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ON THE UNITARY INVARIANCE OF THE NUMERICAL RADIUS

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A characterization is obtained of scalar multiples of unitary matrices in terms of the unitary invariance of a generalized numerical radius. The method of proof involves some rather delicate combinatorial considerations.

1. Introduction. Let n and m be positive integers, $1 \leqq m \leqq n$, and denote by $M_{n,m}(\mathbf{C})$ ($M_n(\mathbf{C})$) the vector space of all n-by-m (n-square) complex matrices. For a matrix $A \in M_n(\mathbf{C})$, define the mth *decomposable numerical range of* A to be the set

(1) $$W_m^\wedge(A) = \{\det(X^*AX) \mid X \in M_{n,m}(\mathbf{C}), \det(X^*X) = 1\}$$

in the complex plane (the reason for this choice of terminology will become apparent in the next section). It is not difficult to verify that $W_m^\wedge(A)$ is compact, so it makes sense to define the mth *decomposable numerical radius of* A by

(2) $$r_m^\wedge(A) = \max_{z \in W_m^\wedge(A)} |z|.$$

When $m = 1$, $W_1^\wedge(A)$ is simply the classical *numerical range*

(3) $$W(A) = \{(Ax, x) \mid x \in \mathbf{C}^n, \|x\| = 1\}$$

(here $(\cdot,\cdot)$ denotes the standard inner product in the space $\mathbf{C}^n$ of complex n-tuples), and $r_1^\wedge(A)$ is the classical *numerical radius*

(4) $$r(A) = \max_{z \in W(A)} |z|.$$

The numerical radius $r(A)$ satisfies the interesting *power inequality*

(5) $$r(A^k) \leqq r(A)^k, \quad k = 1, 2, 3, \cdots$$

[2, §176]. In general, the number $r_m^\wedge(A)$ is an important function of the matrix A. For example, it is a bound for the moduli of all products of m eigenvalues of A. This is an immediate consequence of Proposition 1. Another easy consequence (Corollary 2) of Proposition 1 is that if A

is a scalar multiple of a unitary matrix, then $r_m^\wedge(A)$ remains invariant under pre- and postmultiplication of A by arbitrary unitary matrices. The purpose of the present paper is to prove that in fact this invariance property characterizes scalar multiples of unitary matrices (Theorem 1).

2. Preliminary notions. The mth Grassmann space over $\mathbf{C}^n$, denoted by $\wedge^m \mathbf{C}^n$, provides an appropriate setting for our investigation of the mth decomposable numerical radius. The standard inner product in $\mathbf{C}^n$ induces an inner product in $\wedge^m \mathbf{C}^n$, given on decomposable symmetrized tensors

$$x^\wedge = x_1 \wedge \cdots \wedge x_m, y^\wedge = y_1 \wedge \cdots \wedge y_m \in \overset{m}{\wedge} \mathbf{C}^n$$

by

$$(x^\wedge, y^\wedge) = \det[(x_i, y_j)].$$

The *Grassmannian manifold* $G_m(\mathbf{C}^n)$ is the set of all unit length decomposable symmetrized tensors in $\wedge^m \mathbf{C}^n$:

$$G_m(\mathbf{C}^n) = \left\{ x^\wedge \in \overset{m}{\wedge} \mathbf{C}^n \,\middle|\, \| x^\wedge \| = 1 \right\}.$$

Let $A \in M_n(\mathbf{C})$, and let $C_m(A)$ be the mth compound of A, so that for $x_1, \cdots, x_m \in \mathbf{C}^n$ we have

$$C_m(A) x_1 \wedge \cdots \wedge x_m = A x_1 \wedge \cdots \wedge A x_m.$$

If the columns of a matrix $X \in M_{n,m}(\mathbf{C})$ are $x_1, \cdots, x_m$ in order, then

$$\det(X^* A X) = (C_m(A) x_1 \wedge \cdots \wedge x_m, x_1 \wedge \cdots \wedge x_m).$$

Furthermore, $\det(X^*X) = 1$ if and only if $x_1 \wedge \cdots \wedge x_m \in G_m(\mathbf{C}^n)$. Thus from (1),

(6) $$W_m^\wedge(A) = \{(C_m(A) x^\wedge, x^\wedge) \mid x^\wedge \in G_m(\mathbf{C}^n)\}.$$

