CHAPTER IV

14. Statement of Results Proved in Chapter IV

In this chapter, we begin the proof of the main theorem of this paper. The proof is by contradiction. If the theorem is false, a minimal counterexample is seen to be a non cyclic simple group all of whose proper subgroups are solvable. Such a group is called a *minimal simple group.* Throughout the remainder of this chapter, $\mathfrak{G}$ is a minimal simple group of odd order. We will eventually derive a contradiction from the assumed existence of $\mathfrak{G}$.

In this section, the results to be proved in this chapter are summarized. Several definitions are required.

Let π^* be the subset of $\pi(\mathfrak{G})$ consisting of all primes p such that if $\mathfrak{P}$ is a S_p-subgroup of $\mathfrak{G}$, then either $\mathscr{SCN}_3(\mathfrak{P})$ is empty or $\mathfrak{P}$ contains a subgroup $\mathfrak{A}$ of order p such that $C_{\mathfrak{P}}(\mathfrak{A}) = \mathfrak{A} \times \mathfrak{B}$ where $\mathfrak{B}$ is cyclic. Let π_1^* be the subset of π^* consisting of those p such that if $\mathfrak{P}$ is a S_p-subgroup of $\mathfrak{G}$ and a is the order of a cyclic subgroup of $N(\mathfrak{P})/\mathfrak{P}C(\mathfrak{P})$, then one of the following possibilities occurs:

(i) a divides $p - 1$.

(ii) $\mathfrak{P}$ is abelian and a divides $p + 1$.

(iii) $|\mathfrak{P}| = p^3$ and a divides $p + 1$.

We now define five types of subgroups of $\mathfrak{G}$. The basic property shared by these five types is that they are all maximal subgroups of $\mathfrak{G}$. Thus, for $x = \mathrm{I, II, III, IV, V}$, any group of type x is by definition a maximal subgroup of $\mathfrak{G}$. The remaining properties are more detailed.

We say that $\mathfrak{M}$ is of type I provided

(i) $\mathfrak{M}$ is of Frobenius type with Frobenius kernel $\mathfrak{H}$.

(ii) One of the following conditions is satisfied:

(a) $\mathfrak{H}$ is a T. I. set in $\mathfrak{G}$.

(b) $\pi(\mathfrak{H}) \subseteq \pi_1^*$.

(c) $\mathfrak{H}$ is abelian and $m(\mathfrak{H}) = 2$.

(iii) If $p \in \pi(\mathfrak{M}/\mathfrak{H})$, then $m_p(\mathfrak{M}) \leqq 2$ and a S_p-subgroup of $\mathfrak{M}$ is abelian.

The remaining four types are by definition three step groups. If $\mathfrak{S}$ is a three step group, we use the following notation:

$$\mathfrak{S} = \mathfrak{S}'\mathfrak{W}_1\,, \qquad \mathfrak{S}' \cap \mathfrak{W}_1 = 1\,, \qquad C_{\mathfrak{S}'}(\mathfrak{W}_1) = \mathfrak{W}_2\,.$$

Furthermore, $\mathfrak{H}$ denotes the maximal normal nilpotent S-subgroup of $\mathfrak{S}$. By definition, $\mathfrak{H} \subseteq \mathfrak{S}'$ so we let $\mathfrak{U}$ be a complement for $\mathfrak{H}$ in $\mathfrak{S}'$.

In addition to being a three step group, each of the remaining four types has the property that if $\mathfrak{W}_0$ is any non empty subset of $\mathfrak{W}_1\mathfrak{W}_2 - \mathfrak{W}_1 - \mathfrak{W}_2$, then $N_{\mathfrak{G}}(\mathfrak{W}_0) = \mathfrak{W}_1\mathfrak{W}_2$, by definition. The remaining properties are more detailed.

We say that $\mathfrak{S}$ is of type II provided

(i) $\mathfrak{U} \neq 1$ and $\mathfrak{U}$ is abelian.

(ii) $N_{\mathfrak{G}}(\mathfrak{U}) \not\subseteq \mathfrak{S}$.

(iii) $N_{\mathfrak{G}}(\mathfrak{A}) \subseteq \mathfrak{S}$ for every non empty subset $\mathfrak{A}$ of $\mathfrak{S}'^{\#}$ such that $C_{\mathfrak{H}}(\mathfrak{A}) \neq 1$.

(iv) $|\mathfrak{W}_1|$ is a prime.

(v) For every prime p, if $\mathfrak{A}_0$, $\mathfrak{A}_1$ are cyclic p-subgroups of $\mathfrak{U}$ which are conjugate in $\mathfrak{G}$ but are not conjugate in $\mathfrak{S}$, then either $C_{\mathfrak{H}}(\mathfrak{A}_0) = 1$ or $C_{\mathfrak{H}}(\mathfrak{A}_1) = 1$.

(vi) $\mathfrak{H}C(\mathfrak{H})$ is a T. I. set in $\mathfrak{G}$.

We say that $\mathfrak{S}$ is of type III provided (ii) in the preceding definition is replaced by

(ii)′ $N_{\mathfrak{G}}(\mathfrak{U}) \subseteq \mathfrak{S}$,

and the remaining conditions hold.

We say that $\mathfrak{S}$ is of type IV provided (i) and (ii) in the definition of type II are replaced by

(i)″ $\mathfrak{U}' \neq 1$,

(ii)″ $N_{\mathfrak{G}}(\mathfrak{U}) \subseteq \mathfrak{S}$,

and the remaining conditions hold.

We say that $\mathfrak{S}$ is of type V provided

(i) $\mathfrak{U} = 1$.

(ii) One of the following statements is true:

(a) $\mathfrak{S}'$ is a T. I. set in $\mathfrak{G}$.

(b) $\mathfrak{S}' = \mathfrak{P} \times \mathfrak{S}_0$, where $\mathfrak{S}_0$ is cyclic and $\mathfrak{P}$ is a S_p-subgroup of $\mathfrak{G}$ with $p \in \pi_1^*$.

THEOREM 14.1. *Let $\mathfrak{G}$ be a minimal simple group of odd order. Two elements of a nilpotent S-subgroup $\mathfrak{H}$ of $\mathfrak{G}$ are conjugate in $\mathfrak{G}$ if and only if they are conjugate in $N(\mathfrak{H})$. Either (i) or (ii) is true:*

(i) *Every maximal subgroup of $\mathfrak{G}$ is of type I.*

(ii) (a) *$\mathfrak{G}$ contains a cyclic subgroup $\mathfrak{W} = \mathfrak{W}_1 \times \mathfrak{W}_2$ with the property that $N(\mathfrak{W}_0) = \mathfrak{W}$ for every non empty subset $\mathfrak{W}_0$ of $\mathfrak{W} - \mathfrak{W}_1 - \mathfrak{W}_2$. Also, $\mathfrak{W}_i \neq 1$, $i = 1, 2$.*

(b) *$\mathfrak{G}$ contains maximal subgroups $\mathfrak{S}$ and $\mathfrak{T}$ not of type I such that*

$$\mathfrak{S} \quad \mathfrak{W}_1\mathfrak{S}' , \qquad \mathfrak{T} = \mathfrak{W}_2\mathfrak{T}' , \qquad \mathfrak{S}' \cap \mathfrak{W}_1 = 1 , \qquad \mathfrak{T}' \cap \mathfrak{W}_2 = 1 ,$$
$$\mathfrak{S} \cap \mathfrak{T} = \mathfrak{W} .$$

(c) *Every maximal subgroup of* $\mathfrak{G}$ *is either conjugate to* $\mathfrak{S}$ *or* $\mathfrak{T}$ *or is of type I.*

(d) *Either* $\mathfrak{S}$ *or* $\mathfrak{T}$ *is of type II.*

(e) *Both* $\mathfrak{S}$ *and* $\mathfrak{T}$ *are of type II, III, IV, or V.* (*They are not necessarily of the same type.*)

In order to state the next theorem we need further notation. If $\mathfrak{L}$ is of type I, let

$$\hat{\mathfrak{L}} = \hat{\mathfrak{L}}_1 = \bigcup_{H \in \mathfrak{H}^\sharp} C_{\mathfrak{L}}(H) ,$$

where $\mathfrak{H}$ is the Frobenius kernel of $\mathfrak{L}$.

