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COHOMOLOGY OF DEGREE 1 AND 2 OF THE SUZUKI GROUPS

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Let V be the standard 4-dimensional module for $Sz(q)$, the Suzuki group based on the field of $q = 2^{2n+1}$ elements. In this paper we determine $H^2(Sz(q), V)$. This is usually ($q \geqq 32$) of dimension one (otherwise zero) and is generated by a cocycle which is the restriction of a generator of $H^2(Sp_4(q), V)$. In addition, the well known groups $H^2(Sz(q), GF(q))$ and $H^1(Sz(q), V)$ are calculated. The proof involves the use of the Hochschild–Serre spectral sequence to determine the cohomology of the normalizer of a Sylow 2-subgroup acting on the various one-dimensional modules involved.

Let $K = GF(q)$, $q = 2^{2n+1}$, let $Sz(q)$ (${}^2B_2(q)$) be the Suzuki group based on the field K and let B be a normalizer of a Sylow 2-subgroup of $Sz(q)$. In this paper we use the Hochschild–Serre spectral sequence to determine $H^i(B, V)$ $i = 1, 2$, where V is a one dimensional KB-module, in terms of the solutions to certain equations in $\mathrm{End}(K^*)$. These equations are solved when V is trivial or involved in K^4, the standard four dimensional module for $KSz(q)$. Using this information we determine $H^2(Sz(q), K^4)$ as well as the previously known groups $H^2(Sz(q), K)$ and $H^1(Sz(q), K^4)$. These may be viewed as results concerning conjugacy classes in semi-direct products and concerning exact sequences of groups using the well known group-theoretic interpretation of cohomology of degree 1 and 2 [6].

We will assume all cocycles are normalized, i.e. vanish when any one of their arguments is the identity. When $[f] \in H^2(G, V)$, where G is a group and V is a left G-module, let $E(f)$ denote the extension of V by G using f, that is, $E(f) = \{(v, g) \mid v \in V, g \in G\}$ with multiplication $(v_1, g_1)(v_2, g_2) = (v_1 + g_1(v_2) + f(g_1, g_2), g_1 g_2)$.

We use the explicit description of $Sz(q)$ given in [9]. Let K_0 be the prime subfield of K, $\Gamma = \mathrm{Gal}(K/K_0)$ and $\theta \in \Gamma$ defined by $\theta: x \to x^{2^n}$. For $\alpha, u \in K$ and $t \in K^*$ put

$$(\alpha, u) = \begin{bmatrix} 1 & u^\theta & h & g \\ & 1 & u & \alpha \\ & & 1 & u^\theta \\ & & & 1 \end{bmatrix}, T(t) = \begin{bmatrix} t^\theta & & & \\ & t^{1-\theta} & & \\ & & t^{\theta-1} & \\ & & & t^{-\theta} \end{bmatrix}, \quad J = \begin{bmatrix} & & & 1 \\ & & 1 & \\ & 1 & & \\ 1 & & & \end{bmatrix}$$

where $h = h(\alpha, u) = u^{\theta+1} + \alpha$ and $g = g(\alpha, u) = u^{2\theta+1} + u^{\theta}\alpha + \alpha^{2\theta}$. Set $U = \{(\alpha, u) \mid \alpha, u \in K\}$, $T = \{T(t) \mid t \in K^*\}$, $B = UT$ so $Sz(q) = \langle B, J\rangle \subset SL_4(q)$ (in **[9]**, U^J is used in place of U). Then K^4 (columns) is the standard module on which $Sz(q)$ acts as multiplication on the left. In fact $Sz(q)$ is contained in the Symplectic group defined by J.

Since U is a Sylow 2-subgroup of $Sz(q)$ which is a T. I. set with normalizer B, the Cartan–Eilenberg stability theorem tells us that if V is a $KSz(q)$-module then the restriction maps $H^i(Sz(q), V) \to H^i(B, V) \to H^i(U, V)^T$ are isomorphisms for $i > 0$. Thus (after the case $q = 2$) we shall replace $Sz(q)$ by B. Furthermore these isomorphisms show that when giving explicit cocycles it is sufficient to give their restrictions to U and show they are T-stable.

Assume first $q = 2$. Then $Sz(q)$ is a group of order 20. Its Sylow 5-subgroup is cyclic, normal and a generator acts fixed-point-freely on K^4. This implies $H^i(Sz(2), K^4) = 0$ for $i > 0$ **[7]**. Henceforth we assume $q \geqq 8$.

Throughout we assume α, β, u, $v \in K$ and $t \in K^*$. We identify T with K^* via $T(t) \leftrightarrow t$. It is seen that $(\alpha, u)(\beta, v) = (\alpha + \beta + uv^{\theta}, u + v)$ and $(\alpha, u)^{T(t)} = T(t)(\alpha, u)T(t)^{-1} = (t\alpha, t^{\theta'}u)$ where $\theta' = 2 - 2\theta$. Also $Z = \{(\alpha, 0)\}$ is the center and derived subgroup of U. Set $A = U/Z$ and $X = B/Z$ so X is the semidirect product AT.

When V is a KT-module and $\nu \in \mathrm{End}(K^*)$ we say T acts with weight ν on V provided $T(t)v = t^{\nu}v$ for all $t \in K^*$, $v \in V$. The above formulas show Z and A are KT-modules of weight 1 and θ' respectively. Observe $\mathrm{End}(K^*) \simeq \mathbf{Z}/(q-1)\mathbf{Z}$ and so is a commutative ring.