Given $x^\wedge = x_1 \wedge \cdots \wedge x_m \in G_m(\mathbf{C}^n)$, it may in fact be assumed that the vectors $x_1, \cdots, x_m \in \mathbf{C}^n$ are orthonormal [**4**, p. 1]. Choose, then, a unitary matrix $U \in M_n(\mathbf{C})$ such that

$$U e_k = x_k, \quad k = 1, \cdots, m,$$

where $\{e_1,\cdots,e_n\}$ is the standard orthonormal basis of $\mathbf{C}^n$, and compute that

$$\begin{aligned}(C_m(A)x^\wedge, x^\wedge) &= (C_m(A)C_m(U)e_1\wedge\cdots\wedge e_m, C_m(U)e_1\wedge\cdots\wedge e_m)\\ &= (C_m(U^*AU)e_1\wedge\cdots\wedge e_m, e_1\wedge\cdots\wedge e_m)\\ &= \det(U^*AU)[1,\cdots,m\mid 1,\cdots,m],\end{aligned}$$

where $(U^*AU)[1,\cdots,m\mid 1,\cdots,m]$ indicates the submatrix of U^*AU lying in rows and columns $1,\cdots,m$. In view of (6), this yields yet another formulation of the mth decomposable numerical range: denoting by $U_n(\mathbf{C})$ the multiplicative group of n-square unitary matrices, we have

$$W_m^\wedge(A) = \{\det(U^*AU)[1,\cdots,m\mid 1,\cdots,m]\mid U\in U_n(\mathbf{C})\}. \tag{7}$$

From (6) we obtain

$$W_m^\wedge(A)\subset W(C_m(A)) \tag{8}$$

and hence

$$r_m^\wedge(A)\leqq r(C_m(A)). \tag{9}$$

Strict inequality may hold in (9); e.g., consider

$$A = \begin{bmatrix}0&0&1&0\\0&0&0&1\\0&0&0&0\\0&0&0&0\end{bmatrix}\in M_4(\mathbf{C})$$

with $m=2$ [1].

We define $P_m^\wedge(A)$, the mth *decomposable eigenpolygon of* A, to be the convex polygon in the complex plane spanned by all products of m eigenvalues of A. Thus

$$P_m^\wedge(A) = \mathscr{H}\left(\left\{\prod_{k=1}^m \lambda_{\omega(k)}\,\middle|\,\omega\in Q_{m,n}\right\}\right), \tag{10}$$

where $\lambda_1,\cdots,\lambda_n$ are the eigenvalues of A, $\mathscr{H}$ denotes convex hull, and $Q_{m,n}$ is the set of all strictly increasing sequences of m integers chosen from $\{1,\cdots,n\}$. When $m=1$, $P_1^\wedge(A)$ is simply written $P(A)$ and called the *eigenpolygon of* A. It should be observed that the sets $W_m^\wedge(A)$ and $P_m^\wedge(A)$ are both invariant under transformation of A by a unitary similarity, that is,

$$W_m^\wedge(U^*AU) = W_m^\wedge(A)$$

and

$$P_m^\wedge(U^*AU) = P_m^\wedge(A)$$

for any $U \in U_n(\mathbf{C})$.

PROPOSITION 1. *Let $A \in M_n(\mathbf{C})$ have eigenvalues $\lambda_1, \cdots, \lambda_n$, and let $m \in \{1, \cdots, n\}$. Then*

$$\prod_{k=1}^{m} \lambda_{\omega(k)} \in W_m^\wedge(A),\ \omega \in Q_{m,n}. \tag{11}$$

Moreover, if A is normal then

$$W_m^\wedge(A) \subset P_m^\wedge(A). \tag{12}$$

Proof. Fix $\omega \in Q_{m,n}$. By the Schur triangularization theorem, there exists a matrix $U \in U_n(\mathbf{C})$ such that U^*AU is an upper triangular matrix with first m main diagonal elements $\lambda_{\omega(1)}, \cdots, \lambda_{\omega(m)}$. Then

$$\prod_{k=1}^{m} \lambda_{\omega(k)} = \det(U^*AU)[1, \cdots, m \mid 1, \cdots, m].$$

In view of (7), (11) is established.

