 0,$$

$$(7) \quad \sum_{i \cong r} c_i \lambda(D_i) \neq 0.$$

Since r is restricted by $1 \cong r \cong p$ we may assume that the coefficients c_i have been selected so that r is as large as possible. However, $r = p$ is impossible, since in this case (6) and (7) would reduce to the contradicting relations $c_p D_p^* = 0, c_p \lambda(D_p) \neq 0$. Because of (6) it is impossible to find a point x so that $D_i^*(x) = 1, D_j^*(x) = 0$ for all $i > r$. Hence, every point of D_r is in some D_i with $i > r$, and therefore

$$\begin{aligned} D_r &= D_r \cap (D_{r+1} \cup D_{r+2} \cup \dots \cup D_p) \\ &= (D_r \cap D_{r+1}) \cup (D_r \cap D_{r+2}) \cup \dots \cup (D_r \cap D_p). \end{aligned}$$

Since both λ and the characteristic function are generally additive it follows that

$$\lambda(D_r) = \sum_{r < i < p} \lambda(D_r \cap D_i) - \sum_{r < i < j < p} \lambda(D_r \cap D_i \cap D_j) + \dots$$

and

$$D_r^* = \sum_{r < i < p} (D_r \cap D_i)^* - \sum_{r < i < j < p} (D_r \cap D_i \cap D_j)^* + \dots$$

Each intersection appearing in these two equalities is some D_s with $s > r$. Hence, there are coefficients d_s so that

$$(8) \quad D_r^* = \sum_{s>r} d_s D_s^*$$

and

$$(9) \quad \lambda(D_r) = \sum_{s>r} d_s \lambda(D_s).$$

If (8) and (9) are substituted into (6) and (7) one obtains obviously expressions of the same type as (6) and (7) but with a larger value of r . Since this contradicts the assumption that r be maximal, the proof is finished.

3. Convex sets. We consider now the special case when \mathcal{P} is the class of compact convex polytopes or the class \mathcal{K} of compact convex subsets of E^d . It will be shown that rather weak assumptions on the functional λ enable one to prove that the statements (i)–(iv) of Theorem 1 are valid. The following definitions and notations will be useful for this purpose. If H is a hyperplane in E^d we denote by H^+ and H^- the two closed half-spaces determined by H . A vector valued function λ on \mathcal{P} (or \mathcal{K}) will be said to be *weakly additive* (cf. Sallee [12]) if for every hyperplane H and every $X \in \mathcal{P}$ (or $X \in \mathcal{K}$)

$$(10) \quad \lambda(X) = \lambda(X \cap H^+) + \lambda(X \cap H^-) - \lambda(X \cap H).$$

Clearly, every additive function is weakly additive. But weak additivity is sometimes easier to handle than additivity.

First, we prove a theorem concerning the relationship between weak additivity and the statements (i)–(iv) of Theorem 1. Under the assumption that the pertinent function be additive, and using different methods, essentially the same result has been proved by Volland [14] and Perles and Sallee [10], see also Hadwiger [7, p. 81].

THEOREM 2. *Let λ be a vector valued function on \mathcal{P} such that $\lambda(\emptyset) = 0$. Then, the statements (i)–(iv) of Theorem 1 are true if and only if λ is weakly additive.*

Proof. Since it has already been shown that the statements (i'), (i), (ii), (iii), (iv) are equivalent, and since (iv) implies obviously (10), it suffices to prove that (10) implies (i'). Hence, we have to show that it is impossible that there exist a weakly additive function λ on \mathcal{P} , polytopes P_1, P_2, \dots, P_m in \mathcal{P} , and real numbers p_1, p_2, \dots, p_m so that

$$(11) \quad \sum_{i=1}^m p_i P_i^* = 0$$

and

$$(12) \quad \sum_{i=1}^m p_i \lambda(P_i) \neq 0.$$

For the space E^0 (11) and (12) are clearly contradictory, and we make the induction assumption that the same be the case for E^{d-1} . It will be shown that for the space E^d (11) and (12) will also lead to a contradiction. Let us denote by k the number of d -dimensional polytopes P_i that appear in (11). We may assume that k is minimal; in other words, we suppose that there are no relations of the type (11), (12) with less than k polytopes of dimension d .