When V and W are (finite dimensional) K-modules $\mathrm{Hom}(W, V) = \bigoplus_{\sigma \in \Gamma} H_{\sigma}(W, V)$ where $H_{\sigma}(W, V)$ are the σ-semilinear maps from W to V. If additionally V and W are KT-modules of weight ν and ω then $H_{\sigma}(W, V)$ is a KT-module of weight $\nu - \omega\sigma$.

Now fix V, a one dimensional KB-module on which U acts trivially and T acts with weight ν. We shall often identify V with K. From the (nonsplit) exact sequence of groups $1 \to Z \to B \xrightarrow{\pi} X \to 1$ the Hochschild–Serre spectral sequence gives us the exact sequences of K-modules

$$0 \to H^2(B, V)_0 \to H^2(B, V) \xrightarrow{\mathrm{Res}} H^2(Z, V)^X$$

$$(*) \qquad 0 \to H^1(X, V) \to H^1(B, V) \to H^1(Z, V)^X \to H^2(X, V)$$

$$\to H^2(B, V)_0 \xrightarrow{\Phi} H^1(X, H^1(Z, V)) \to H^3(X, V).$$

Our aim is to determine $H^2(B, V)$. In Lemmas 1, 2 and 3 we determine most of the other terms in $(*)$ and study the maps Res and Φ.

LEMMA 1. *Let W and V* (each identified with K) *be one dimensional KT-modules of weight ω and ν respectively and regard V as a trivial W-module. For $\sigma, \tau \in \Gamma$ define $h_\sigma: W \to V$ by $h_\sigma(w) = w^\sigma$ and $f_{(\sigma,\tau)}: W \times W \to V$ by $f_{(\sigma,\tau)}: (w_1, w_2) \to w_1^\sigma w_2^\tau$.*

(a) $\{[h_\sigma] \mid \nu = \omega\sigma\}$ $\sigma \in \Gamma$ *is a K-base for $H^1(W, V)^T$.*

(b) $\{[f_{(\sigma,\tau)}] \mid \nu = \omega(\sigma + \tau)\}\{\sigma, \tau\} \subseteq \Gamma$ *is a K-base for $H^2(W, V)^T$.*

Proof. (a) This statement is immediate since $H^1(W, V)^T = \mathrm{Hom}(W, V)^T \simeq \bigoplus H_\sigma(W, V)^T$ and T acts on $H_\sigma(W, V) = Kh_\sigma$ with weight $\nu - \omega\sigma$.

(b) Since W is abelian and trivial on V we have an exact sequence of KT-modules $0 \to H^2_{ab}(W, V) \to H^2(W, V) \xrightarrow{\Psi} \mathrm{Alt}^2(W, V) \to 0$ where $\mathrm{Alt}^2(W, V)$ is the group of alternate 2-forms: $W \times W \to V$ and $\Psi[f]: (w_1, w_2) \to f(w_1, w_2) - f(w_2, w_1)$. Furthermore $H^2_{ab}(W, V) \simeq \mathrm{Hom}(W, V)$. See [7] for the proofs of these statements. Taking T-cohomology of the above sequence gives the exact sequence of K-modules $0 \to \mathrm{Hom}(W, V)^T \to H^2(W, V)^T \to \mathrm{Alt}^2(W, V)^T \to 0 = H^1(T, \mathrm{Hom}(W, V))$. We have seen $\dim_K \mathrm{Hom}(W, V)^T = \#\{\sigma \in \Gamma \mid \nu = \omega\sigma\}$ and it can be seen that when $\nu = \omega\sigma$ then $f_{(\sigma/2,\sigma/2)}$ is a corresponding cocycle in $H^2_{ab}(W, V)^T \simeq \mathrm{Hom}(W, V)^T$.

In [5] it is shown that $\mathrm{Alt}^2(W, V) = \bigoplus KF_{\{\sigma,\tau\}}$ where we sum over all sets $\{\sigma, \tau\} \subseteq \Gamma$, $\sigma \neq \tau$ and $F_{\{\sigma,\tau\}}: (w_1, w_2) \to w_1^\sigma w_2^\tau - w_1^\tau w_2^\sigma$. Since T acts with weight $\nu - \omega(\sigma + \tau)$ on $KF_{\{\sigma,\tau\}}$, we have $\mathrm{Alt}^2(W, V)^T = \bigoplus KF_{\{\sigma,\tau\}}$ summed over those $\{\sigma, \tau\}$ such that $\nu = \omega(\sigma + \tau)$. For such $\{\sigma, \tau\}$ it can be seen that $[f_{(\sigma,\tau)}] \in H^2(W, V)^T$ with $\Psi[f_{(\sigma,\tau)}] = F_{\{\sigma,\tau\}}$. Note $[f_{(\sigma,\tau)}] + [f_{(\tau,\sigma)}] = 0$ since $f_{(\sigma,\tau)} + f_{(\tau,\sigma)} = \delta g$ where $g(w) = w^{\sigma+\tau}$. This completes the proof.