Next, assume $A \in M_n(\mathbf{C})$ is normal. Let $\{u_1, \cdots, u_n\}$ be an orthonormal basis of $\mathbf{C}^n$ such that

$$Au_i = \lambda_i u_i, \quad i = 1, \cdots, n.$$

Then

$$\{u_\omega^\wedge = u_{\omega(1)} \wedge \cdots \wedge u_{\omega(m)} \in G_m(\mathbf{C}^n) \mid \omega \in Q_{m,n}\}$$

is an orthonormal basis of $\wedge^m \mathbf{C}^n$ [3, p. 132]. Given $x^\wedge \in G_m(\mathbf{C}^n)$, we have

$$\begin{aligned}(C_m(A)x^\wedge, x^\wedge) &= \Big(C_m(A) \sum_{\omega \in Q_{m,n}} (x^\wedge, u_\omega^\wedge)u_\omega^\wedge, \sum_{\omega \in Q_{m,n}} (x^\wedge, u_\omega^\wedge)u_\omega^\wedge\Big) \\ &= \sum_{\omega \in Q_{m,n}} \left|(x^\wedge, u_\omega^\wedge)\right|^2 \prod_{k=1}^{m} \lambda_{\omega(k)}.\end{aligned} \tag{13}$$

Since

$$\sum_{\omega \in Q_{m,n}} \left|(x^\wedge, u_\omega^\wedge)\right|^2 = \|x^\wedge\|^2 = 1,$$

(13) expresses the element $(C_m(A)x^\wedge, x^\wedge)$ of $W_m^\wedge(A)$ as a convex combination of all products of m eigenvalues of A. This establishes (12).

COROLLARY 1. *Let $A \in M_n(\mathbf{C})$ be normal and $m \in \{1,\cdots,n\}$. Then $r_m^\wedge(A)$ is the maximum modulus of a product of m eigenvalues of A.*

COROLLARY 2. *Let $A = cZ \in M_n(\mathbf{C})$, where $Z \in U_n(\mathbf{C})$ and $c \in \mathbf{C}$, and let $m \in \{1,\cdots,n\}$. Then*

$$r_m^\wedge(UAV) = r_m^\wedge(A)$$

for all $U, V \in U_n(\mathbf{C})$.

3. Some lemmas. In the following discussion let $A \in M_n(\mathbf{C})$ be a fixed matrix, $m \in \{1,\cdots,n\}$ a fixed positive integer, and assume the rank of A is at least m. Denote the singular values of A by $\alpha_1,\cdots,\alpha_n$, arranged so that

$$\alpha_1 \geqq \cdots \geqq \alpha_n \geqq 0,$$

and set

$$D = \operatorname{diag}(\alpha_1,\cdots,\alpha_n) \in M_n(\mathbf{C}).$$

It is well known that there exist matrices $U_1, V_1 \in U_n(\mathbf{C})$ such that

$$A = U_1DV_1.$$

Suppose momentarily that

$$r_m^\wedge(UAV) = r_m^\wedge(A) \tag{14}$$

for all $U, V \in U_n(\mathbf{C})$. Then clearly

$$r_m^\wedge(UDV) = r_m^\wedge(D) \tag{15}$$

for all $U, V \in U_n(\mathbf{C})$:

$$\begin{aligned} r_m^\wedge(UDV) &= r_m^\wedge(UU_1^*AV_1^*V) \\ &= r_m^\wedge(A) \qquad \text{(by (14))} \\ &= r_m^\wedge(U_1^*AV_1^*) \qquad \text{(by (14))} \\ &= r_m^\wedge(D). \end{aligned}$$

Fix $U_0 \in U_n(\mathbf{C})$ and choose $x_0^\wedge \in G_m(\mathbf{C}^n)$ so that

$$r_m^\wedge(U_0D) = |(C_m(U_0D)x_0^\wedge, x_0^\wedge)|. \tag{16}$$

Set

$$y_0^\wedge = C_m(U_0^*)x_0^\wedge \in G_m(\mathbf{C}^n). \tag{17}$$

Let $\{e_1, \cdots, e_n\}$ be the standard orthonormal basis of $\mathbf{C}^n$; then

$$\{e_\omega^\wedge = e_{\omega(1)} \wedge \cdots \wedge e_{\omega(m)} \in G_m(\mathbf{C}^n) \mid \omega \in Q_{m,n}\}$$

is the induced orthonormal basis of $\wedge^m \mathbf{C}^n$. Write

$$x_0^\wedge = \sum_{\omega \in Q_{m,n}} \chi_\omega e_\omega^\wedge, \chi_\omega \in \mathbf{C}, \omega \in Q_{m,n} \tag{18}$$

and

$$y_0^\wedge = \sum_{\omega \in Q_{m,n}} \eta_\omega e_\omega^\wedge, \eta_\omega \in \mathbf{C}, \omega \in Q_{m,n}. \tag{19}$$