If $\mathfrak{L}$ is of type II, III, IV, or V, we write $\mathfrak{L} = \mathfrak{L}'\mathfrak{W}_1$, $\mathfrak{L}' \cap \mathfrak{W}_1 = 1$. Let $\mathfrak{H}$ be the maximal normal nilpotent S-subgroup of $\mathfrak{L}$, let $\mathfrak{U}$ be a complement for $\mathfrak{H}$ in $\mathfrak{L}'$ and set $\mathfrak{W} = C_{\mathfrak{L}}(\mathfrak{W}_1)$, $\mathfrak{W}_2 = \mathfrak{W} \cap \mathfrak{L}'$, $\hat{\mathfrak{W}} = \mathfrak{W} - \mathfrak{W}_1 - \mathfrak{W}_2$.

If $\mathfrak{L}$ is of type II, let

$$\hat{\mathfrak{L}} = \bigcup_{H \in \mathfrak{H}^\sharp} C_{\mathfrak{L}'}(H) .$$

If $\mathfrak{L}$ is of type III, IV, or V, let

$$\hat{\mathfrak{L}} = \mathfrak{L}' .$$

If $\mathfrak{L}$ is of type II, III, IV, or V, let

$$\hat{\mathfrak{L}}_1 = \hat{\mathfrak{L}} \cup \bigcup_{L \in \mathfrak{L}} L^{-1}\hat{\mathfrak{W}}L .$$

We next define a set $\mathscr{A} = \mathscr{A}(\mathfrak{L})$ of subgroups associated to $\mathfrak{L}$. Namely, $\mathfrak{M} \in \mathscr{A}$ if and only if $\mathfrak{M}$ is a maximal subgroup of $\mathfrak{G}$ and there is an element L in $\hat{\mathfrak{L}}^\sharp$ such that $C(L) \nsubseteq \mathfrak{L}$ and $C(L) \subseteq \mathfrak{M}$. Let $\{\mathfrak{N}_1, \cdots, \mathfrak{N}_n\}$ be a subset of $\mathscr{A}$ which is maximal with the property that $\mathfrak{N}_i$ and $\mathfrak{N}_j$ are not conjugate if $i \neq j$. For $1 \leqq i \leqq n$, let $\mathfrak{H}_i$ be the maximal normal nilpotent S-subgroup of $\mathfrak{N}_i$.

THEOREM 14.2. *If* $\mathfrak{L}$ *is of type I, II, III, IV, or V, then* $\hat{\mathfrak{L}}$ *and* $\hat{\mathfrak{L}}_1$ *are tamely imbedded subsets of* $\mathfrak{G}$ *with*

$$N(\hat{\mathfrak{L}}) = N(\hat{\mathfrak{L}}_1) = \mathfrak{L} .$$

If $\mathscr{A}(\mathfrak{L})$ *is empty,* $\hat{\mathfrak{L}}$ *and* $\hat{\mathfrak{L}}_1$ *are* T. I. *sets in* $\mathfrak{G}$. *If* $\mathscr{A}(\mathfrak{L})$ *is non empty, the subgroups* $\mathfrak{H}_1, \cdots, \mathfrak{H}_n$ *are a system of supporting subgroups for* $\hat{\mathfrak{L}}$ *and for* $\hat{\mathfrak{L}}_1$.

The purpose of Chapter IV is to provide proofs for these two theorems.

15. A Partition of $\pi(\mathfrak{G})$

We partition $\pi(\mathfrak{G})$ into four subsets, some of which may be empty:

$\pi_1 = \{p \mid$ A S_p-subgroup of $\mathfrak{G}$ is a non identity cyclic group.$\}$

$\pi_2 = \{p \mid$ 1. A S_p-subgroup of $\mathfrak{G}$ is non cyclic.
2. $\mathfrak{G}$ does not contain an elementary subgroup of order p^3.$\}$

$\pi_3 = \{p \mid$ 1. $\mathfrak{G}$ contains an elementary subgroup of order p^3.
2. If $\mathfrak{P}$ is a S_p-subgroup of $\mathfrak{G}$, then $И(\mathfrak{P})$ contains a non identity subgroup.$\}$

$\pi_4 = \{p \mid$ 1. $\mathfrak{G}$ contains an elementary subgroup of order p^3.
2. If $\mathfrak{P}$ is a S_p-subgroup of $\mathfrak{G}$, then $И(\mathfrak{P})$ contains only $\langle 1 \rangle$.$\}$

It is immediate that the sets partition $\pi(\mathfrak{G})$. The purpose of Lemma 8.4 (i) is that condition 2 defining π_2 is equivalent to the statement that $\mathscr{SCN}_3(\mathfrak{P})$ is empty if $\mathfrak{P}$ is a S_p-subgroup of $\mathfrak{G}$. Lemma 8.5 implies that $3 \notin \pi_1 \cup \pi_2$.

16. Lemmas about Commutators

Following P. Hall [19], we adopt the notation $\gamma\mathfrak{A}\mathfrak{B} = [\mathfrak{A}, \mathfrak{B}]$, $\gamma^{n+1}\mathfrak{A}\mathfrak{B}^{n+1} = [\gamma^n\mathfrak{A}\mathfrak{B}^n, \mathfrak{B}]$, $n = 1, 2, \cdots$, and $\gamma^2\mathfrak{A}\mathfrak{B}\mathfrak{C} = [\mathfrak{A}, \mathfrak{B}, \mathfrak{C}]$.

If $\mathfrak{X}$ is a group, $\mathscr{NA}(\mathfrak{X})$ denotes the set of normal abelian subgroups of $\mathfrak{X}$.

The following lemmas parallel Lemma 5.6 of [27] and in the presence of (B) absorb much of the difficulty of the proof of Theorem 14.1.

LEMMA 16.1. *Let $\mathfrak{P}$ be a S_p-subgroup of $\mathfrak{G}$ and $\mathfrak{A}$ an element of $\mathscr{NA}(\mathfrak{P})$. If $\mathfrak{F}$ is a subgroup of $\mathfrak{G}$ such that*
(i) *$\langle \mathfrak{A}, \mathfrak{F} \rangle$ is a p-group,*
(ii) *$\mathfrak{F}$ centralizes some element of $Z(\mathfrak{P}) \cap \mathfrak{A}^\#$,*
then $\gamma^3\mathfrak{F}\mathfrak{A}^3 = \langle 1 \rangle$.

Proof. Let $Z \in C(\mathfrak{F}) \cap Z(\mathfrak{P}) \cap \mathfrak{A}^\#$, and let $\mathfrak{C} = C(Z)$. By Lemma 7.2 (1) we have $\mathfrak{A} \subseteq O_{p',p}(\mathfrak{C}) = \mathfrak{H}$. As $\mathfrak{P}$ is a S_p-subgroup of $\mathfrak{C}$, $\mathfrak{P}_1 = \mathfrak{P} \cap \mathfrak{H}$ is a S_p-subgroup of $\mathfrak{H}$. Since $\mathfrak{A} \lhd \mathfrak{P}$, so also $\mathfrak{A} \lhd \mathfrak{P}_1$, and since $\mathfrak{A}$ is abelian, we see that $\gamma^2\mathfrak{H}\mathfrak{A}^2 \subseteq O_{p'}(\mathfrak{C})$. Since $\mathfrak{H} \lhd \mathfrak{C}$, we have $\gamma\mathfrak{F}\mathfrak{A} \subseteq \mathfrak{H}$ and so $\gamma^3\mathfrak{F}\mathfrak{A}^3 \subseteq O_{p'}(\mathfrak{C})$. Since $\langle \mathfrak{A}, \mathfrak{F} \rangle$ is assumed to be a p-group, the lemma follows.

If $\mathfrak{P}$ is a non cyclic p-group, we define $\mathscr{U}(\mathfrak{P})$ as follows: in case $Z(\mathfrak{P})$ is non cyclic, $\mathscr{U}(\mathfrak{P})$ consists of all subgroups of $Z(\mathfrak{P})$ of type (p, p); in case $Z(\mathfrak{P})$ is cyclic, $\mathscr{U}(\mathfrak{P})$ consists of all normal abelian subgroups of $\mathfrak{P}$ of type (p, p).

LEMMA 16.2. *Let $\mathfrak{P}$ be a non cyclic S_p-subgroup of $\mathfrak{G}$, $\mathfrak{A} \in \mathscr{NA}(\mathfrak{P})$,* *and let* $\mathfrak{F}$ *be a subgroup such that*

(i) $\langle \mathfrak{A}, \mathfrak{F} \rangle$ *is a p-group,*

(ii) $\mathfrak{A}$ *contains a subgroup* $\mathfrak{B}$ *of* $\mathscr{U}(\mathfrak{P})$ *such that* $\mathfrak{B}_0 = C_{\mathfrak{B}}(\mathfrak{F}) \neq \langle 1 \rangle$.