Using Lemma 1 and the Cartan–Eilenberg stability theorem we can determine the terms of (*). We have $H^1(X, V) \simeq H^1(A, V)^T = \mathrm{Hom}(A, V)^T \simeq \mathrm{Hom}(U, V)^T = H^1(U, V)^T \simeq H^1(B, V)$ has K-dimension $\#\{\sigma \in \Gamma \mid \nu = \theta'\sigma\}$. Also $H^1(X, H^1(Z, V)) \simeq \bigoplus H_\sigma(A, H_\tau(Z, V))^T$ (summed over $(\sigma, \tau) \in \Gamma \times \Gamma$) has K-dimension $\#\{(\sigma, \tau) \in \Gamma \times \Gamma \mid \nu = \sigma\theta' + \tau\}$ and $H^2(X, V) \simeq H^2(A, V)^T$ has K-dimension $\#\{\{\sigma, \tau\} \subseteq \Gamma \mid \nu = \theta'(\sigma + \tau)\}$. Since A acts trivially on Z and V we have $H^i(Z, V)^X \simeq H^i(Z, V)^T$ has K-dimension $\#\{\sigma \in \Gamma \mid \nu = \sigma\}$ when $i = 1$, and $\#\{\{\sigma, \tau\} \subseteq \Gamma \mid \nu = \sigma + \tau\}$ when $i = 2$.

LEMMA 2. *If $\nu = \sigma + \tau$ for some $\sigma, \tau \in \Gamma$ assume ν is invertible in* $\mathrm{End}(K^*)$. *Then* $\mathrm{Res} = 0$ *in* (*).

Proof. First we claim $\dim_K H^2(Z, V)^X \leqq 1$. By the previous remarks this is evident if we show $\sigma + \tau = \varphi + \rho$ in $\mathrm{End}(K^*)$, where $\sigma, \tau, \varphi, \rho \in \Gamma$, implies $\{\sigma, \tau\} = \{\varphi, \rho\}$. For this apply both sides to $(x + 1)$, expand, cancel and see the same equality holds in $\mathrm{End}(K^+)$. The claim follows from Dedekind's lemma.

Thus if $H^2(Z,V)^X \neq 0$ it is generated by some $\bar{f}$ of the form $\bar{f}((\alpha,0),(\beta,0)) = \alpha^\sigma\beta^\tau$ with $\nu = \sigma+\tau$. If $\mathrm{Res} \neq 0$ we can find $f \in Z^2(B,V)$ with $\mathrm{Res} f = \bar{f}$, that is, $f(\alpha,0,\beta,0) = \alpha^\sigma\beta^\tau$ (we use $f(\alpha,u,\beta,v)$ for $f((\alpha,u),(\beta,v))$). Let $E = E(f)$, the extension using f, and let $\tilde{U}$ be its Sylow 2-subgroup. We show $\tilde{U}$ is a Suzuki 2-group of exponent 8 contradicting a theorem of G. Higman **[3]**. A Suzuki 2-group is a non-abelian 2-group with more than one involution and an automorphism φ with $\langle\varphi\rangle$ transitive on the involutions.

Writing (a,α,u) for $(a,(\alpha,u)) \in \tilde{U}$ we see $(0,0,0) = (a,\alpha,u)^2 = (f(\alpha,u,\alpha,u), u^{\theta+1}, 0)$ implies $u = 0$. Now $f(\alpha,u,\alpha,u) = \alpha^{\sigma+\tau} = 0$ implies $\alpha = 0$. Thus $V^* = \{(a,0,0) \mid a \in K^*\}$ is the set of involutions. There are $q-1>1$ of them. It is easily seen that (a,α,u) is of exponent 8 when $u \neq 0$.

Choose t with $\langle t\rangle = K^*$. Since ν is invertible in $\mathrm{End}(K^*)$, we have $(1,0,0)^{\langle T(t)\rangle} = \{(t^\nu,0,0) \mid t \in K^*\} = V^*$. Thus $T(t) \in \mathrm{Aut}(\tilde{U})$ will serve as the required automorphism showing $\tilde{U}$ is a Suzuki 2-group. This completes the proof.

LEMMA 3. *In* $(*)$ *the map* Φ *is a surjection* $\Leftrightarrow H^1(X, H^1(Z,V)) = 0$.

Proof. First we give the description of Φ as found in **[7]**. Choose a set splitting $S\colon X \to B$ with $\pi S = 1_X$, $S(1) = 1$. For $f \in Z^2(B,V)_0 = \{f \in Z^2(B,V) \mid f \mid Z\times Z = 0\}$ define $\tilde{\Phi} f \in C^1(X, Z^1(Z,V))$ by $\tilde{\Phi} f(x)(\alpha) = f(S(x), \alpha^{x^{-1}}) - f(\alpha, S(x))$. Now $\tilde{\Phi}$ induces a well defined map Φ on the classes (this uses only the fact that Z is abelian).