LEMMA 1. *Assume*

$$r_m^\wedge(U_0D) = r_m^\wedge(D).$$

Then

$$\alpha_1 \cdots \alpha_m = \alpha_{\omega(1)} \cdots \alpha_{\omega(m)}$$

for every $\omega \in Q_{m,n}$ *for which* $\chi_\omega \neq 0$. *Moreover,*

$$|\chi_\omega| = |\eta_\omega|, \omega \in Q_{m,n}.$$

Proof. Notice that

$$\alpha_1 \cdots \alpha_m > 0$$

since A has rank at least m. We compute

$$\begin{aligned} \alpha_1 \cdots \alpha_m &= r_m^\wedge(D) && \text{(by Corollary 1)} \\ &= r_m^\wedge(U_0D) && \text{(by hypothesis)} \\ &= |(C_m(U_0D)x_0^\wedge, x_0^\wedge)| && \text{(by (16))} \end{aligned}$$

$$
\begin{aligned}
&= \left| \sum_{\omega \in Q_{m,n}} \alpha_\omega \chi_\omega \bar{\eta}_\omega \right| \qquad (\alpha_\omega = \alpha_{\omega(1)} \cdots \alpha_{\omega(m)}) \\
(20) \qquad &\leqq \sum_{\omega \in Q_{m,n}} \alpha_\omega |\chi_\omega| |\eta_\omega| \\
&\leqq \alpha_1 \cdots \alpha_m \sum_{\omega \in Q_{m,n}} |\chi_\omega| |\eta_\omega| \\
&\leqq \alpha_1 \cdots \alpha_m \left(\sum_{\omega \in Q_{m,n}} |\chi_\omega|^2 \right)^{\frac{1}{2}} \left(\sum_{\omega \in Q_{m,n}} |\eta_\omega|^2 \right)^{\frac{1}{2}} \\
&= \alpha_1 \cdots \alpha_m \|x_0^\wedge\| \|y_0^\wedge\| \\
&= \alpha_1 \cdots \alpha_m.
\end{aligned}
$$

The last inequality in (20) is the Cauchy–Schwarz inequality. Since equality holds throughout, $\alpha_1 \cdots \alpha_m > 0$, and $x_0^\wedge, y_0^\wedge \neq 0$, we conclude that

$$|\chi_\omega| = c|\eta_\omega|, \omega \in Q_{m,n}$$

for some $c > 0$. But then $\|x_0^\wedge\| = 1 = \|y_0^\wedge\|$ implies $c = 1$. Thus

$$|\chi_\omega| = |\eta_\omega|, \omega \in Q_{m,n}.$$

It follows from equality in the second inequality in (20) that

$$\alpha_1 \cdots \alpha_m = \alpha_{\omega(1)} \cdots \alpha_{\omega(m)}$$

for every $\omega \in Q_{m,n}$ for which $\chi_\omega \neq 0$.

Suppose now that σ is a permutation in S_n, the symmetric group of degree n, and $U_0^* \in U_n(\mathbf{C})$ is the permutation matrix corresponding to σ:

$$U_0^* = P(\sigma) = [\delta_{i\sigma(j)}].$$

In this situation, continuing with the above notation, we have

$$
\begin{aligned}
y_0^\wedge &= C_m(P(\sigma))x_0^\wedge \\
&= \sum_{\omega \in Q_{m,n}} \chi_\omega C_m(P(\sigma)) e_\omega^\wedge \\
(21) \qquad &= \sum_{\omega \in Q_{m,n}} \chi_\omega e_{\sigma\omega(1)} \wedge \cdots \wedge e_{\sigma\omega(m)} \text{ (since } P(\sigma)e_i = e_{\sigma(i)},\ i = 1, \cdots, n) \\
&= \sum_{\omega \in Q_{m,n}} \epsilon_\omega \chi_\omega e_{\omega\sigma}^\wedge.
\end{aligned}
$$

Here $\omega_\sigma \in Q_{m,n}$ is the strictly increasing rearrangement of the sequence

$$(\sigma\omega(1),\cdots,\sigma\omega(m)),$$

and $\epsilon_\omega = \pm 1$ is the sign of the permutation

$$\begin{pmatrix} \sigma\omega(1)\cdots\sigma\omega(m) \\ \omega_\sigma(1)\cdots\omega_\sigma(m) \end{pmatrix}.$$