If $p \geqq 5$, *then* $\gamma^4 \mathfrak{F}\mathfrak{A}^4 = \langle 1 \rangle$, *while if* $p = 3$, *then* $\gamma^6 \mathfrak{F}\mathfrak{A}^6 = \langle 1 \rangle$. *Also, if* $\mathfrak{A}_1 = \mathfrak{A} \cap Z(\mathfrak{P})$ *and* $p \geqq 5$, *then* $\gamma^3 \mathfrak{F}\mathfrak{A}_1^3 = \langle 1 \rangle$.

Proof. If $\mathfrak{B}_0 \subseteqq Z(\mathfrak{P})$, the lemma follows from Lemma 16.1. If $\mathfrak{B}_0 \not\subseteqq Z(\mathfrak{P})$, then $\mathfrak{P}_0 = C_{\mathfrak{P}}(\mathfrak{B}_0)$ is of index p in $\mathfrak{P}$ so is of index at most p in a suitable S_p-subgroup $\mathfrak{P}^*$ of $C(\mathfrak{B}_0) = \mathfrak{C}$. In particular, $\mathfrak{P}_0 \lhd \mathfrak{P}^*$.

Let $\mathfrak{H} = O_{p',p}(\mathfrak{C})$, $\mathfrak{P}_1^* = \mathfrak{P}^* \cap \mathfrak{H}$, and $\mathfrak{P}_1 = \mathfrak{P}_0 \cap \mathfrak{H}$. Since $\mathfrak{P}_0 \lhd \mathfrak{P}^*$, so also $\mathfrak{P}_1 \lhd \mathfrak{P}_1^*$. Hence $\gamma \mathfrak{P}_1^* \mathfrak{A} \subseteqq \mathfrak{P}_0 \cap \mathfrak{H} \subseteqq \mathfrak{P}_1$, and so $\gamma^3 \mathfrak{P}_1^* \mathfrak{A}^3 = \langle 1 \rangle$, $\mathfrak{A}$ being in $\mathscr{N}\mathscr{A}(\mathfrak{P}_0)$. If $p \geqq 5$, we conclude from (B) that $\mathfrak{A} \subseteqq \mathfrak{H}$, and so $\gamma^3 \mathfrak{H}\mathfrak{A}^3 \subseteqq O_{p'}(\mathfrak{C})$. Since $\gamma \mathfrak{F}\mathfrak{A} \subseteqq \mathfrak{H}$, the lemma follows in this case. (Since $\mathfrak{P}_0$ centralizes $\mathfrak{A}_1$, we have $\gamma^3 \mathfrak{F}\mathfrak{A}_1^3 = \langle 1 \rangle$.)

Suppose now that $p = 3$. If $\mathfrak{P}_1^* = \mathfrak{P}_1$, then $\gamma^2 \mathfrak{P}_1^* \mathfrak{A}^2 = \langle 1 \rangle$, and so by (B), $\mathfrak{A} \subseteqq \mathfrak{H}$ and the lemma follows. If $\mathfrak{P}_1^* \neq \mathfrak{P}_1$, then $\mathfrak{P}^* = \mathfrak{P}_0 \mathfrak{P}_1^*$, since $|\mathfrak{P}^* : \mathfrak{P}_0| = p$. In this case, letting $\bar{\mathfrak{A}} = \mathfrak{A}\mathfrak{H}/\mathfrak{H}$, $\bar{\mathfrak{P}}^* = \mathfrak{P}^*\mathfrak{H}/\mathfrak{H}$, we see that $\bar{\mathfrak{A}} \in \mathscr{N}\mathscr{A}(\bar{\mathfrak{P}}^*)$ and so $\bar{\mathfrak{A}} \subseteqq O_{p',p}(\mathfrak{C}/\mathfrak{H})$, that is, $\mathfrak{A} \subseteqq O_{p',p,p',p}(\mathfrak{C}) = \mathfrak{K}$. Hence, $\gamma \mathfrak{F}\mathfrak{A} \subseteqq \mathfrak{K}$ and since $\bar{\mathfrak{A}} \lhd \bar{\mathfrak{P}}^*$, we see that $\gamma^3 \mathfrak{F}\mathfrak{A}^3 \subseteqq O_{p',p,p'}(\mathfrak{C})$, and so $\gamma^3 \mathfrak{F}\mathfrak{A}^3 \subseteqq \mathfrak{H}$. Continuing, we see that $\gamma^4 \mathfrak{F}\mathfrak{A}^4 \subseteqq O_{p'}(\mathfrak{C})\mathfrak{P}_1$ and so $\gamma^6 \mathfrak{F}\mathfrak{A}^6 \subseteqq O_{p'}(\mathfrak{C})$, from which the lemma follows.

LEMMA 16.3. *Let* $\mathfrak{P}$ *be a* S_3*-subgroup of* $\mathfrak{G}$ *and let* $\mathfrak{E} \in \mathscr{U}(\mathfrak{P})$. *Let* $\mathfrak{F}$ *be a subgroup of* $\mathfrak{G}$ *such that*

(i) $\langle \mathfrak{F}, \mathfrak{E} \rangle$ *is a 3-group.*

(ii) $\mathfrak{E}_1 = C_{\mathfrak{E}}(\mathfrak{F}) \neq \langle 1 \rangle$.

If $\gamma^2 \mathfrak{F}\mathfrak{E}^2 \neq \langle 1 \rangle$, *then* $\gamma^2 \mathfrak{F}\mathfrak{E}^2 = \mathfrak{E}_1$, *and* $\mathfrak{E}_1 = \Omega_1(Z(\mathfrak{P}))$.

Proof. First suppose $\mathfrak{E}_1 \subseteqq Z(\mathfrak{P})$. Let $\mathfrak{H} = C(\mathfrak{E}_1) \supseteqq \langle \mathfrak{P}, \mathfrak{F} \rangle$. Since $\mathfrak{P}$ is a S_3-subgroup of $\mathfrak{H}$, (B) implies that $\mathfrak{E} \subseteqq O_{3',3}(\mathfrak{H})$. Setting $\mathfrak{P}_1 = O_{3',3}(\mathfrak{H}) \cap \mathfrak{P}$, we have $O_{3',3}(\mathfrak{H}) = O_{3'}(\mathfrak{H})\mathfrak{P}_1$. If $\mathfrak{E} \subseteqq Z(\mathfrak{P})$, then $\mathfrak{E} \subseteqq Z(\mathfrak{P}_1)$ and so $\gamma^2 \mathfrak{F}\mathfrak{E}^2 \subseteqq O_{3'}(\mathfrak{H}) \cap \langle \mathfrak{F}, \mathfrak{E} \rangle = \langle 1 \rangle$, since $\langle \mathfrak{F}, \mathfrak{E} \rangle$ is a 3-group. If $\mathfrak{E} \not\subseteqq Z(\mathfrak{P})$, then the definition of $\mathscr{U}(\mathfrak{P})$ implies that $\gamma^2 \mathfrak{F}\mathfrak{E}^2 \subseteqq \mathfrak{E}_1 O_{3'}(\mathfrak{H})$, so if $\gamma^2 \mathfrak{F}\mathfrak{E}^2 \neq \langle 1 \rangle$, we must have $\gamma^2 \mathfrak{F}\mathfrak{E}^2 = H^{-1}\mathfrak{E}_1 H$ for suitable H in $O_{3'}(\mathfrak{H})$. By definition of $\mathfrak{H}$ it follows that $H^{-1}\mathfrak{E}_1 H = \mathfrak{E}_1$.