Now assume $\mathrm{Im}\,\Phi = H^1(X,H^1(Z,V)) \neq 0$ and choose a nonzero $[d] \in H^1(X,H^1(Z,V)) \simeq \bigoplus H_\sigma(A, H_\tau(Z,V))^T$ of the form $d(u)(\alpha) = u^\sigma\alpha^\tau$ where $u \in A$, $\alpha \in Z$, $\sigma,\tau \in \Gamma$. Find $[f] \in H^2(B,V)_0$ with $\Phi[f] = [d]$. We no longer need the action of T so replace f by $f \mid U\times U$. We use S defined by $S(u) = (0,u)$. Since $B^1(A,H^1(Z,V)) = 0$ we may assume $\tilde{\Phi} f = d$, that is

$$(1) \qquad f(0,u,\alpha,0) + f(\alpha,0,0,u) = u^\sigma\alpha^\tau.$$

Let $E = E(f) = \{(a,\alpha,u) \mid a,\alpha,u \in K\}$, the extension of V by U using f, and let $\tilde{Z} = \{(a,\alpha,0)\}$. Then $\tilde{Z} \lhd E$ and $\tilde{Z}$ is abelian since $f \mid Z\times Z = 0$. We have an exact sequence of groups $1 \to \tilde{Z} \to E \to A \to 1$. Define $\rho\colon A \to E$ by $\rho(u) = (0,0,u)$ and let $g \in Z^2(A,\tilde{Z})$ be the corresponding cocycle, that is, $g(u,v) = \rho(u)\rho(v)\rho(u+v)^{-1}$. All multiplication in E can be performed in terms of f and it can be computed that $g = (g_1, g_2, 0)$ where $g_1(u,v) = f(uv^\theta, u+v, (u+v)^{\theta+1}, u+v)$ and $g_2(u,v) = uv^\theta$.

Similarly it can be computed that $(b,\alpha,0)^{\rho(u)} = (b+f(0,u,\alpha,0)+f(0,u,u^{\theta+1},u)+f(\alpha,u,u^{\theta+1},u),\alpha,0)$. Since $f \in Z^2(U,V)$ we have $0=\delta f((\alpha,0),(0,u),(u^{\theta+1},u)) = f(\alpha,0,0,u)+f(\alpha,u,u^{\theta+1},u)+f(0,u,u^{\theta+1},u)+f(\alpha,0,0,0)$. Now use $f(\alpha,0,0,0)=0$, equation (1) and the above expression for $(b,\alpha,0)^{\rho(u)}$ to obtain $(b,\alpha,0)^{\rho(u)} = (b+u^{\sigma}\alpha^{\tau},\alpha,0)$.

Using this expression for the action of A on $\tilde{Z}$ the first slot of the equation $0=\delta g(u,v,w)$ implies

$$0 = g_1(u,v)+g_1(u+v,w)+g_1(v,w)+u^{\sigma}g_2(v,w)^{\tau}+g_1(u,v+w).$$

Take $u=v=w=1$ and use the fact that g_1 vanishes when either of its arguments is 0 to obtain $0 = 1g_2(1,1)^{\tau} = 1$, a contradiction. This completes the proof.

Let $\{e_i\}$, $i=1,2,3,4$ be the standard base for K^4 (columns) and put $V_i = \langle e_1,\cdots,e_i\rangle/\langle e_1,\cdots,e_{i-1}\rangle$ as KB-module. Then V_i is a KB-module on which U acts trivially and T acts with weight ν_i where $\nu_1=\theta$, $\nu_2 = 1-\theta$, $\nu_3=\theta-1$, $\nu_4 = -\theta$. For convenience we set $\nu_0=0$. In the following lemma we determine the terms occuring in (*) when $\nu=\nu_i$, $i=0,1,2,3,4$ by solving the equations following Lemma 1.

LEMMA 4. *The solutions are as indicated when* $q>2$ *and* $i\in\{0,1,2,3,4\}$.

(a) $\nu_i = \theta'\sigma$: $(i,q,\sigma) = (2,q,1/2)$; $(4,8,1)$.

(b) $\nu_i = \sigma$: $(i,q,\sigma)=(1,q,\theta)$; $(3,8,1)$.

(c) $\nu_i = \sigma\theta'+\tau$: $(i,q,\sigma,\tau) = (0,8,\sigma,2\sigma)$ (any $\sigma\in\Gamma$); $(1,q,\theta/2,1/2)$; $(2,8,1,1)$; $(3,8,4,2)$; $(4,8,2,2)$; $(4,32,2,8)$; $(4,32,1,2)$.

(d) $\nu_i = \theta'(\sigma+\tau)$: $(i,q,\{\sigma,\tau\}) = (1,q,\{1/2,\theta\})$; $(2,q,\{1/4\})$; $(3,8,\{1,2\})$; $(4,8,\{1/2\})$.

(e) $\nu_i=\sigma+\tau$: $(i,q,\{\sigma,\tau\}) = (1,q,\{\theta/2\})$; $(2,8,\{2,3\})$; $(3,8,\{4\})$; $(3,32,\{2,1\})$; $(4,8,\{1,4\})$.

The following will be useful for solving these equations.

LEMMA 5. *Let* $\varphi_i \in \Gamma \hookrightarrow \mathrm{End}(K^*)$ $i=1,2,\cdots,m$. *The following is arithmetic in* $\mathrm{End}(K^*)$.

(a) *If* $\varphi_1+\varphi_2=\varphi_3+\varphi_4$ *then* $\{\varphi_1,\varphi_2\}=\{\varphi_3,\varphi_4\}$.

(b) *If the* φ_i's *are distinct then* $\Sigma_{i=1}^m \varphi_i \notin \Gamma$.

(c) *If* $\Sigma_{i=1}^m \varphi_i = 0$ *then* $m \geqq |\Gamma|$, *and* $m=|\Gamma| \Leftrightarrow \{\varphi_i\}_{i=1}^m = \Gamma$.

Proof of Lemma 5. (a) A proof is included in the proof of Lemma 2.