If $k = 0$ the polytopes P_i are contained in a finite number of hyperplanes, say H_1, H_2, \dots, H_l , where we may assume that l be minimal in the sense that there are no relations of the form (11), (12) with all P_i in less than l hyperplanes. If $l = 1$ the plane H_1 can be interpreted as E^{d-1} and one obtains a contradiction to our induction assumption. Thus, $l > 1$. If we define polytopes \bar{P}_i by $\bar{P}_i = P_i \cap H_l$, then it follows from (11) that $\sum_{i=1}^m p_i \bar{P}_i^* = 0$ and therefore, using again the induction assumption, $\sum_{i=1}^m p_i \lambda(\bar{P}_i) = 0$. Hence, we obtain from (11) and (12)

$$(13) \quad \sum_{i=1}^m p_i (P_i^* - \bar{P}_i^*) = 0$$

and

$$(14) \quad \sum_{i=1}^m p_i (\lambda(P_i) - \lambda(\bar{P}_i)) \neq 0.$$

Since $P_i \subset H_l$ implies $P_i = \bar{P}_i$, the terms $p_i (P_i^* - \bar{P}_i^*)$ and $p_i (\lambda(P_i) - \lambda(\bar{P}_i))$ may be deleted from (13) and (14) whenever $P_i \subset H_l$. These deletions yield relations of the form (11), (12) where all the remaining polytopes are contained in the planes H_1, H_2, \dots, H_{l-1} . This however is impossible since l was assumed to be minimal.

We consider now the case $k \geq 1$. It can be assumed that $\dim P_i = d$ for $1 \leq i \leq k$, and $\dim P_i < d$ for $k < i \leq m$. Let H be a hyperplane that contains a $((d - 1)$ -dimensional) face of P_1 and let us assume that the notation of the half-spaces H^+, H^- , has been selected so that $P_1 \subset H^+$. (11) implies obviously

$$(15) \quad \sum_{i=1}^m p_i (P_i \cap H^-)^* = 0$$

and

$$(16) \quad \sum_{i=1}^m p_i (P_i \cap H)^* = 0.$$

Because of $P_1 \subset H^+$ the half-space H^- contains at most $k - 1$ polytopes $P_i \cap H^-$ of dimension d , and it follows therefore from (15) and the minimal property of k that

$$(17) \quad \sum_{i=1}^m p_i \lambda(P_i \cap H^-) = 0.$$

Also, from (16) and the induction assumption we obtain

$$(18) \quad \sum_{i=1}^m p_i \lambda(P_i \cap H) = 0.$$

If (10) is applied to each P_i it follows from (12), (17) and (18) that

$$(19) \quad \sum_{i=1}^m p_i \lambda(P_i \cap H^+) \neq 0.$$

We note also that (11) implies

$$(20) \quad \sum_{i=1}^m p_i (P_i \cap H^+)^* = 0.$$

Hence, from the relations (11) and (12) one can derive an analogous pair (19), (20). If this procedure is repeated for all the other hyperplanes that contain faces of P_1 one obtains

$$(21) \quad \sum_{i=1}^m p_i \lambda(P_i \cap P_1) \neq 0, \quad \sum_{i=1}^m p_i (P_i \cap P_1)^* = 0.$$

If one of the polytopes $P_i \cap P_1$ ($i = 1, 2, \dots, k$) is less than d -dimensional we have reached a contradiction to the minimal property of k . If all these polytopes are again d -dimensional we may start with (21) and repeat the same arguments with $P_1 \cap P_2$ accepting the role of P_1 . This leads to relations of the same kind as (21) with $P_i \cap P_1$ replaced by $P_i \cap P_1 \cap P_2$, and again either one has reached a contradiction or the process can be repeated. If one continues in this way, and if the polytope $D = P_1 \cap P_2 \cap \dots \cap P_k$ is less than d -dimensional one will meet after a suitable number of repetitions a contradiction to the minimal property of k . If $\dim D = d$ one arrives at the following relations (setting $\sum_{i=1}^m p_i = c$):

$$(22) \quad \sum_{i=1}^m p_i \lambda(P_i \cap D) = c \lambda(D) + \sum_{i>k} p_i \lambda(P_i \cap D) \neq 0,$$

$$(23) \quad \sum_{i=1}^m p_i (P_i \cap D)^* = c D^* + \sum_{i>k} p_i (P_i \cap D)^* = 0.$$

Because of $\dim D = d$ and $\dim(P_i \cap D) < d$ ($i > k$), there is a point x with $x \in D$, $x \notin P_i \cap D$ ($i > k$). Consequently, $D^*(x) = 1$, $(P_i \cap D)^*(x) = 0$ ($i > k$) and it follows from (23) that $c = 0$. Hence, in (22) and (23) there appear only polytopes of dimension less than d and this has already been shown to be impossible.

Since general additivity implies additivity we can state the following corollary to Theorem 2. It has also been noted by Sallee [12].