We can suppose now that $\mathfrak{E}_1 \not\subseteqq Z(\mathfrak{P})$. In this case, the definition of $\mathscr{U}(\mathfrak{P})$ implies that $\mathfrak{E} = \langle \mathfrak{D}, \mathfrak{E}_1 \rangle$, where $\mathfrak{D} = \Omega_1(Z(\mathfrak{P}))$. Let $\mathfrak{P}_0 = C_{\mathfrak{P}}(\mathfrak{E}_1)$ and let $\mathfrak{P}^*$ be a S_3-subgroup of $\mathfrak{H} = C(\mathfrak{E}_1)$ containing $\mathfrak{P}_0$ and let $\mathfrak{P}_0^* = \mathfrak{P}^* \cap O_{3',3}(\mathfrak{H})$. Since $\mathfrak{P}_0$ is of index at most 3 in $\mathfrak{P}^*$ and since $\mathfrak{P}_0$ centralizes $\mathfrak{E}$, we have $\gamma^2 \mathfrak{P}^* \mathfrak{E}^2 = \langle 1 \rangle$, and so $\mathfrak{E} \subseteqq \mathfrak{P}_0^*$. If $\mathfrak{P}_0^* \subseteqq \mathfrak{P}_0$, it follows that $\gamma^2 \mathfrak{F}\mathfrak{E}^2 \subseteqq O_{3'}(\mathfrak{H}) \cap \langle \mathfrak{E}, \mathfrak{F} \rangle = \langle 1 \rangle$ and we are done. Hence, we can suppose that $\mathfrak{P}_0^* \not\subseteqq \mathfrak{P}_0$. In this case, it follows that $\mathfrak{P}^* = \mathfrak{P}_0 \mathfrak{P}_0^*$, since $|\mathfrak{P}^* : \mathfrak{P}_0| = 3$. We also have $D(\mathfrak{P}_0^*) \subseteq \mathfrak{P}_0$, and so $\mathfrak{E} \subseteq C_{\mathfrak{P}_0^*}(D(\mathfrak{P}_0^*)) = \mathfrak{F}$. If $\mathfrak{F} \subseteq \mathfrak{P}_0$, we have $\mathfrak{E} \subseteq Z(\mathfrak{F})$, and since $Z(\mathfrak{F})$ char $\mathfrak{F}$ char $\mathfrak{P}_0^*$, it follows that $\gamma^2\mathfrak{J}\mathfrak{E}^2 \subseteq O_{3'}(\mathfrak{H}) \cap \langle \mathfrak{E}, \mathfrak{J} \rangle = \langle 1 \rangle$ and we are done. We can therefore suppose that $\mathfrak{E} \not\subseteq Z(\mathfrak{F})$. Choose E in $\mathfrak{F} - C_{\mathfrak{F}}(\mathfrak{E})$. Since $\mathfrak{P}^*$ centralizes $\mathfrak{E}_1$ it follows that E does not centralize $\mathfrak{D} = \langle D \rangle$. Consider $[D, E] = F \neq 1$. Now $\mathfrak{E} \subseteq Z(\mathfrak{P}_0) \lhd \mathfrak{P}^*$, and so $F \in Z(\mathfrak{P}_0)$. On the other hand, F lies in $D(\mathfrak{P}_0^*)$ since both E and D are in $\mathfrak{P}_0^*$. Since $E \in \mathfrak{F}$, it follows that E centralizes F. Since $\langle \mathfrak{P}_0, E \rangle = \mathfrak{P}^*$, it follows that F is in $Z(\mathfrak{P}^*)$. But F is of order 3 and $\mathfrak{E}_1 = \Omega_1(Z(\mathfrak{P}^*))$, since $Z(\mathfrak{P}^*)$ is cyclic. It follows that $\langle F \rangle = \mathfrak{E}_1$, and so E normalizes $\mathfrak{E}$ and with respect to the basis (D, F) of $\mathfrak{E}$ has the matrix $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$. On the other hand, $\mathfrak{P}$ possesses an element which normalizes $\mathfrak{E}$ and with respect to the basis (D, F) has the matrix $\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$. Since these two matrices generate a group of even order, we have the desired contradiction which completes the proof of this lemma.

17. A Domination Theorem and Some Consequences

In view of other applications, Theorem 17.1 is proved in greater generality than is required for this paper.

Let $\mathfrak{P}$ be a S_p-subgroup of the minimal simple group $\mathfrak{X}$ and let $\mathfrak{A}$ be an element of $\mathscr{S}\mathscr{C}\mathscr{N}(\mathfrak{P})$. Let q be a prime different from p.

THEOREM 17.1. *Let $\mathfrak{Q}, \mathfrak{Q}_1$ be maximal elements of $\mathscr{M}(\mathfrak{A}; q)$.*

(i) *Suppose that $\mathfrak{Q}$ is not conjugate to $\mathfrak{Q}_1$ by any element of $C_{\mathfrak{X}}(\mathfrak{A})$. Then for each element A in $\mathfrak{A}^{\#}$, either $C_{\mathfrak{Q}}(A) = 1$ or $C_{\mathfrak{Q}_1}(A) = 1$.*

(ii) *If $\mathfrak{A} \in \mathscr{S}\mathscr{C}\mathscr{N}_3(\mathfrak{P})$, then $\mathfrak{Q}$ and $\mathfrak{Q}_1$ are conjugate by an element of $C(\mathfrak{A})$.*

Proof. The proof of (i) proceeds by a series of reductions. If $\mathfrak{A} = 1$, the theorem is vacuously true, so we may assume $\mathfrak{A} \neq 1$.

Choose Z in $Z(\mathfrak{P})$, and let $\mathfrak{Q}^*$ be any element of $\mathscr{M}(\mathfrak{A}; q)$ which is centralized by Z. By Lemmas 7.4 and 7.8, if $\mathfrak{L}$ is any proper subgroup of $\mathfrak{X}$ containing $\mathfrak{A}\mathfrak{Q}^*$, then $\mathfrak{Q}^* \subseteq O_{p'}(\mathfrak{L})$.

Now let $\mathfrak{Q}^*$ denote any element of $\mathscr{M}(\mathfrak{A}; q)$ and let $\mathfrak{L}$ be a proper subgroup of $\mathfrak{X}$ containing $\mathfrak{A}\mathfrak{Q}^*$. We will show that $\mathfrak{Q}^* \subseteq O_{p'}(\mathfrak{L})$. First, suppose $Z(\mathfrak{P})$ is non cyclic. Then $\mathfrak{Q}^* = \langle C_{\mathfrak{Q}^*}(Z) \mid Z \in Z(\mathfrak{P})^{\#} \rangle$, so by the preceding paragraph, $\mathfrak{Q}^* \subseteq O_{p'}(\mathfrak{L})$. We can suppose that $Z(\mathfrak{P})$ is cyclic. Let Z be an element of $Z(\mathfrak{P})$ of order p. We only need to show that $[\mathfrak{Q}^*, Z] \subseteq O_{p'}(\mathfrak{L})$, by the preceding paragraph. Replacing $\mathfrak{Q}^*$ by $[\mathfrak{Q}^*, Z]$, we may suppose that $\mathfrak{Q}^* = [\mathfrak{Q}^*, Z]$. Furthermore, we may suppose that $\mathfrak{A}$ acts irreducibly on $\mathfrak{Q}^*/D(\mathfrak{Q}^*)$.

Suppose $Z \in O_{p',p}(\mathfrak{L})$. Then $\mathfrak{Q}^* = [\mathfrak{Q}^*, Z] \subseteq O_{p',p}(\mathfrak{L}) \cap \mathfrak{Q}^* \subseteq O_{p'}(\mathfrak{L})$ and we are done. If $\mathfrak{A}$ is cyclic, then Z is necessarily in $O_{p',p}(\mathfrak{L})$, since $\mathfrak{A} \cap O_{p',p}(\mathfrak{L}) \neq 1$. Thus, we can suppose that $\mathfrak{A}$ is non cyclic.

Let $\mathfrak{A}_1 = C_{\mathfrak{A}}(\mathfrak{Q}^*) = C_{\mathfrak{A}}(\mathfrak{Q}^*/D(\mathfrak{Q}^*))$, so that $\mathfrak{A}/\mathfrak{A}_1$ is cyclic and $Z \notin \mathfrak{A}_1$. We now choose W of order p in $\mathfrak{A}_1$ such that $\langle Z, W\rangle \lhd \mathfrak{P}$.