For (b) and (c) write $\varphi_i\colon x \to x^{2^{n_i}}$ for $0 \leqq n_i < |\Gamma|$.

(b) Here we assume the n_i's are distinct. Then $\Sigma\varphi_i\in\Gamma$ implies for all $x \in K$ we have $(x^{\Sigma\varphi_i}+1) = (x+1)^{\Sigma\varphi_i} = \Pi(x^{\varphi_i}+1) = \Sigma x^{\Sigma_{j\in J}\varphi_j}$ where we sum over all $J \subseteq \{1, 2, \cdots, m\}$. Cancelling the terms on the left with the corresponding terms on the right there remains a polynomial of degree less than $2^{|\Gamma|}$ with $2^{|\Gamma|} = |K|$ solutions.

(c) Assume m is minimal with $\Sigma\varphi_i = 0$. Then the φ_i's are distinct since $\varphi_i + \varphi_i = 2\varphi_i \in \Gamma$. Then $\Sigma\varphi_i = 0$ implies $(q-1)|\Sigma 2^{n_i}$. Thus $q - 1 = 2^{|\Gamma|} - 1 = \Sigma_{i=0}^{|\Gamma|} 2^i \leqq \Sigma_{i=1}^{m} 2^{n_i}$ implying $m = |\Gamma|$ and $\{\varphi_i\} = \Gamma$.

We now indicate a proof of Lemma 4. Observe first that from their definitions we have $\theta \neq 1$, $2\theta^2 = 1$, $\theta'(\theta+1) = 1$. Thus θ', $\theta+1$, $1-\theta = \theta'/2$ are invertible in $\mathrm{End}(K^*)$. Using these facts the equations can be manipulated to take advantage of Lemma 5 and reduce the problem to a few case by case investigations. We illustrate with the solution of $\nu_i = \sigma\theta' + \tau$.

$i = 0$: $0 = \sigma\theta' + \tau \Rightarrow \tau\sigma^{-1} = -\theta' = 2\theta - 2 \Rightarrow 2\theta = 2 + \tau\sigma^{-1}$. Now Lemma 5 (b) says $2 = \tau\sigma^{-1}$ so $\theta = 2$, $q = 8$, $\tau = 2\sigma$.

$i = 1$: $\theta = \sigma\theta' + \tau = 2\sigma - 2\sigma\theta + \tau \Rightarrow \theta + 2\sigma\theta = 2\sigma + \tau$ and Lemma 5 (a) implies $\{\theta, 2\sigma\theta\} = \{2\sigma, \tau\}$. Now $\theta \neq 1 \Rightarrow (\sigma, \tau) = (\theta/2, \theta^2) = (\theta/2, 1/2)$.

$i = 2$: Multiplying by $1 + \theta$ we obtain $1/2 = \sigma + \tau\theta + \tau$ and Lemma 5 (b) says σ, $\tau\theta$, τ are not distinct. $\theta \neq 1 \Rightarrow \tau\theta \neq \tau$. $\sigma = \tau\theta \Rightarrow 1/2 = 2\tau\theta + \tau \Rightarrow 2\theta = 1 \Rightarrow 1 = 2\theta^2 = \theta$, a contradiction. $\sigma = \tau \Rightarrow 1/2 = 2\tau + 2\theta \Rightarrow 2 = \theta$, $q = 8$ and it may be seen $\sigma = \tau = 1$.

$i = 3$: Since $\nu_3 = -\nu_2$ we obtain $0 = 1/2 + \sigma + \tau\theta + \tau$ and Lemma 5 (c) implies $|\Gamma| \leqq 4$. Thus $q = 8$, $\sigma = \tau = 1$.

$i = 4$: Since $\nu_4 = -\nu_2$ we obtain $2\sigma\theta = 2\sigma + \theta + \tau$ implying 2σ, θ, τ are not distinct. $2\sigma = \theta \Rightarrow \theta^2 = 2\theta + \tau \Rightarrow 2\theta = \tau \Rightarrow \theta^2 = 4\theta$, $\theta = 4$, $q = 32$, $(\sigma, \tau) = (2, 8)$. $2\sigma = \tau \Rightarrow 2\sigma\theta = 4\sigma + \theta \Rightarrow 4\sigma = \theta$, $\theta = 4$, $q = 32$, $(\sigma, \tau) = (1, 2)$. $\tau = \theta \Rightarrow 2\sigma\theta = 2\sigma + 2\theta \Rightarrow \sigma = \theta$, $\theta = 2$, $q = 8$, $(\sigma, \tau) = (2, 2)$.

LEMMA 6. *When $i \in \{1, 2, 3, 4\}$ we have*

$$\dim_K H^1(B, V_i) = \begin{cases} 1 & (i, q) = (2, q);\ (4, 8) \\ 0 & \textit{otherwise}, \end{cases}$$

$$\dim_K H^2(B, V_i) = \begin{cases} 1 & (i, q) = (2, q);\ (4, 8);\ (4, 32) \\ 0 & \textit{otherwise}. \end{cases}$$

Proof. The first statement is immediate from Lemma 4 and the remarks following Lemma 1. For the second observe $\nu_i \in \{\pm\theta, \pm\theta'/2\}$ and so ν_i is invertible in $\mathrm{End}(K^*)$. Now Lemmas 1, 2, 3 and 4 may be used to determine the relevant terms of sequences (*) when $\nu = \nu_i$. These considerations prove the claim except to show $H^2(B, V_4) \neq 0$ when $q = 32$. In this case it may be seen that $(\alpha, u), (\beta, v) \to u^2\beta^8 + u\beta^2 + u^3v^8 + u^2v^9$ gives a nonzero class in $H^2(U, V_4)^T \simeq H^2(B, V_4)$.

