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RATIONAL HOMOTOPY AND UNIQUE FACTORIZATION

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Decompositions of various kinds of mathematical objects as products or coproducts are considered, and the uniqueness of these decompositions is discussed. For instance, the following topological result is proved. Let Δ be the set of formal, simply-connected, rational homotopy types having finitely generated rational homotopy. Of course, Δ is a commutative semigroup with respect to the usual product space construction. Then Δ is a *free* commutative semigroup.

This is a statement about "unique factorization" in Δ, and it follows from our main result (Theorem 4) concerning the unique factorization of certain differential graded Q-algebras called formal minimal algebras.

These results are reminiscent of the Krull–Schmidt Theorem (cf. [7], page 58), which is a unique factorization result for suitable classes of "M-groups". However, the Krull–Schmidt Theorem discusses (categorical) products, while our algebraic results are concerned with tensor product decompositions where the tensor product is the (categorical) coproduct.

In contrast to Corollary 8, negative results in finer topological contexts are obtained from the interesting noncancellation example of Hilton and Roitberg [5]: E is a compact, simply-connected manifold such that $S^3 \times \mathrm{Sp}(2)$ and $S^3 \times E$ are diffeomorphic; however, $\mathrm{Sp}(2)$ and E have distinct homotopy types. Thus, differentiable manifolds, topological spaces, and homotopy types each fail to satisfy the unique factorization property (cf. definition just before Theorem 2).

Several of the proofs in this paper require a discussion of rational homotopy theory, in the form of Sullivan's theory of "minimal models". A demonstration of the scope and depth of this beautiful theory may be found in [3] and [9], while [4] is a clear, self-contained introduction to this view of rational homotopy theory.

2. Splittings of minimal algebras. For the purpose of this paper, the term "minimal algebra" is restricted to the following (somewhat limited) definition (cf. [4]): A minimal algebra M consists of a simply-connected, free, associative, graded-commutative Q-algebra M, together with a degree one derivation $d\colon M \to M$ (the differential) such that:

(i) $d \cdot d = 0$.

(ii) $d(M) \subset D(M) =$ the ideal of decomposable elements of M.

(iii) $H^*(M)$, the cohomology of M (with respect to d) is a finitely generated graded Q-algebra.

If $f\colon M \to N$ is a map of differential graded Q-algebras (a degree preserving algebra homomorphism which commutes with differentials), then $f^*\colon H^*(M) \to H^*(N)$ will denote the map of graded Q-algebras induced on cohomology. Let M^+ be the ideal of elements of positive degree, let $D(M) = M^+ \wedge M^+$, the ideal of decomposables, and define $\pi(M) = M^+/D(M)$, the graded vector space of indecomposables of M (cf. Theorem 3.12 of [4]). Define $f^*\colon \pi(M) \to \pi(N)$ to be the graded vector space homomorphism induced by $f\colon M \to N$.

End(M) will denote the set of all differential graded Q-algebra (D.G.A.) endomorphisms of M. End(M) is a semigroup under composition and Aut(M) will denote the group of invertible endomorphisms (i.e. automorphisms) of M. Thus, Aut(M) acts on End(M) by conjugation, producing the action of a transformation group. Recall that the (graded) tensor product, $\otimes$ is the coproduct in the category of minimal algebras. This category has an obvious zero object [8] and the zero endomorphisms are denoted by 0. A useful characterization of the coproduct for minimal algebras follows.

PROPOSITION 1. *If* $\{e_1, \cdots, e_n\} \subset \mathrm{End}(M)$ *satisfying*

(i) $e_k \cdot e_k = e_k$ *for all* $k = 1, \cdots, n$

(ii) $e_k \cdot e_l = 0$ *if* $k \neq l$

(iii) $\Sigma_{k=1}^n e_k^* =$ *identity,*

then $M \cong \bigotimes_{k=1}^n M_k$, *on* $\pi(M)$ *where the minimal algebra* $M_k = e_k(M)$.

The converse of this proposition is an obvious remark; the proof of Proposition 1 is sufficiently routine to require only an outline here.