COROLLARY 1. *If λ is a weakly additive vector valued function on P then λ is additive.*

In this context it is not necessary to assume that $\lambda(\emptyset) = 0$; if $\lambda(\emptyset) = c \neq 0$ one can apply Theorem 2 to the function $\lambda - c$.

Instead of \mathcal{P} we consider now the larger class \mathcal{K} , i.e. the compact convex subsets of E^d . Under a kind of continuity assumption it will be possible to prove an analogue of Theorem 2. Let λ be a mapping of \mathcal{K} into a topological (Hausdorff) vector space T (over the real numbers). We shall say that λ is continuous if for every decreasing sequence of convex bodies A_i one can infer that $\lim_{i \rightarrow \infty} \lambda(A_i) = \lambda(\bigcap_{i=1}^{\infty} A_i)$. If λ is continuous in the usual sense with respect to the Hausdorff-Blaschke topology in \mathcal{K} and the given topology in T , then it is also continuous in the sense just mentioned.

THEOREM 3. *Let λ be a continuous function that maps \mathcal{K} into a topological vector space so that $\lambda(\emptyset) = 0$. Then the statements (i)–(iv) of Theorem 1 are true if and only if λ is weakly additive.*

Proof. We note that it is not possible to deduce this theorem from Theorem 2 and known approximation properties of polytopes, since a relation of the form $\sum x_i X_i^* = 0$ with $X_i \in \mathcal{K}$ does not imply a corresponding relation for the approximating polytopes.

As in the proof of Theorem 2 it suffices to show that the above assumptions on λ imply that (i') holds. To prove this by contradiction we assume that (i') is not true for some weakly additive continuous function λ . This means that there exist sets K_1, K_2, \dots, K_m in \mathcal{K} and real numbers k_1, k_2, \dots, k_m so that

$$(24) \quad \sum_{i=1}^m k_i K_i^* = 0,$$

$$(25) \quad \sum_{i=1}^m k_i \lambda(K_i) = a, \quad \text{where } a \neq 0.$$

We may assume that m is the smallest number with the property that there are relations of the form (24), (25). Obviously, $m \geq 2$. Let H be a

hyperplane in E^d so that $K_1 \subset \text{int } H^+$. Then, it follows from (24) that

$$(26) \quad \sum_{i=1}^m k_i (K_i \cap H^-)^* = 0,$$

$$(27) \quad \sum_{i=1}^m k_i (K_i \cap H)^* = 0.$$

Because of $K_1 \cap H^- = \emptyset$ and $K_1 \cap H = \emptyset$ these two sums have at most $m - 1$ nonzero terms. Therefore, it follows from the minimal property of m (together with $\lambda(\emptyset) = 0$) that

$$(28) \quad \sum_{i=1}^m k_i \lambda(K_i \cap H^-) = 0,$$

$$(29) \quad \sum_{i=1}^m k_i \lambda(K_i \cap H) = 0.$$

If (10) is applied to each set K_i , it follows from (25), (28), and (29) that

$$(30) \quad \sum_{i=1}^m k_i \lambda(K_i \cap H^+) = a.$$

We also note that (24) implies

$$(31) \quad \sum_{i=1}^m k_i (K_i \cap H^+)^* = 0.$$

Instead of one plane we consider now an infinite sequence H_1, H_2, \dots of hyperplanes in E^d with the property that $K_1 \subset \text{int } H_j^+$ and

$$(32) \quad K_1 = \bigcap_{j=1}^{\infty} H_j^+.$$

The existence of such a sequence of planes is an immediate consequence of well-known approximation properties of convex bodies (see Bonnesen-Fenchel [1] or Hadwiger [7]). If the procedure that led from (24), (25) to (30), (31) is repeated n times one obtains

$$(33) \quad \sum_{i=1}^m k_i \lambda \left(K_i \cap \bigcap_{j=1}^n H_j^+ \right) = a.$$

(32) and the continuity of λ imply that for every fixed i

$$\lim_{n \rightarrow \infty} \lambda \left(K_i \cap \bigcap_{j=1}^n H_j^+ \right) = \lambda(K_i \cap K_1).$$

From this relation and (33) we obtain therefore

$$(34) \quad \sum_{i=1}^m k_i \lambda(K_i \cap K_1) = a.$$

Also, from (24) we get the corresponding equation

$$(35) \quad \sum_{i=1}^m k_i (K_i \cap K_1)^* = 0.$$

The process which enabled us to derive (34) and (35) from (24) and (25) can now be repeated using successively the sets $K_i, K_i \cap K_1, K_i \cap K_1 \cap K_2, \dots, K_i \cap K_1 \cap K_2 \cap \dots \cap K_m$. If we write $k_1 + k_2 + \dots + k_m = b, K_1 \cap K_2 \cap \dots \cap K_m = L$, and note that $K_i \cap L = L$ we arrive at the relations

$$b\lambda(L) = a, \quad bL^* = 0.$$

Because of $a \neq 0$, and since the second of these equations implies that $b = 0$ or $L = \emptyset$ we have reached a contradiction, and the proof of the theorem is finished.