We are now ready to proceed to the main results of this paper.

THEOREM 1. *Let K be the trivial module for $Sz(q)$, $q \geqq 8$. Then $\dim_K H^2(Sz(q), K)$ is 0 if $q > 8$, and is 2 if $q = 8$ with generators (on a Sylow 2-subgroup) any two of f_σ: $(\alpha, u), (\beta, v) \to (\alpha^2 v + u^2\beta^4)^\sigma$, $\sigma \in \Gamma$.*

Proof. We use B in place of $Sz(q)$ and sequences (*) with $\nu = 0$ and $V = K$. According to Lemma 4 we have $H^2(Z, V)^X = H^2(X, V) = 0$ and $\dim_K H^1(X, H^1(Z, V))$ is 0 if $q > 8$, and $|\Gamma| = 3$ if $q = 8$. Now sequences (*) with Lemma 3 give the upperbound. For the lowerbound it is easily checked that f_σ as given is a T-stable cocycle and when $\sigma \neq \tau$, $\Phi[f_\sigma]$ and $\Phi[f_\tau]$ are independent in $H^1(A, H^1(Z, V))^T \simeq H^1(X, H^1(Z, V))$.

THEOREM 2. *Assume $q \geqq 8$ and K^4 is the standard module for $Sz(q)$. Then $H^1(Sz(q), K^4)$ is of dimension one and is generated by the restriction of a generator of $H^1(Sp_4(q), K^4)$.*

Proof. Define $[d] \in H^1(U, K^4)^T \simeq H^1(B, K^4)$ by $d(\alpha, u) = (\alpha^\theta, u^{1/2}, 0, 0)^*$ (* denotes transpose). It can be checked explicitly that d is a nontrivial T-stable cocycle defined on U giving the claimed lowerbound. Furthermore it can be seen that if $v \in K^4$, $x \in U$, then $v^*x^*Jd(x) = (v^*J_0v + v^*x^{**}J_0xv)^{1/2}$ where J_0 is the 4×4 matrix with all entries 0 except $(J_0)_{41} = (J_0)_{32} = 1$. This means d is the restriction of Dickson's derivation which generates $H^1(Sp_4(q), K^4)$ **[8]**.

For the upperbound we use Lemma 6 to conclude $\dim_K H^1(B, K^4) \leqq \sum_{i=1}^4 \dim_K H^1(B, V_i) = 1$ if $q > 8$, and 2 if $q = 8$. We are done at $q > 8$ and continue at $q = 8$.

Define $V_{12} = \langle e_1, e_2 \rangle$, $V_{34} = K^4/V_{12}$. We obtain the exact sequence of K-modules

$$0 \to H^1(B, V_{12}) \to H^1(B, K^4) \xrightarrow{(\pi_1)_*} H^1(B, V_{34})$$

(2)

$$\to H^2(B, V_{12}) \to H^2(B, K^4) \xrightarrow{(\pi_2)_*} H^2(B, V_{34}) \to .$$

The given cocycle shows $\dim_K H^1(B, V_{12}) = 1$ so it suffices to see $(\pi_1)_* =$

0. Lemma 6 implies $\dim_K H^1(B, V_{34}) \leqq 1$. It can be seen that $(\alpha, u) \to (\_,\_, \alpha + u^3, u)^*$ is a nontrivial T-fixed cocycle in $Z^1(U, V_{34})^T$ so its class generates $H^1(U, V_{34})^T \simeq H^1(B, V_{34})$. If $(\pi_1)_* \neq 0$ we can find $f \in Z^1(U, K^4)$ of the form $f(\alpha, u) = (f_1(\alpha, u), f_2(\alpha, u), \alpha + u^3, u)^*$. The e_2 coordinate of the equation $\delta f((\alpha, u), (\beta, v)) = 0$ gives the equation $f_2(\alpha + \beta + uv^\theta, u + v) = f_2(\alpha, u) + f_2(\beta, v) + u(\beta + v^3) + \alpha v$. Set $u = v = 0$ to obtain $f_2(\alpha + \beta, 0) = f_2(\alpha, 0) + f_2(\beta, 0)$; and set $(\alpha, u) = (\beta, v)$ to obtain $f_2(u^3, 0) = u^4$, that is, $f_2(u, 0) = u^6$. This is a contradiction as $u \to u^6$ is not an additive function.

THEOREM 3. *Let K^4 be the standard module for $Sz(q)$. Then $H^2(Sz(q), K^4)$ is zero if $q = 8$, and is of dimension one if $q > 8$ generated by a cocycle which is the restriction of a generator of $H^2(Sp(q), K^4)$.*

Proof. Landázuri (see [7]) has explicitly constructed (on a Sylow 2-subgroup) a nontrivial cocycle in $Z^2(Sp_4(2^m), GF(2^m)^4)$ and further (see [5]) has shown $H^2(Sp_4(2^m), (GF(2^m))^4)$ is of dimension one when $m > 1$. Restricting his cocycle gives

$$f\colon (\alpha, u), (\beta, v) \to$$
$$((\alpha^\theta u^\theta v^{1/2} + \alpha^\theta \beta^\theta + u^\theta \beta + u^\theta v^{\theta+1} + u^\theta \beta^\theta v^{1/2})^{1/2}, (uv)^{1/4}, 0, 0)^*.$$

We will see f is a coboundary only at $q = 8$. McLaughlin [7] has given a somewhat different argument to see $\operatorname{Res}(Sz(q), Sp_4(q))$ is nonzero when $q > 8$ using the sufficient condition of Griess [2].