Proof. For each *D.G.*-subalgebra $M_k = e_k(M)$,

$$D(M_k) = M_k \cap D(M) \tag{1}$$

(because e_k is an idempotent endomorphism). Of course, (1) is equivalent to the requirement that $i_k\colon M_k \hookrightarrow M$, the inclusion, induces a monomorphism $i_k^*\colon \pi(M_k) \to \pi(M)$ on the graded vector spaces of indecomposables. In fact, the monomorphisms give a direct sum (coproduct) decomposition

$$\pi(M) \cong \bigoplus_{k=1}^n \pi(M_k). \tag{2}$$

$d(M_k) \subset M_k \cap d(M) \subset M_k \cap D(M) = D(M_k)$, and so M_k has a decomposable differential.

Choose $\{m_{kj} \mid j \in R_k\} \subset M_k$ such that $\{\bar{m}_{kj} \mid j \in R_k\}$ is a basis for $\pi(M_k)$, where $\bar{m}_{kj} = m_{kj} + D(M_k)$. Let $\hat{m}_{kj} = i_k^*(\bar{m}_{kj}) = m_{kj} + D(M)$. By (2), $\{\hat{m}_{kj} \mid j \in R_k; k = 1, \cdots, n\}$ is then a basis for $\pi(M)$. Now observe that M is the free algebra generated by the set $\{m_{kj} \mid j \in R_k; k = 1, \cdots, n\}$. Therefore, M_k is the free subalgebra generated by $\{m_{kj} \mid j \in R_k\}$, and so M_k is a minimal algebra. Moreover, it is now easy to see that the obvious D.G.A. map $\bigotimes_{k=1}^n M_k \to M$ is an isomorphism of minimal algebras.

Regrettably, the following definitions are required in order to state the main result of this section (Theorem 2).

A minimal algebra is *irreducible*, if it is neither isomorphic to a tensor product of two nontrivial minimal algebras, nor isomorphic to the trivial minimal algebra. Let M be a minimal algebra. A set of endomorphisms, $\{e_1, \cdots, e_n\} \subset \mathrm{End}(M)$ satisfying conditions (i), (ii) and (iii) of Proposition 1, will be called a *splitting* of M. A splitting will be called *irreducible*, if each $M_k = \mathrm{Image}(e_k)$ is irreducible ($k = 1, \cdots, n$).

Observe that if $\alpha \in \mathrm{Aut}(M)$ and $\{e_1, \cdots, e_n\} \subset \mathrm{End}(M)$ is a splitting of M, then conjugation by α gives another splitting of $M, \{\alpha \cdot e_1 \cdot \alpha^{-1}, \cdots, \alpha \cdot e_n \cdot \alpha^{-1}\}$, equivalent to the first splitting, in the sense that $\mathrm{Image}(\alpha \cdot e_k \cdot \alpha^{-1})$ is isomorphic to $\mathrm{Image}(e_k)$, for each k.

Two splittings of $M, \{e_1, \cdots, e_n\}$ and $\{f_1, \cdots, f_m\}$ will be said to *commute*, if $e_i \cdot f_j = f_j \cdot e_i$ for all $i = 1, \cdots, n$ and $j = 1, \cdots, m$. Two splittings will be called *compatible*, if a conjugate of one commutes with the other. A minimal algebra M is called *flexible*, if any two irreducible splittings of M are compatible. A category of minimal algebras will be termed *flexible*, if each of its objects is flexible.

A category of minimal algebras Γ will be called *productive*, whenever the following two conditions are satisfied.

(i) $M \in \mathrm{obj}(\Gamma)$ and $M \cong N \Rightarrow N \in \mathrm{obj}(\Gamma)$.

(ii) $M, N \in \mathrm{obj}(\Gamma) \Leftrightarrow M \otimes N \in \mathrm{obj}(\Gamma)$.

A category will be said to satisfy the *unique factorization* (respectively, *cofactorization*) *property*, if the set of equivalence classes of objects is a free, commutative semigroup, under the binary operation induced by the categorical product (respectively, coproduct). In other words, a productive category of minimal algebras Γ satisfies the *unique cofactorization property* if each object of Γ is isormorphic to the tensor product of a uniquely determined, finite set of nontrivial, irreducible objects of Γ (unique, up to isomorphism).