Suppose by way of contradiction that $\mathfrak{Q}^* \not\subseteq O_{p'}(\mathfrak{L})$. Then by Lemma 7.8, we can find a subgroup $\mathfrak{K}$ of $\mathfrak{A}C(\mathfrak{A}_1)$ which contains $\mathfrak{A}\mathfrak{Q}^*$ and such that $\mathfrak{Q}^* \not\subseteq O_{p'}(\mathfrak{K})$. In particular, $\mathfrak{Q}^* \not\subseteq O_{p'}(C(W))$. Thus, we suppose without loss of generality that $\mathfrak{L} = C(W)$. Let $\mathfrak{P}^*$ be a S_p-subgroup of $\mathfrak{L}$ which contains $\tilde{\mathfrak{P}} = \mathfrak{P} \cap C(W)$. If $\mathfrak{P}^* = \tilde{\mathfrak{P}}$, then $Z \in O_{p',p}(\mathfrak{L})$, by Lemma 1.2.3 of [21], which is not the case. Hence, $\tilde{\mathfrak{P}}$ is of index p in $\mathfrak{P}^*$. Clearly, $\mathfrak{A} \subseteq \tilde{\mathfrak{P}}$ and $Z \in Z(\tilde{\mathfrak{P}})$. Hence, $[\mathfrak{P}^*, Z] \subseteq Z(\tilde{\mathfrak{P}}) \subseteq \mathfrak{A}$. Let $\mathfrak{P}_1^* = \mathfrak{P}^* \cap O_{p',p}(\mathfrak{L})$ so that $\mathfrak{P}_1^*$ is a S_p-subgroup of $O_{p',p}(\mathfrak{L})$. Then $[\mathfrak{P}_1^*, \langle Z\rangle, \mathfrak{Q}^*] \subseteq [\mathfrak{A}, \mathfrak{Q}^*] \cap O_{p',p}(\mathfrak{L}) \subseteq \mathfrak{Q}^* \cap O_{p',p}(\mathfrak{L})$, so that $[\mathfrak{P}_1^*, \langle Z\rangle, \mathfrak{Q}^*] \subseteq O_{p'}(\mathfrak{L})$. Let $\mathfrak{B} = O_{p',p}(\mathfrak{L})/O_{p'}(\mathfrak{L})$ and let $\mathfrak{B}_1 = C_{\mathfrak{B}}(\mathfrak{Q}^*)$. The preceding containment implies that $[\mathfrak{B}, \langle Z\rangle] \subseteq \mathfrak{B}_1$. Let $\mathfrak{B}_2 = N_{\mathfrak{B}}(\mathfrak{B}_1)$. Then Z acts trivially on the $\mathfrak{Q}^*\mathfrak{A}$-admissible group $\mathfrak{B}_2/\mathfrak{B}_1$. Hence, so does $[\langle Z\rangle, \mathfrak{Q}^*] = \mathfrak{Q}^*$, that is, $\mathfrak{B}_2 \subseteq \mathfrak{B}_1$. This implies that $\mathfrak{B} = \mathfrak{B}_1$ is centralized by $\mathfrak{Q}^*$ so $\mathfrak{Q}^* \subseteq O_{p'}(\mathfrak{L})$. We have succeeded in showing that if $\mathfrak{Q}^*$ is in $\mathsf{V}(\mathfrak{A}; q)$ and $\mathfrak{L}$ is any proper subgroup of $\mathfrak{X}$ containing $\mathfrak{A}\mathfrak{Q}^*$, then $\mathfrak{Q}^* \subseteq O_{p'}(\mathfrak{L})$.

Now let $\mathscr{O}_1, \cdots, \mathscr{O}_t$ be the orbits under conjugation by $C(\mathfrak{A})$ of the maximal elements of $\mathsf{V}(\mathfrak{A}; q)$. We next show that if $\mathfrak{Q} \in \mathscr{O}_i$, $\mathfrak{Q}_1 \in \mathscr{O}_j$ and $i \neq j$, then $\mathfrak{Q} \cap \mathfrak{Q}_1 = 1$. Suppose false and $i, j, \mathfrak{Q}, \mathfrak{Q}_1$ are chosen so that $|\mathfrak{Q} \cap \mathfrak{Q}_1|$ is maximal. Let $\mathfrak{Q}^* = N_{\mathfrak{Q}}(\mathfrak{Q} \cap \mathfrak{Q}_1)$ and $\mathfrak{Q}_1^* = N_{\mathfrak{Q}_1}(\mathfrak{Q} \cap \mathfrak{Q}_1)$. Since $\mathfrak{Q}$ and $\mathfrak{Q}_1$ are distinct maximal elements of $\mathsf{V}(\mathfrak{A}; q)$, $\mathfrak{Q} \cap \mathfrak{Q}_1$ is a proper subgroup of both $\mathfrak{Q}^*$ and $\mathfrak{Q}_1^*$. Let $\mathfrak{L} = N(\mathfrak{Q} \cap \mathfrak{Q}_1)$. By the previous argument, $\langle \mathfrak{Q}^*, \mathfrak{Q}_1^*\rangle \subseteq O_{p'}(\mathfrak{L})$. Let $\mathfrak{R}$ be a S_q-subgroup of $O_{p'}(\mathfrak{L})$ containing $\mathfrak{Q}^*$ and permutable with $\mathfrak{A}$ and let $\mathfrak{R}_1$ be a S_q-subgroup of $O_{p'}(\mathfrak{L})$ containing $\mathfrak{Q}_1^*$ and permutable with $\mathfrak{A}$. The groups $\mathfrak{R}$ and $\mathfrak{R}_1$ are available by $D_{p,q}$ in $\mathfrak{A}O_{p'}(\mathfrak{L})$. By the conjugacy of Sylow systems, there is an element C in $O_{p'}(\mathfrak{L})\mathfrak{A}$ such that $\mathfrak{A}^C = \mathfrak{A}$ and $\mathfrak{R}^C = \mathfrak{R}_1$. As $\mathfrak{A}$ has a normal complement in $O_{p'}(\mathfrak{L})\mathfrak{A}$, it follows that C centralizes $\mathfrak{A}$. Let $\hat{\mathfrak{Q}}$ be a maximal element of $\mathsf{V}(\mathfrak{A}; q)$ containing $\mathfrak{R}_1$. Then $\hat{\mathfrak{Q}} \cap \mathfrak{Q}_1 \supseteq \mathfrak{Q}_1^* \supset \mathfrak{Q} \cap \mathfrak{Q}_1$, and so $\hat{\mathfrak{Q}} \in \mathscr{O}_j$. Also, $\hat{\mathfrak{Q}} \cap \mathfrak{Q}^C \supseteq \mathfrak{Q}^{*C} \supset (\mathfrak{Q} \cap \mathfrak{Q}_1)^C$ so that $\hat{\mathfrak{Q}} \in \mathscr{O}_i$ and $i = j$.

To complete the proof of (i), let $\mathfrak{Q}$, $\mathfrak{Q}_1$ be maximal elements of $\mathsf{V}(\mathfrak{A}; q)$ with $\mathfrak{Q} \in \mathscr{O}_i$, $\mathfrak{Q}_1 \in \mathscr{O}_j$. Suppose $A \in \mathfrak{A}^\#$ and $C_{\mathfrak{Q}}(A) \neq 1$, $C_{\mathfrak{Q}_1}(A) \neq 1$. Let $\mathfrak{L} = C(A)$, let $\mathfrak{R}$ be a S_q-subgroup of $O_{p'}(\mathfrak{L})$ containing $C_{\mathfrak{Q}}(A)$ and permutable with $\mathfrak{A}$, and let $\mathfrak{R}_1$ be a S_q-subgroup of $O_{p'}(\mathfrak{L})$ containing $C_{\mathfrak{Q}_1}(A)$ and permutable with $\mathfrak{A}$. Then $\mathfrak{R}^C = \mathfrak{R}_1$ for suitable C in $C(\mathfrak{A})$. Let $\mathfrak{Q}^*$ be a maximal element of $\mathsf{V}(\mathfrak{A}; q)$ containing $\mathfrak{R}_1$. Then $\mathfrak{Q}^* \cap \mathfrak{Q}_1 \supseteq C_{\mathfrak{Q}_1}(A) \neq 1$ so $\mathfrak{Q}^* \in \mathscr{O}_j$. Also, $\mathfrak{Q}^* \cap \mathfrak{Q}^C \supseteq (C_{\mathfrak{Q}}(A))^C \neq 1$ so $\mathfrak{Q}^* \in \mathscr{O}_i$ and $i = j$. This completes the proof of (i).

As for (ii), if $\mathfrak{A} \in \mathscr{SCN}_3(\mathfrak{P})$, then there is an element A in $\mathfrak{A}^\#$

such that $C_{\mathfrak{Q}}(A) \neq 1$ and $C_{\mathfrak{Q}_1}(A) \neq 1$. By (i), $\mathfrak{Q}$ and $\mathfrak{Q}_1$ are conjugate under $C(\mathfrak{A})$.