Similarly as in the case regarding convex polytopes we can state the relationship between additivity and weak additivity as a corollary.

COROLLARY 2. *If λ is a weakly additive continuous function that maps \mathcal{K} into a topological vector space, then λ is additive.*

It is possible to construct examples which show that Theorem 3 and Corollary 2 are not valid without the assumption of continuity.

4. Examples. One of the most simple continuous additive functionals on \mathcal{K} is defined by $\chi(K) = 1$ if $K \neq \emptyset$, and $\chi(\emptyset) = 0$. According to Theorem 3 there is a unique additive extension of λ to $U(\mathcal{K})$ and a unique linear extension to $V(\mathcal{K})$. This extended functional is called the Euler characteristic on $U(\mathcal{K})$ or $V(\mathcal{K})$. Various other geometric methods have been developed to define the Euler characteristic on these structures, and even more such methods exist for the smaller classes $U(\mathcal{P})$ and $V(\mathcal{P})$. The pertinent literature is cited in [2], [3], [7], and [8].

The Euler characteristic, along with volume and surface area, are

(aside from constant factors) special cases of the Minkowskian projection integrals (Quermassintegrale). These projection integrals are easily seen to be additive and continuous, and are zero on the empty set (see [1] or [7]). Consequently, Theorem 3 shows that the projection integrals have unique additive extensions to $U(\mathcal{K})$ and unique linear extensions to $V(\mathcal{K})$. More special methods that can be used for the construction of these extensions are described in [2], [7] and [8].

As an example of a function that maps \mathcal{K} into a d -dimensional vector space one may take the Steiner point of a convex body. It can be proved that this function is continuous (with respect to the usual topology of E^d) and additive, and can be defined to be zero on the empty set. Theorem 3 enables one again to infer the existence of a unique additive extension to $U(\mathcal{K})$ and a unique linear extension to $V(\mathcal{K})$. For particular proofs of these facts see [13] and [11].

The following examples show that in some cases the class $V(\mathcal{K})$ rather than $U(\mathcal{K})$ is the domain of principal interest.

Let H be a hyperplane in E^d and let p denote the orthogonal projection of the sets from \mathcal{K} onto H . If \mathcal{K}_H denotes the class of compact convex subsets of H then p can also be viewed as a mapping of \mathcal{K} onto \mathcal{K}_H^* , i.e. the class $\{Y^*: Y \in \mathcal{K}_H\}$. It is easily seen that p is additive and continuous (with respect to the topology induced by pointwise convergence of the functions from $V(\mathcal{K}_H)$). Also, $p(\emptyset) = 0$. Hence, Theorem 3 can be applied. The resulting extension of p to $U(\mathcal{K})$ maps $U(\mathcal{K})$ into $V(\mathcal{K}_H)$ so that to each $X \in U(\mathcal{K})$ and $x \in H$ there corresponds a certain multiplicity $p(x)$, determined by the number of intervals of $X \cap L$ where L is a line through x that is orthogonal to H . Since many applications that involve projections use induction with respect to dimension it is usually advantageous to work with the extension of p that maps $V(\mathcal{K})$ into $V(\mathcal{K}_H)$. This generalization of the concept of a projection has been introduced by more special methods in [2], and can be used as a basis for a theory of projection integrals of nonconvex sets. Obviously, Theorem 2 can also be applied to projections onto general k -flats (instead of hyperplanes) and to arbitrary parallel (instead of orthogonal) projections.