THEOREM 2. *Flexible, productive categories of minimal algebras satisfy the unique cofactorization property.*

Proof. Assume that Γ is a flexible, productive category of minimal algebras. The existence of tensor product factorizations into irreducible factors is obvious, and so it suffices to prove "uniqueness".

Suppose M_k, $N_l \in \mathrm{obj}(\Gamma)$, $(k = 1, \cdots, m$ and $l = 1, \cdots, n)$ are irreducible factors in $M \cong \bigotimes_{k=1}^m M_k \cong \bigotimes_{l=1}^n N_l$. This situation produces two nontrivial, irreducible splittings of M, say $\{e_1, \cdots, e_m\}$ and $\{f_1, \cdots, f_n\}$, with $M_k \cong e_k(M)$ and $N_l \cong f_l(M)$. Because these splittings are compatible, it is possible to assume, without loss of generality, that they commute.

Define $e_{k,l}: M_k \to M_k$ by $e_{k,l} = e_k \cdot f_l \cdot i_k$, where $M_k = e_k(M)$ and $i_k: M_k \hookrightarrow M$. Similarly, define $f_{l,k}: N_l \to N_l$ by $f_{l,k} = f_l \cdot e_k \cdot j_l$, where $N_l = f_l(M)$ and $j_l: N_l \hookrightarrow M$. Next, apply Proposition 1, because $\{e_{k,h} \mid h = 1, \cdots, n\}$ and $\{f_{l,p} \mid p = 1, \cdots, m\}$ are splittings of the irreducible minimal algebras M_k and N_l, respectively. Therefore, for each $k = 1, \cdots, m$, there is an integer $\phi(k), 1 \leqq \phi(k) \leqq n$, such that $e_{k,\phi(k)}$ is an automorphism and $e_{k,l}$ is trivial for $l \neq \phi(k)$. Similarly, for each $l = 1, \cdots, n$, we have an integer $\psi(l), 1 \leqq \psi(l) \leqq m$, such that $f_{l,\psi(l)}$ is an automorphism and $f_{l,k}$ is trivial for $k \neq \psi(l)$. Now, an easy argument shows that $\psi = \phi^{-1}, m = n$, and $M_k \cong N_{\phi(k)}$.

3. Formal rational homotopy types. A minimal algebra M is called *formal*, if and only if there exists a graded algebra map $\phi: M \to H^*(M)$ extending the canonical algebra epimorphism $Z^*(M) \to H^*(M)$ from cocycles to cohomology (cf. §4 of [3]). If $H^*(M)$ is viewed as a differential graded algebra, with differential $d = 0$, then M is the *minimal model* of $H^*(M)$ (by the above definition), because the D.G.A. map $\phi: M \to H^*(M)$ induces an isomorphism on cohomology. By the uniqueness and existence result for minimal models ([4], Theorem 2.5, or [3], Theorem 1.1) we have the following:

LEMMA 3. *Cohomology is a bijection from the set of (isomorphism classes of) formal minimal algebras to the set of (isomorphism classes of) all simply-connected, associative, graded-commutative, finitely generated Q-algebras. Moreover, this bijection preserves tensor products [Kunneth relation].*

The main result in this section is:

THEOREM 4. *The category of formal, finitely generated minimal algebras satisfies the unique cofactorization property.*

(Theorem 4 will follow from Theorem 2 and from Propositions 5 and 7 below.)

PROPOSITION 5. *The category of formal minimal algebras is productive.*

Proof. If M and N are formal, then there exist $D.G.A.$ maps $\sigma: M \to H^*(M)$ and $\tau: N \to H^*(N)$ which induce isomorphisms on cohomology. Moreover, $\sigma \otimes \tau: M \otimes N \to H^*(M) \otimes H^*(N)$ is a D.G.A. map, which induces an epimorphism on cohomology. This induced epimorphism must be an isomorphism, because the Kunneth relation tells us that $H^*(M \otimes N) \cong H^*(M) \otimes H^*(N)$. Thus, $M \otimes N$ is formal, when M and N are formal.