The mapping

$$\omega \mapsto \omega_\sigma, \omega \in Q_{m,n}$$

is clearly a bijection of $Q_{m,n}$. Hence from (19) and (21),

$$\begin{aligned} y_0^\wedge &= \sum_{\omega \in Q_{m,n}} \eta_\omega e_\omega^\wedge \\ &= \sum_{\omega \in Q_{m,n}} \eta_{\omega_\sigma} e_{\omega_\sigma}^\wedge \\ &= \sum_{\omega \in Q_{m,n}} \epsilon_\omega \chi_\omega e_{\omega_\sigma}^\wedge \end{aligned}$$

so that

$$\eta_{\omega_\sigma} = \epsilon_\omega \chi_\omega, \omega \in Q_{m,n}. \tag{22}$$

LEMMA 2. *Assume*

$$r_m^\wedge(P(\sigma)^T D) = r_m^\wedge(D).$$

Then

$$\alpha_1\cdots\alpha_m = \alpha_{\omega_\sigma(1)}\cdots\alpha_{\omega_\sigma(m)}$$

for every $\omega \in Q_{m,n}$ for which $\chi_{\omega_\sigma} \neq 0$. Moreover,

$$|\chi_{\omega_\sigma}| = |\chi_\omega|, \omega \in Q_{m,n}.$$

Proof. The first assertion is immediate from Lemma 1, as is the second:

$$\begin{aligned} |\chi_{\omega_\sigma}| &= |\eta_{\omega_\sigma}| \\ &= |\epsilon_\omega \chi_\omega| \qquad \text{(by (22))} \\ &= |\chi_\omega|, \omega \in Q_{m,n}. \end{aligned}$$

4. The main result.

THEOREM 1. *Let $A \in M_n(\mathbf{C})$ and let m be a positive integer, $1 \leqq m < n$. Assume the rank of A is at least m. Then*

$$r_m^{\wedge}(UAV) = r_m^{\wedge}(A) \tag{23}$$

for all $U, V \in U_n(\mathbf{C})$ if and only if A is a scalar multiple of a unitary matrix.

Proof. We have observed in Corollary 2 that the condition is sufficient.

To see that the condition is necessary, assume (23) holds for all $U, V \in U_n(\mathbf{C})$. Since there exist matrices $U_1, V_1 \in U_n(\mathbf{C})$ such that

$$A = U_1 D V_1,$$

where

$$D = \operatorname{diag}(\alpha_1, \cdots, \alpha_n) \in M_n(\mathbf{C})$$

and

$$\alpha_1 \geqq \cdots \geqq \alpha_n$$

are the singular values of A, it suffices to show that

$$\alpha_1 = \alpha_n.$$

Consider the full cycle

$$\varphi = (1\,2 \cdots n) \in S_n.$$

Choose $x_0^{\wedge} \in G_m(\mathbf{C}^n)$ so that

$$r_m^{\wedge}(P(\varphi)^T D) = |(C_m(P(\varphi)^T D) x_0^{\wedge}, x_0^{\wedge})|$$

and write

$$x_0^{\wedge} = \sum_{\omega \in Q_{m,n}} \chi_\omega e_\omega^{\wedge}, \chi_\omega \in \mathbf{C}, \omega \in Q_{m,n}.$$

Since

$$\sum_{\omega \in Q_{m,n}} |\chi_\omega|^2 = \|x_0^{\wedge}\|^2 = 1,$$

there exists $\omega \in Q_{m,n}$ for which

(24) $$\chi_\omega \neq 0.$$

Set

(25) $$\gamma = \omega_{\varphi^{n-\omega(1)+1}} \in Q_{m,n}.$$

By (15) and Lemma 2 (with $\sigma = \varphi^{n-\omega(1)+1}$), $|\chi_\gamma| = |\chi_\omega|$ and hence by (24)

(26) $$\chi_\gamma \neq 0.$$

Also observe that

$$\begin{aligned}\varphi^{n-\omega(1)+1}\omega(1) &= \varphi(\omega(1)+n-\omega(1))\\ &= \varphi(n)\\ &= 1\end{aligned}$$

implies $\omega_{\varphi^{n-\omega(1)+1}}(1) = 1$, i.e.,

$$\gamma(1) = 1.$$

The argument now splits into two cases.