Consider now sequence (2). We have seen $(\pi_1)_* = 0$ and $\dim_K H^1(B, V_{34}) = 0$ if $q > 8$, and 1 if $q = 8$. Next we show $\dim_K H^2(B, V_{12}) = 1$. The upper bound follows from Lemma 6 and the lower bound follows from the displayed cocycle f. Also from Lemma (6), $H^2(B, V_{34}) = 0$ when $q > 32$. Using sequence (2) the proof is now complete when $q > 32$. Furthermore, the cases $q = 8$, 32 follow if we show there is no $f \in Z^2(B, K^4)$ which has a nontrivial projection onto V_4.

Assuming we have such an f, a contradiction is obtained by using the following: Let $L = K^4/V_1$ as KB-module.

(a) $H^2(Z, L)^X \simeq K$ generated by $(\alpha, \beta) \to (\_, \alpha^2\beta^4, 0, 0)^*$ when $q = 8$ and by $(\alpha, \beta) \to (\_, 0, \alpha\beta^2, 0)^*$ when $q = 32$.

(b) $H^2(X, L^Z) \simeq K$ generated by $(u, v) \to (\_, (uv)^{1/4}, 0, 0)^*$.

(c) $H^1(X, H^1(Z, L)) = 0$.

We now assume (a), (b), (c). From the exact sequence of groups $1 \to Z \to B \to X \to 1$ the Hochschild–Serre spectral sequence gives the exact sequences

$$0 \to H^2(B, L)_0 \to H^2(B, L) \xrightarrow{\text{Res}} H^2(Z, L)^X$$
$$\to H^2(X, L^Z) \to H^2(B, L)_0 \to H^1(X, H^1(Z, L)) \to .$$

In general when we have a function whose range is K^4 let the subscript i denote its projection onto V_i. Thus $f = (f_1, f_2, f_3, f_4)^*$. We are assuming $0 \neq [f_4] \in H^2(B, V_4)$. Let $\tilde{f}$ denote the projection of f onto L. We write this as $\tilde{f} = (\_, f_2, f_3, f_4)$. Thus $\tilde{f} \in Z^2(B, L)$.

Assume first $\text{Res}[\tilde{f}] = 0$. Then using (c) and the above sequences $\tilde{f}$ is cohomologous to the image under the inflation map of a generator of $H^2(X, L^Z)$, i.e. there is a $g \in C^1(B, L)$ with $(\tilde{f} - \delta f)((\alpha, u), (\beta, v)) = (\_, (uv)^{1/4}, 0, 0)^*$. Using the fact that (α, u) is an upper triangular matrix it is easily seen that this equation implies $f_4 = \delta g_4 \in B^2(B, V_4)$, contradicting present assumptions.

Now we assume $\text{Res}[\tilde{f}] \neq 0$. Let $\bar{f} = \text{Res}(f)$ so $[\bar{f}] \in H^2(Z, K^4)^X$. Assume first $q = 8$. Now (a) tells us we may assume $\bar{f}(\alpha, \beta) = (\bar{f}_1(\alpha, \beta), \alpha^2\beta^4, 0, 0)^*$. Let $u = (0, 1) \in U$. Then $(u - 1).\bar{f} = \delta g$ for some $g \in C^1(Z, K^4)$. Apply both sides to (α, β) and obtain

$$\begin{bmatrix} \alpha^2\beta^4 \\ 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} g_1(\alpha+\beta) \\ g_2(\alpha+\beta) \\ g_3(\alpha+\beta) \\ g_4(\alpha+\beta) \end{bmatrix} + \begin{bmatrix} 1 & 0 & \alpha & \alpha^4 \\ & 1 & 0 & \alpha \\ & & 1 & 0 \\ & & & 1 \end{bmatrix} \begin{bmatrix} g_1(\beta) \\ g_2(\beta) \\ g_3(\beta) \\ g_4(\beta) \end{bmatrix} + \begin{bmatrix} g_1(\alpha) \\ g_2(\alpha) \\ g_3(\alpha) \\ g_4(\alpha) \end{bmatrix}.$$

The third and fourth rows tell us g_3 and g_4 are additive; $\alpha = \beta$ in the second row tells us $0 = \alpha g_4(\alpha)$ implying $g_4 = 0$; $\alpha = \beta$ in the first row tells us $\alpha^5 = g_3(\alpha)$, contradicting the additivity of g_3.

Assume now $q = 32$. Here (a) tells us we may assume $\bar{f}(\alpha, \beta) = (\bar{f}_1(\alpha, \beta), 0, \alpha\beta^2, 0)^*$. Now, with $u = (0, 1) \in U$, the equation $(u - 1).\bar{f} = \delta g$ implies $(\alpha\beta^2, \alpha\beta^2, 0, 0)^* = \delta g(\alpha, \beta)$. As before g_3 and g_4 are additive. Set $\alpha = \beta$. The second coordinate implies $g_4(\alpha) = \alpha^2$; the first implies $\alpha^2 = g_3(\alpha) + \alpha^{2\theta+1}$; these imply $\alpha \to \alpha^{2\theta+1} = \alpha^9$ is additive, a contradiction.