Conversely, suppose $M \otimes N$ is formal. Then there exists an algebra map

$$\phi: M \otimes N \to H^*(M \otimes N)$$

which extends the canonical map

$$Z^*(M \otimes N) \to H^*(M \otimes N).$$

Let $i: M \to M \otimes N$ and $p: M \otimes N \to M$ be the usual inclusion and projection, respectively. Define $\psi: M \to H^*(M)$ by the commutative square:

$$\begin{array}{ccc} M & \xrightarrow{\psi} & H^*(M) \\ {\scriptstyle i}\downarrow & & \uparrow{\scriptstyle p^*} \\ M \otimes N & \xrightarrow[\phi]{} & H^*(M \otimes N) \end{array}$$

An easy diagram chase shows that the algebra map $\psi: M \to H^*(M)$ extends the canonical map $Z^*(M) \to H^*(M)$.

Finally, a second characterization of formality will be needed.

DEFINITION. If each degree component of a minimal algebra M decomposes as a direct sum of sub vector spaces, $M^k = \bigoplus_{n \in Z} {}_nM^k$, for each gradation $k \geqq 0$ (N.B. n may take any integer values), such that $d({}_nM^k) \subset {}_nM^{k+1}$ and ${}_nM^k \wedge {}_mM^l \subset {}_{n+m}M^{k+l}$, then it will be said that the elements of $\bigoplus_{k \geq 0} {}_nM^k = {}_nM$ are *of weight n* in this *weight decomposition* $\bigoplus_{n \in Z} {}_nM$ of M.

Observe that a weight decomposition on M makes $H^*(M) = \bigoplus_{n \in Z} H^*({}_nM)$, the cohomology algebra, a bigraded algebra.

LEMMA 6. *A minimal algebra M is formal, if and only if it possesses a weight decomposition for which*

(3) $$H^p({}_nM) = 0 \quad \text{if} \quad p \neq n.$$

Moreover, in this case,

(4) $${}_nM^p = 0, \quad \text{if} \quad p > n.$$

(*cf. Theorem* 4.1 *in* **[3]**.)

Proof. The "if" part is proved by inductively showing that (3) implies (4), and then observing that the algebra projection $M \to \bigoplus_{n \geq 0} {}_nM^n \to H^*(M)$ extends the canonical projection $Z^*(M) \to H^*(M)$.

Conversely, if M is formal, then the homomorphism of groups

$$\begin{array}{rcl} H^*\colon \mathrm{Aut}(M) & \to & \mathrm{Aut}(H^*(M)) \\ f & \to & H^*(f) = f^* \end{array}$$

is an epimorphism (by Theorem 2.13 of **[4]**). Consider the grading automorphism $\alpha_t \in \mathrm{Aut}(H^*(M))$, where $\alpha_t(x) = t^n \cdot x, x \in H^n(M)$ and t is any nonzero rational number. Since M is formal, there exists an automorphism $f \in \mathrm{Aut}(M)$ for which $f^* = \alpha_t$. Moreover, f may be chosen to be semisimple (by the remarks immediately following Lemma B on page 96 of **[6]**, f may be chosen to be the semisimple Jordan part of any lifting of α_t into $\mathrm{Aut}(M)$). An induction argument (on the Postnikov sections of M) shows that if f is semisimple and $f^* = \alpha_t$, then f is diagonalizable, with eigenvalues t^n, for various integers n. If $t \neq 1$, then the eigenspaces of f in M^k give a direct sum decomposition

$$M^k = \bigoplus_{n \in Z} {}_nM^k,$$

where ${}_nM^k$ is the eigenspace, with eigenvalue t^n. Moreover, this is a weight decomposition of M satisfying condition (3).

4. Formal, finitely generated, minimal algebras are flexible. If M is formal, with weight decomposition $M^k = \bigoplus_{n \geq k} {}_nM^k$ (by (4)), and $t \in Q^* = \{$all nonzero rational numbers$\}$, then define $\alpha_t \in \mathrm{Aut}(M)$ by setting $\alpha_t(x) = t^n \cdot x$, for $x \in {}_nM^k$ (for x of weight n). Moreover, $\alpha_s \cdot \alpha_t = \alpha_{st} = \alpha_t \cdot \alpha_s$.