Case I. $\gamma(m) < n$. Apply the permutation $\varphi^{n-\gamma(m)}$ to

$$\gamma = (1, \gamma(2), \cdots, \gamma(m))$$

to obtain

(27) $$\begin{aligned}\varphi^{n-\gamma(m)}\gamma &= (1+n-\gamma(m), \gamma(2)+n-\gamma(m), \cdots, \gamma(m-1)+n-\gamma(m), n)\\ &= \gamma_{\varphi^{n-\gamma(m)}}\end{aligned}$$

Since $\gamma(m) < n$, we have

$$\begin{aligned}2 &\leqq 1+n-\gamma(m),\\ 3 &\leqq \gamma(2)+n-\gamma(m),\\ &\vdots\\ m &\leqq \gamma(m-1)+n-\gamma(m).\end{aligned}$$

Therefore

$$\alpha_2\alpha_3\cdots\alpha_m \geqq \alpha_{1+n-\gamma(m)}\alpha_{\gamma(2)+n-\gamma(m)}\cdots\alpha_{\gamma(m-1)+n-\gamma(m)}. \tag{28}$$

By (15) and Lemma 2 (with $\sigma = \varphi^{n-\gamma(m)}$), $|\chi_{\gamma_{\varphi^{n-\gamma(m)}}}| = |\chi_\gamma|$ and hence by (26)

$$\chi_{\gamma_{\varphi^{n-\gamma(m)}}} \neq 0.$$

Then Lemma 2 together with (27) implies

$$\alpha_1\alpha_2\alpha_3\cdots\alpha_m = \alpha_{1+n-\gamma(m)}\alpha_{\gamma(2)+n-\gamma(m)}\cdots\alpha_{\gamma(m-1)+n-\gamma(m)}\alpha_n. \tag{29}$$

Since $\alpha_1\cdots\alpha_m > 0$ (A has rank at least m), it follows from (28) and (29) that

$$\alpha_1 = \alpha_n.$$

Case II. $\gamma(m) = n$. In this case

$$\gamma = (1, \gamma(2), \cdots, \gamma(m-1), n).$$

Now $m < n$ by hypothesis, so there exists a least positive integer $k \in \{2, \cdots, m\}$ such that

$$k < \gamma(k).$$

Apply the permutation φ^{1-k} to

$$\gamma = (1, \cdots, k-1, \gamma(k), \cdots, \gamma(m-1), n)$$

to obtain

$$\varphi^{1-k}\gamma = (n-k+2, n-k+3, \cdots, n-1, n, \gamma(k)-k+1, \cdots, \\ \gamma(m-1)-k+1, n-k+1).$$

Then

$$\gamma_{\varphi^{1-k}} = (\gamma(k)-k+1, \cdots, \gamma(m-1)-k+1, n-k+1, n-k+2, \\ n-k+3, \cdots, n-1, n). \tag{30}$$

Since $k < \gamma(k)$, we have

$$
\begin{aligned}
2 &\leqq \gamma(k) - k + 1, \\
3 &\leqq \gamma(k+1) - k + 1, \\
&\vdots \\
m - k + 1 &\leqq \gamma(m-1) - k + 1, \\
m - k + 2 &\leqq n - k + 1, \\
m - k + 3 &\leqq n - k + 2, \\
&\vdots \\
m &\leqq n - 1.
\end{aligned}
$$

Therefore

$$\alpha_2\alpha_3 \cdots \alpha_m \geqq \alpha_{\gamma(k)-k+1}\alpha_{\gamma(k+1)-k+1} \cdots \alpha_{n-1}. \tag{31}$$

By (15) and Lemma 2 (with $\sigma = \varphi^{1-k}$), $|\chi_{\gamma_{\varphi^{1-k}}}| = |\chi_\gamma|$ and hence by (26)

$$\chi_{\gamma_{\varphi^{1-k}}} \neq 0.$$

Then Lemma 2 together with (30) implies

$$\alpha_1\alpha_2\alpha_3 \cdots \alpha_m = \alpha_{\gamma(k)-k+1}\alpha_{\gamma(k+1)-k+1} \cdots \alpha_{n-1}\alpha_n. \tag{32}$$

Once again, since $\alpha_1 \cdots \alpha_m > 0$ it follows from (31) and (32) that

$$\alpha_1 = \alpha_n.$$

This completes the proof.

We remark that the restriction $m \neq n$ in Theorem 1 is inevitable. Indeed, for any matrix $A \in M_n(\mathbf{C})$,

$$
\begin{aligned}
r_n^\wedge(A) &= |\det(A)| \\
&= |\det(UAV)| \\
&= r_n^\wedge(UAV)
\end{aligned}
$$

for all $U, V \in U_n(\mathbf{C})$. The hypothesis that A have rank at least m is equally essential, since any matrix $A \in M_n(\mathbf{C})$ of rank less than m satisfies

$$r_m^\wedge(A) = 0 = r_m^\wedge(UAV)$$

for all $U, V \in U_n(\mathbf{C})$.

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