COROLLARY 17.1. *If $p \in \pi_3 \cup \pi_4$, $\mathfrak{P}$ is a S_p-subgroup of $\mathfrak{G}$ and $\mathfrak{A} \in \mathscr{SCN}_3(\mathfrak{P})$, then for each prime $q \neq p$ and each maximal element $\mathfrak{Q}$ of $\mathsf{N}(\mathfrak{A}; q)$, there is a S_p-subgroup of $N(\mathfrak{A})$ which normalizes $\mathfrak{Q}$.*

Proof. Let $G \in N(\mathfrak{A})$. Then $\mathfrak{Q}^G$ is a maximal element of $\mathsf{N}(\mathfrak{A}; q)$, since any two maximal elements of $\mathsf{N}(\mathfrak{A}; q)$ have the same order, so $\mathfrak{Q}^G = \mathfrak{Q}^C$ for suitable $C = C(G)$ in $C(\mathfrak{A})$. Hence, GC^{-1} normalizes $\mathfrak{Q}$. Setting $\mathfrak{J} = N(\mathfrak{Q}) \cap N(\mathfrak{A})$, we see that $\mathfrak{J}$ covers $N(\mathfrak{A})/C(\mathfrak{A})$, that is, $N(\mathfrak{Q})$ dominates $\mathfrak{A}$. Now we have $\mathfrak{J}C(\mathfrak{A}) = N(\mathfrak{A})$ and $\mathfrak{J}$ contains $\mathfrak{A}$. Since $C(\mathfrak{A}) = \mathfrak{A} \times \mathfrak{D}$ where $\mathfrak{D}$ is a p'-group, we have $N(\mathfrak{A}) = \mathfrak{J}C(\mathfrak{A}) = \mathfrak{J}\mathfrak{A}\mathfrak{D} = \mathfrak{J}\mathfrak{D}$, and $\mathfrak{J}$ contains a S_p-subgroup of $N(\mathfrak{A})$ as required.

COROLLARY 17.2. *If $p \in \pi_3 \cup \pi_4$, $\mathfrak{P}$ is a S_p-subgroup of $\mathfrak{G}$, $\mathfrak{A} \in \mathscr{SCN}_3(\mathfrak{P})$ and q is a prime different from p, then $\mathfrak{P}$ normalizes some maximal element $\mathfrak{Q}$ of $\mathsf{N}(\mathfrak{A}; q)$. Furthermore if G is an element of $\mathfrak{G}$ such that $\mathfrak{A}^G \subseteq \mathfrak{P}$, then $\mathfrak{A}^G = \mathfrak{A}^N$ for some N in $N(\mathfrak{Q})$.*

Proof. Applying Corollary 17.1, some S_p-subgroup $\mathfrak{P}^*$ of $N(\mathfrak{A})$ normalizes $\mathfrak{Q}_1$, a maximal element of $\mathsf{N}(\mathfrak{A}; q)$. Since $\mathfrak{P}$ is a S_p-subgroup of $N(\mathfrak{A})$, $\mathfrak{P} = \mathfrak{P}^{*X}$ for suitable X in $N(\mathfrak{A})$, and so $\mathfrak{P}$ normalizes $\mathfrak{Q} = \mathfrak{Q}_1^X$, a maximal element of $\mathsf{N}(\mathfrak{A}; q)$.

Suppose $G \in \mathfrak{G}$ and $\mathfrak{A}^G \subseteq \mathfrak{P}$. Then $\mathfrak{A}^G$ normalizes $\mathfrak{Q}$ since $\mathfrak{P}$ does, so $\mathfrak{A}$ normalizes $\mathfrak{Q}^{G^{-1}}$. Now $\mathfrak{Q}^{G^{-1}}$ is a maximal element of $\mathsf{N}(\mathfrak{A}; q)$ since any two such have the same order. Hence, $\mathfrak{Q}^{G^{-1}} = \mathfrak{Q}^C$ for some C in $C(\mathfrak{A})$, by Theorem 17.1 and so $CG = N$ is in $N(\mathfrak{Q})$. Since $\mathfrak{A}^N = \mathfrak{A}^{CG} = \mathfrak{A}^G$, the corollary follows.

COROLLARY 17.3. *If $p \in \pi_4$, $\mathfrak{P}$ is a S_p-subgroup of $\mathfrak{G}$ and $\mathfrak{A} \in \mathscr{SCN}_3(\mathfrak{P})$, then $\mathsf{N}(\mathfrak{A})$ is trivial.*

Proof. Otherwise, $\mathsf{N}(\mathfrak{A}; q)$ is non trivial for some prime $q \neq p$, by Lemma 7.4, and so $\mathsf{N}(\mathfrak{P}; q)$ is non trivial, contrary to the definition of π_4.

Hypothesis 17.1.

(i) *$p \in \pi_3$, $\mathfrak{P}$ is a S_p-subgroup of $\mathfrak{G}$ and $\mathfrak{A} \in \mathscr{SCN}_3(\mathfrak{P})$.*

(ii) *q is a prime different from p, $\mathsf{N}(\mathfrak{A}; q)$ is non trivial and $\mathfrak{Q}$ is a maximal element of $\mathsf{N}(\mathfrak{A}; q)$ normalized by $\mathfrak{P}$.*

REMARK. Most of Hypothesis 17.1 is notation. The hypothesis is that $p \in \pi_3$, for in this case a prime q is available such that (ii) is satisfied. Furthermore, we let

$$\mathfrak{B} = V(ccl_{\mathfrak{G}}(\mathfrak{A}); \mathfrak{P}), \quad \mathfrak{N} = N(\mathfrak{Q}), \quad \text{and} \quad \mathfrak{N}_1 = N(Z(\mathfrak{B})).$$

LEMMA 17.1. *Under Hypothesis* 17.1 *if* $G \in \mathfrak{G}$ and $\mathfrak{A}^G \subseteq \mathfrak{P}$, *then* $\mathfrak{A}^G = \mathfrak{A}^N$ *for some element* N *in* $N(\mathfrak{Q}) \cap N(\mathfrak{B})$.

Proof. By Corollary 17.2, $\mathfrak{A}^G = \mathfrak{A}^X$ for some element X in $\mathfrak{N}$. Since $\mathfrak{N}$ is solvable, Lemma 7.2 (1) and Corollary 17.2 imply that $\mathfrak{N} = O_{p'}(\mathfrak{N}) \cdot N_{\mathfrak{N}}(\mathfrak{B})$, so we can write $X = N_1N$ where $N_1 \in O_{p'}(\mathfrak{N})$ and $N \in N_{\mathfrak{N}}(\mathfrak{B})$. Now $\mathfrak{A}^N$ is in $\mathfrak{B}$, so in particular is in $\mathfrak{P}$. Also $\mathfrak{A}^{N_1N} = \mathfrak{A}^X$ is in $\mathfrak{P}$. Hence, if A is in $\mathfrak{A}$, then $A^{-N} \cdot A^{N_1N} = [A, N_1]^N$ is in $\mathfrak{P}$, and in particular is a p-element. Since $[A, N_1]$ is a p'-element, we see that $N_1 \in C(\mathfrak{A})$. Hence $\mathfrak{A}^{N_1N} = \mathfrak{A}^N$, and the lemma follows.

LEMMA 17.2. *Under Hypothesis* 17.1, $\mathfrak{N}_1 = O^p(\mathfrak{N}_1)$.

Proof. Since $Z(\mathfrak{B})$ char $\mathfrak{B}$, and $\mathfrak{B}$ is weakly closed in $\mathfrak{P}$, $\mathfrak{N}_1$ contains $N(\mathfrak{P})$, so Theorem 14.4.1 of [12] applies. We consider the double cosets $\mathfrak{N}_1X\mathfrak{P}$ distinct from $\mathfrak{N}_1$. Denote by $\mathfrak{K}(X)$ the kernel of the homomorphism of $\mathfrak{P}$ onto the permutation representation of $\mathfrak{P}$ on the cosets of $\mathfrak{N}_1$ in $\mathfrak{N}_1X\mathfrak{P}$. Let $P = P(X)$ be an element of $\mathfrak{P}$ such that $\mathfrak{K}(X)P$ is of order p in $Z(\mathfrak{P}/\mathfrak{K}(X))$.

Suppose we are able to show that P can always be taken to lie in $\mathfrak{A}$. In this case, we have $[U, P, P] = 1$ for all U in $\mathfrak{P}$. Since $p \geqq 3$ and $\mathfrak{G}$ is simple we conclude from Theorem 14.4.1 in [12] that $\mathfrak{N}_1 = O^p(\mathfrak{N}_1)$.

We now proceed to show that P can always be taken to lie in $\mathfrak{A}$. The only restriction on the element X is that $X \notin \mathfrak{N}_1$, that is, we must have $\mathfrak{K}(X) \neq \mathfrak{P}$.

Now $\mathfrak{A} \subseteq \mathfrak{B}$, so $Z(\mathfrak{B})$ centralizes $\mathfrak{A}$. Since $\mathfrak{A} \in \mathscr{SCN}(\mathfrak{P})$, we have $Z(\mathfrak{B}) \subseteq \mathfrak{A}$. It follows that $\mathfrak{N}_1$ contains $C(\mathfrak{A})$.