In fact, if M is finitely generated, then $\mathrm{End}(M)$ is an affine variety, $\mathrm{Aut}(M)$ is an algebraic group, as well as a subspace of $\mathrm{End}(M)$ with the Zariski topology, and $\mathrm{Aut}(M)$ acts morphically on $\mathrm{End}(M)$, by conjugation (cf. §8.2 of **[6]**; also Proposition M.7 and §A in **[9]**); all defined over the rational field Q.

Suppose S is a subset of a topological space T, and suppose $x \in T$ is the only accumulation point of S which does not lie in S. In this case, we will simply say that *S converges to x in T.*

In case M is formal, $\alpha: Q^* \to \mathrm{Aut}(M)$ $(t \to \alpha_t)$ has been constructed as a one parameter multiplicative subgroup of $\mathrm{Aut}(M)$ converging to 0 in $\mathrm{End}(M)$ in the Zariski topology (cf. **[6]**, page 103). (Alternatively, see Theorem K.2 of **[9]** and Theorems 1 and 2 of **[1]**).

At last we are ready to prove:

PROPOSITION 7. *Formal, finitely generated, minimal algebras are flexible.*

Proof. Suppose M is formal and $\{e_1, \cdots, e_n\}$ is a splitting of M, with $e_k(M) = M_k$. Each M_k is formal, by Proposition 5, and there exist one parameter subgroups (for each $k = 1, \cdots, n$)

$$\begin{aligned} \alpha^k : Q^* &\to \mathrm{Aut}(M_k) \\ t &\to \alpha^k_t \end{aligned}$$

converging to the basepoint endomorphism $0 \in \mathrm{End}(M_k)$.

Let $\phi_k(t) \in \mathrm{Aut}(M)$ be defined by

$$\phi_k(t) = \alpha^1_t \otimes \cdots \otimes \alpha^{k-1}_t \otimes 1 \otimes \alpha^{k+1}_t \otimes \cdots \otimes \alpha^n_t$$

for $t \in Q^*$. Then $\phi_k(t) \cdot \phi_l(s) = \phi_l(s) \cdot \phi_k(t)$ for all $s, t \in Q^*$ and $k, l = 1, \cdots, n$. Moreover, $\{\phi_k(t) \mid k = 1, \cdots, n; t \in Q^*\}$ is contained in a maximal Q-split torus in $\mathrm{Aut}(M)$ (see §34.3 in **[6]**), and $\phi_k(Q^*)$ converges to e_k in $\mathrm{End}(M)$. Thus, we know that each splitting $\{e_1, \cdots, e_n\}$ of a formal minimal algebra M lies in the Zariski closure of a maximal Q-split torus T in $\mathrm{Aut}(M)$, $\{e_1, \cdots, e_n\} \subset \bar{T} \subset \mathrm{End}(M)$.

However, any two maximal Q-split tori in $\mathrm{Aut}(M)$ are conjugate (**[2]**, Theorem 15.9). Thus, given any two splittings of M, $\{e_1, \cdots, e_n\}$ and $\{f_1, \cdots, f_m\}$, there exists $\beta \in \mathrm{Aut}(M)$ and a maximal Q-split torus $T \subset \mathrm{Aut}(M)$, such that $\{e_1, \cdots, e_n, \beta \cdot f_1 \cdot \beta^{-1}, \cdots, \beta \cdot f_m \cdot \beta^{-1}\} \subset \bar{T} \subset \mathrm{End}(M)$. Now, $\{e_1, \cdots, e_n\}$ and $\{\beta \cdot f_1 \cdot \beta^{-1}, \cdots, \beta \cdot f_m \cdot \beta^{-1}\}$ commute, since the Zariski closure of a commutative set in $\mathrm{End}(M)$ is again commutative.

5. An application to topology. Finally, the topological interpretation claimed is recorded.

COROLLARY 8. *The unique factorization property is satisfied by the rational homotopy category of formal, simply-connected topological spaces with finitely generated rational homotopy.*

Proof. This follows from Theorem 4 by Theorem 3.3 of **[3]**.

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