We now prove (a), (b), (c). Note that if x is an involution in some group and d and f are 1 and 2-cocycles from that group to some module then $\delta d(x, x) = 0$ and $\delta f(x, x, x) = 0$ imply $d(x) = -xd(x)$ and $f(x, x) = xf(x, x)$. Regard $L = K^3$ (columns) $= \langle e_2, e_3, e_4 \rangle$ on which (α, u) acts as multiplication by

$$\begin{bmatrix} 1 & u & \alpha \\ & 1 & u^\theta \\ & & 1 \end{bmatrix}.$$

(a) Take $[f] \in H^2(Z, L)^X$ and using our convention we have $f = (f_2, f_3, f_4)^*$. Since $[f_4] \in H^2(Z, V_4)^T$, by Lemma 4 (e) we may assume

$f_4(\alpha, \beta) = \alpha\beta^4 k_4$ and $k_4 = 0$ when $q = 32$. The relation $f(\alpha, \alpha) = \alpha f(\alpha, \alpha)$ implies $k_4 = 0$. Now $[f_3] \in H^2(Z, V_3)^T$ and we may assume $f_3(\alpha, \beta) = \alpha^\sigma \beta^\tau k_3$ where $\{\sigma, \tau\} = \{4\}$ if $q = 8$ and $\{\sigma, \tau\} = \{2, 1\}$ if $q = 32$. Set $u = (0, 1) \in U$. Then $(u-1).f = \delta g$ for $g \in C^1(Z, L)$. In the usual way this equation implies g_3 and g_4 are additive. Setting $\alpha = \beta$ we obtain $\alpha^{\sigma+\tau} k_3 = \alpha g_4(\alpha)$ implying $k_3 = 0$ or $\alpha \to \alpha^{\sigma+\tau-1}$ is additive. At $q = 8$ the latter is false implying $k_3 = 0$.

Since $k_4 = 0$ it follows that $[f_2] \in H^2(Z, V_2)^T$ and by Lemma 4 (e) we may assume $f_2(\alpha, \beta) = \alpha^2 \beta^4 k_2$ with $k_2 = 0$ when $q = 32$. This proves (a).

(b) We see $L^Z = \langle e_2, e_3 \rangle \simeq K^2$ (columns) on which $(0, u)$ $(-: U \to U/Z)$ acts as multiplication by $\begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix}$. Take $[f] \in H^2(X, K^2)$. By Lemma 4 (d) we may assume $f_3(u, v) = uv^2 k_3$ with $k_3 = 0$ when $q = 32$. Now the relation $\bar{u} f(\bar{u}, \bar{u}) = f(\bar{u}, \bar{u})$ implies $f_3 = 0$. Thus $f_2 \in Z^2(X, V_2)$ and (b) follows from Lemma 4 (d).

(c) Take $f \in Z^1(Z, L)$. Then $f(\alpha) = \alpha f(\alpha)$ implies the image of f lies in $L^\alpha = L^Z = \langle e_2, e_3 \rangle$. Thus $f_4 = 0$. Taking Z-cohomology of the exact sequence $0 \to L^Z \to L \to V_4 \to 0$ gives the exact sequence of KX-modules $0 \to V_4 \xrightarrow{\delta_*} H^1(Z, L^Z) \to H^1(Z, L) \xrightarrow{\pi_*} H^1(Z, V_4) \to$. We have just seen $\pi_* = 0$. Set $V_{23} = L^Z$. It is easily seen that $\operatorname{Im} \delta_* = \operatorname{Hom}_K(Z, V_2) \subset \operatorname{Hom}_K(Z, V_{23}) \subset \operatorname{Hom}(Z, L^Z) = H^1(Z, L^Z)$ showing $H^1(Z, L) = \bigoplus_{\tau \neq 1} H_\tau(Z, V_{23}) \bigoplus H$ where

$$H = \operatorname{Hom}_K(Z, V_{23}) / \operatorname{Hom}_K(Z, V_2) \simeq \operatorname{Hom}_K(Z, V_3).$$

Now $H^1(X, H) = \bigoplus H_\sigma(A, \operatorname{Hom}_K(Z, V_3))^T = 0$ since by Lemma 4 (c) there is no $\sigma \in \Gamma$ with $\nu_3 = \sigma\theta' + 1$. Finally, we show $H^1(X, H_\tau(Z, V_{23})) = 0$ when $\tau \neq 1$. Take $[f] \in H^1(A, H_\tau(Z, V_{23}))^T$. Taking $u = v$ in the cocycle condition on f we see $0 = u f_3(u)(\alpha)$ showing $f_3 = 0$. Thus

$$H^1(X, H_\tau(Z, V_{23})) \simeq H^1(X, H_\tau(Z, V_2)) = \bigoplus H_\sigma(A, H_\tau(Z, V_2))^T = 0$$

since by Lemma 4 (c) there is no $\sigma \in \Gamma$ with $\nu_2 = \theta'\sigma + \tau$ when $\tau \neq 1$.

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