It suffices to show that $\mathfrak{A} \not\subseteq \mathfrak{K}(X)$. For if $\mathfrak{A} \not\subseteq \mathfrak{K}(X)$, choose A in $\mathfrak{A}$ so that $(\mathfrak{K}(X) \cap \mathfrak{A})A$ is of order p in $Z(\mathfrak{P}/\mathfrak{K}(X) \cap \mathfrak{A})$. It follows that $\mathfrak{K}(X)A$ is of order p in $Z(\mathfrak{P}/\mathfrak{K}(X))$.

Suppose by way of contradiction that $\mathfrak{A} \subseteq \mathfrak{K}(X)$. Then $\mathfrak{A} \subseteq \mathfrak{N}_1^X$ so $\mathfrak{A} \subseteq \mathfrak{P}^{*X}$ for $\mathfrak{P}^*$ a suitable S_p-subgroup of $\mathfrak{N}_1$. But $\mathfrak{P}^* = \mathfrak{P}^Y$ for some Y in $\mathfrak{N}_1$. Setting $X_1 = YX$, we have $\mathfrak{N}_1X\mathfrak{P} = \mathfrak{N}_1X_1\mathfrak{P}$ and $\mathfrak{A} \subseteq \mathfrak{P}^{X_1}$. Hence, $\mathfrak{A}^{X_1^{-1}} \subseteq \mathfrak{P}$, so by Lemma 17.1, $\mathfrak{A}^{X_1^{-1}} = \mathfrak{A}^W$ for some W in $\mathfrak{N} \cap N(\mathfrak{B})$. Since $N(\mathfrak{B}) \subseteq \mathfrak{N}_1$, we have $\mathfrak{A} = \mathfrak{A}^{WX_1}$ and $W \in \mathfrak{N} \cap \mathfrak{N}_1$. Let $WX_1 = X_2$. Since $W \in \mathfrak{N}_1$, we have $\mathfrak{N}_1X_1\mathfrak{P} = \mathfrak{N}_1X_2\mathfrak{P}$.

Since X_2 normalizes $\mathfrak{A}$, $\mathfrak{A}$ normalizes $\mathfrak{Q}^{X_2^{-1}}$. By Theorem 17.1,

$\mathfrak{Q}^{X_2^{-1}} = \mathfrak{Q}^C$ for some C in $C(\mathfrak{A})$. Hence $X_2^{-1}C^{-1} = X_3^{-1}$ (this defines X_3) normalizes $\mathfrak{Q}$. Since X_2 and C normalize $\mathfrak{A}$, we see that $X_3 \in \mathfrak{N} \cap N(\mathfrak{A})$. Since C centralizes $\mathfrak{A}$ and $C(\mathfrak{A}) \subseteq \mathfrak{N}_1$, we have $\mathfrak{N}_1 X_2 \mathfrak{P} = \mathfrak{N}_1 X_3 \mathfrak{P}$.

We now write $X_3 = X_3' X_4$, where $X_3' \in \mathfrak{N} \cap N(\mathfrak{B})$ and $X_4 \in O_{p'}(\mathfrak{N})$. Such a representation is possible since $X_3 \in \mathfrak{N}$. Consider the equation $X_4 = X_3'^{-1} X_3$. Since $N(\mathfrak{B}) \subseteq \mathfrak{N}_1$, we have $\mathfrak{N}_1 X_3 \mathfrak{P} = \mathfrak{N}_1 X_4 \mathfrak{P}$. If $A \in \mathfrak{A}$, then $[A, X_4^{-1}]$ is a p'-element since $X_4 \in O_{p'}(\mathfrak{N})$. But $[A, X_3^{-1} X_3'] = [A, X_3'][A, X_3^{-1}]^{X_3'}$, an identity holding in all groups. Since $X_3' \in N(\mathfrak{B})$, $[A, X_3'] \in \mathfrak{B}$. Since $X_3 \in N(\mathfrak{A})$, $[A, X_3^{-1}] \in \mathfrak{A} \subseteq \mathfrak{B}$, so $[A, X_3^{-1}]^{X_3'} \in \mathfrak{B}$, a p-group. Hence

$$[A, X_4^{-1}] = [A, X_3^{-1} X_3'] = 1 .$$

Since A is an arbitrary element of $\mathfrak{A}$, we have $X_4 \in C(\mathfrak{A}) \subseteq \mathfrak{N}_1$. Now, however, we have

$$\mathfrak{N}_1 X \mathfrak{P} = \mathfrak{N}_1 X_1 \mathfrak{P} = \mathfrak{N}_1 X_2 \mathfrak{P} = \mathfrak{N}_1 X_3 \mathfrak{P} = \mathfrak{N}_1 X_4 \mathfrak{P} = \mathfrak{N}_1 ,$$

so $X \in \mathfrak{N}_1$, contrary to assumption.

LEMMA 17.3. *Under Hypothesis* 17.1, $\mathfrak{N}_1 = O_{p'}(\mathfrak{N}_1) \cdot (\mathfrak{N}_1 \cap \mathfrak{N})$, *and* $\mathfrak{N} = O^p(\mathfrak{N})$.

Proof. We must show that $\mathfrak{N}$ contains at least one element from each coset $\mathfrak{C} = O_{p'}(\mathfrak{N}_1) W$, $W \in \mathfrak{N}_1$, from which the lemma follows directly.

Let $\mathfrak{H} = \mathfrak{P} \cap O_{p',p}(\mathfrak{N}_1)$, $\mathfrak{K} = N_{\mathfrak{N}_1}(\mathfrak{H})$, and $C(\mathfrak{A}) = \mathfrak{A} \times \mathfrak{D}$, $\mathfrak{D}$ being a p'-group. Notice that $\mathfrak{D} \subseteq O_{p'}(\mathfrak{N}_1)$ by Lemma 7.4 together with $C(\mathfrak{A}) \subseteq \mathfrak{N}_1$. (This was the point in taking $Z(\mathfrak{B})$ in place of $\mathfrak{B}$.)

By Sylow's theorem, $\mathfrak{K}$ contains some element of $\mathfrak{C}$, so suppose $W \in \mathfrak{K}$. Since $\mathfrak{A}$ is contained in $\mathfrak{H}$ by Lemma 7.2 (1), we have $\mathfrak{A}^W \subseteq \mathfrak{H} \subseteq \mathfrak{P}$, and $\mathfrak{A}^W$ normalizes $\mathfrak{Q}$. Hence, $\mathfrak{A}$ normalizes $\mathfrak{Q}^{W^{-1}}$ and by Theorem 17.1, $\mathfrak{Q}^{W^{-1}} = \mathfrak{Q}^S$ for some S in $C(\mathfrak{A})$. Write $S = AD$ where $A \in \mathfrak{A}$, $D \in \mathfrak{D}$, so that $\mathfrak{Q}^S = \mathfrak{Q}^D$, since $\mathfrak{A}$ normalizes $\mathfrak{Q}$. Hence, DW normalizes $\mathfrak{Q}$. But $DW \in \mathfrak{C}$, since $D \in O_{p'}(\mathfrak{N}_1)$, so $DW \in \mathfrak{N} \cap \mathfrak{N}_1$ and $\mathfrak{N}$ contains an element of $\mathfrak{C}$.

LEMMA 17.4. *Under Hypothesis* 17.1, *if* $\mathfrak{H}$ *is a subgroup of* $\mathfrak{P}$ *which contains* $\mathfrak{A}$, *then* $N(\mathfrak{H}) \subseteq \mathfrak{N}_1$.

Proof. Let $G \in N(\mathfrak{H})$. Since $\mathfrak{P}$ normalizes $\mathfrak{Q}$, so does $\mathfrak{H}$. Hence, $\mathfrak{H}^G$ normalizes $\mathfrak{Q}^G$. But $\mathfrak{H}^G = \mathfrak{H}$ and $\mathfrak{H}$ contains $\mathfrak{A}$, so $\mathfrak{A}$ normalizes $\mathfrak{Q}^G$. By Theorem 17.1, $\mathfrak{Q}^G = \mathfrak{Q}^C$ for some C in $C(\mathfrak{A})$. Let $GC^{-1} = N \in \mathfrak{N}$. Now $N = N_1 N_2$ where $N_1 \in O_{p'}(\mathfrak{N})$ and $N_2 \in \mathfrak{N} \cap \mathfrak{N}_1$. Consider the equation $GC^{-1}N_2^{-1} = N_1$. Let $Z \in Z(\mathfrak{B})$.