As another application of this type we consider the Minkowski sum $K + L = \{x + y : x \in K, y \in L\}$ of two compact convex sets. The same definition of $K + L$ can be used if K, L are not convex bodies, but then many desirable properties, particularly those that are essential for the development of the theory of mixed volumes, are lost. The following extension procedure is more useful in this respect. For a fixed set $L \in \mathcal{K}$ and any $X \in \mathcal{K}$ let us define a function λ by $\lambda(X, L) = (X + L)^*$. Then λ as a function of X maps \mathcal{K} into the vector space $V(\mathcal{K})$. It is easily proved that this mapping is additive, continuous (with respect to the topology in $V(\mathcal{K})$) that is induced by pointwise convergence of the functions in $V(\mathcal{K})$),

and that $\lambda(\emptyset, L) = 0$. Hence, by Theorem 3 there is a unique additive extension to $U(\mathcal{K})$ and a unique linear extension to $V(\mathcal{K})$. For any fixed function v in $V(\mathcal{K})$ (or any $T \in U(\mathcal{K})$) the same procedure can be repeated for $\lambda(v, Y)$ (or $\lambda(T, Y)$) as a function in Y . The final result is that the Minkowski sum, originally defined on $\mathcal{K} \times \mathcal{K}$, has a unique bilinear extension to $V(\mathcal{K}) \times V(\mathcal{K})$ (and a unique "biadditive" extension to $U(\mathcal{K}) \times U(\mathcal{K})$). By the use of the Euler characteristic and more special methods this result has been proved in [5], where further details can be found.

Similarly as for Minkowski sums it is possible to extend the mixed volumes. If $v_{i,n-i}(K, L)$ denotes the mixed volume $V(K, \dots, K, L, \dots, L)$ (where K appears i times and L $n-i$ times) one can consider first $v_{i,n-i}(X, L)$ as a continuous additive function of X and extend to $V(\mathcal{K})$. Then this process can be repeated for $v_{i,n-i}(u, Y)$ as a function in Y (for any fixed $u \in V(\mathcal{K})$). The result of these two extensions is a bilinear function on $V(\mathcal{K}) \times V(\mathcal{K})$. Another approach and a more detailed presentation of results on mixed volumes of nonconvex sets can be found in [5] and, regarding integral geometric aspect, in [6].

5. Spherically convex sets. Let S^n denote the n -dimensional unit sphere, and let \mathcal{C} be the class of spherically compact convex subsets of S^n (in the restricted sense that every $C \in \mathcal{C}$ is contained in an open semisphere). \mathcal{C} is obviously intersectional and Theorem 1 applies if \mathcal{S} is taken to be \mathcal{C} . Theorems 2 and 3 do not immediately apply to \mathcal{C} , but the proofs of these theorems are still valid for spherically convex sets if hyperplanes are replaced by $(n-1)$ -dimensional unit spheres on S^n , and closed half-spaces by closed semispheres. Consequently, those functionals of the preceding section which have analogues for spherically convex sets have corresponding extension properties. For example, one can use this approach for the introduction of the Euler characteristic for sets on the sphere. For other geometric possibilities to define this functional on the sphere see Hadwiger and Mani [9], and Groemer [3], [4].

REFERENCES

1. T. Bonnesen and W. Fenchel, *Theorie der konvexen Körper* *Ergebn. d. Math.* Bd. 3, Springer Verl., Berlin 1934.
2. H. Groemer, *Eulersche Charakteristik, Projektionen und Quermassintegrale*, *Math. Ann.*, **198** (1972), 26-56.
3. ———, *On the Euler characteristic in spaces with a separability property*, *Math. Ann.*, **211** (1974), 315-321.
4. ———, *The Euler characteristic and related functionals on convex surfaces*, *Geom. Dedic.*, **4** (1974), 91-104.
5. ———, *Minkowski addition and mixed volumes*, *Geom. Dedic.* (in print).

6. ———, *Translative integral geometry*, Arch. d. Math., **29** (1977), 324–330.
7. H. Hadwiger, *Vorlesungen über Inhalt, Oberfläche und Isoperimetrie*, Grundle. d. math. Wiss. Bd. **93**, Springer Verl., Berlin-Göttingen-Heidelberg, 1957.
8. ———, *Normale Körper im euklidischen Raum und ihre topologischen und metrischen Eigenschaften*, Math. Zeitschr., **71** (1959), 124–14.
9. H. Hadwiger and P. Mani, *On the Euler characteristic of spherical polyhedra and the Euler relation*, Matematika, **19** (1972), 139–143.
10. M. A. Perles and G. T. Sallee, *Cell complexes, valuations, and the Euler relation*, Canad. J. Math., **22** (1970), 235–241.
11. G. T. Sallee, *A valuation property of Steiner points*, Matematika, **13** (1966), 76–82.
12. ———, *Polytopes, valuations, and the Euler relation*, Canad. J. Math., **18** (1968), 1412–1424.
13. G. C. Shephard, *The Steiner point of a convex polytope*, Canad. J. Math., **18** (1966), 1294–1300.
14. W. Volland, *Ein Fortsetzungssatz für additive Eipolyederfunktionale im euklidischen Raum*, Arch. d. Math., **8** (1957), 144–149.

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