We have $GC^{-1}N_2^{-1}ZN_2CG^{-1} = GZ_1G^{-1}$, where $Z_1 = Z^{N_2C}$ is in $Z(\mathfrak{B})$;

hence, $Z^{-1}GC^{-1}N_2^{-1}ZN_2CG^{-1} = [Z, N_2CG^{-1}] = Z^{-1}GZ_1G^{-1}$ is a p-element of $\mathfrak{H}$, since $Z_1 \in \boldsymbol{Z}(\mathfrak{B}) \subseteq \mathfrak{A} \subseteq \mathfrak{H}$, so that $GZ_1G^{-1} \in G\mathfrak{H}G^{-1} = \mathfrak{H}$. But $Z^{-1}N_1ZN_1^{-1} \in \boldsymbol{O}_{p'}(\mathfrak{N})$. Hence, $[Z, N_2CG^{-1}] = [Z, N_1^{-1}] = 1$. Since Z is an arbitrary element of $\boldsymbol{Z}(\mathfrak{B})$, it follows that N_1 centralizes $\boldsymbol{Z}(\mathfrak{B})$, so N_1 is contained in $\mathfrak{N}_1$. But now the elements N_1, N_2 and C normalize $\boldsymbol{Z}(\mathfrak{B})$. Since $G = N_1N_2C$, the lemma follows.

LEMMA 17.5. *Under Hypothesis* 17.1, *if* $\mathfrak{K}$ *is a proper subgroup of* $\mathfrak{G}$ *which contains* $\mathfrak{P}$, *then* $\mathfrak{B} \subseteq \boldsymbol{O}_{p',p}(\mathfrak{K})$.

Proof. If $\mathfrak{P}_1 = \mathfrak{P} \cap \boldsymbol{O}_{p',p}(\mathfrak{K})$, and $\mathfrak{K}_1 = \boldsymbol{N}_{\mathfrak{K}}(\mathfrak{P}_1)$, it suffices to show that $\mathfrak{B} \subseteq \mathfrak{P}_1$. By Lemma 7.2 (1), we have $\mathfrak{A} \subseteq \mathfrak{P}_1$, and so by Lemma 17.4, $\mathfrak{K}_1 \subseteq \mathfrak{N}_1$. Thus it suffices to show that $\mathfrak{B} \subseteq \boldsymbol{O}_{p',p}(\mathfrak{N}_1)$. By Lemma 17.3, it suffices to show that $\mathfrak{B} \subseteq \boldsymbol{O}_{p',p}(\mathfrak{N})$. However, this last containment follows from Lemma 7.2 (1) and Corollary 17.1.

LEMMA 17.6. *Under Hypothesis* 17.1, *if* $\mathfrak{K}$ *is a proper subgroup of* $\mathfrak{G}$, *and* $\mathfrak{P}_0$ *is a* S_p*-subgroup of* $\mathfrak{K}$, *then* $\boldsymbol{V}(ccl_{\mathfrak{G}}(\mathfrak{A}); \mathfrak{P}_0) \subseteq \boldsymbol{O}_{p',p}(\mathfrak{K})$.

Proof. Suppose false, and that $\mathfrak{K}$ is chosen to maximize $|\mathfrak{K}|_p$ and with this restriction to minimize $|\mathfrak{K}|_{p'}$. Let $\mathfrak{P}_1 = \mathfrak{P}_0 \cap \boldsymbol{O}_{p',p}(\mathfrak{K})$. By minimality of $|\mathfrak{K}|_{p'}$ we have $\mathfrak{P}_1 \lhd \mathfrak{K}$. By maximality of $|\mathfrak{K}|_p$, $\mathfrak{P}_0$ is a S_p-subgroup of $\boldsymbol{N}(\mathfrak{P}_1)$. We assume without loss of generality that $\mathfrak{P}_0 \subseteq \mathfrak{P}$. In this case, Lemma 7.9 implies that $\mathfrak{A} \subseteq \mathfrak{P}_1$. Since $\mathfrak{A} \subseteq \mathfrak{P}_1$, by Lemma 17.4 we have $\mathfrak{K} \subseteq \mathfrak{N}_1$; by Lemma 17.5, $\mathfrak{B} \subseteq \boldsymbol{O}_{p',p}(\mathfrak{N}_1)$, so in particular, $\boldsymbol{V}(ccl_{\mathfrak{G}}(\mathfrak{A}); \mathfrak{P}_0) \subseteq \mathfrak{P}_1$, as required.

18. Configurations

The necessary E-theorems emerge from a study of the following objects:

(C)

1. A proper subgroup $\mathfrak{K}$ of $\mathfrak{G}$.
2. A S_p-subgroup $\mathfrak{P}$ of $\mathfrak{K}$.
3. A p-subgroup $\mathfrak{A}$ of $\mathfrak{G}$.
4. $\mathfrak{B} = \boldsymbol{V}(ccl_{\mathfrak{G}}(\mathfrak{A}); \mathfrak{P})$, $\mathfrak{M} = [\boldsymbol{O}_{p',p,p'}(\mathfrak{K}), \mathfrak{B}]$, $\mathfrak{W} = \boldsymbol{O}_{p',p}(\mathfrak{K})/\boldsymbol{O}_{p'}(\mathfrak{K})$.

DEFINITION 18.1. A *configuration* is any 6-tuple $(\mathfrak{K}, \mathfrak{P}, \mathfrak{A}; \mathfrak{B}, \mathfrak{M}, \mathfrak{W})$ satisfying (C). The semi-colon indicates that $\mathfrak{B}$, $\mathfrak{M}$, $\mathfrak{W}$ are determined when $\mathfrak{K}$, $\mathfrak{P}$, $\mathfrak{A}$ are given.

DEFINITION 18.2.

$$\mathscr{C}(p) = \{\mathfrak{A} \mid$$

(i) $\mathfrak{A}$ is a p-subgroup of $\mathfrak{G}$.
(ii) for every configuration $(\mathfrak{K}, \mathfrak{P}, \mathfrak{A}; \mathfrak{B}, \mathfrak{M}, \mathfrak{W})$,
 (a) $\mathfrak{M}$ centralizes $\boldsymbol{Z}(\mathfrak{W})$.
 (b) If $\boldsymbol{Z}(\mathfrak{W})$ is cyclic, then $\mathfrak{M}$ centralizes $\boldsymbol{Z}_2(\mathfrak{W})/\boldsymbol{Z}(\mathfrak{W})$.}

DEFINITION 18.3.

$$\mathscr{SCN}_m(p) = \cup \mathscr{SCN}_m(\mathfrak{P}) , \qquad \mathscr{U}(p) = \cup \mathscr{U}(\mathfrak{P}) ,$$

$\mathfrak{P}$ ranging over all S_p-subgroups of $\mathfrak{G}$ in both unions.

LEMMA 18.1. *If $p \geqq 5$, then $\mathscr{U}(p) \cup \mathscr{SCN}_3(p) \subseteq \mathscr{C}(p)$.*

Proof. Let $\mathfrak{A} \in \mathscr{U}(p) \cup \mathscr{SCN}_3(p)$, and let $(\mathfrak{K}, \mathfrak{P}, \mathfrak{A}; \mathfrak{B}, \mathfrak{M}, \mathfrak{W})$ be a configuration. Suppose by way of contradiction that either $\mathfrak{M}$ fails to centralize $\boldsymbol{Z}(\mathfrak{W})$ or $\boldsymbol{Z}(\mathfrak{W})$ is cyclic and $\mathfrak{M}$ fails to centralize $\boldsymbol{Z}_2(\mathfrak{W})/\boldsymbol{Z}(\mathfrak{W})$. Since $\boldsymbol{O}_{p',p}(\mathfrak{K})$ centralizes both $\boldsymbol{Z}(\mathfrak{W})$ and $\boldsymbol{Z}_2(\mathfrak{W})/\boldsymbol{Z}(\mathfrak{W})$, it follows that some element of $\mathfrak{M}$ induces a non identity p'-automorphism of either $\boldsymbol{Z}(\mathfrak{W})$ or $\boldsymbol{Z}_2(\mathfrak{W})/\boldsymbol{Z}(\mathfrak{W})$, so in both cases, some non identity p'-automorphism is induced on $\boldsymbol{Z}_2(\mathfrak{W})$ by some element of $\mathfrak{M}$. By 3.6, some non identity p'-automorphism is induced on $\Omega_1(\boldsymbol{Z}_2(\mathfrak{W})) = \mathfrak{W}_1$ by some element of $\mathfrak{M}$. Let $\mathfrak{W}_0 = \Omega_1(\boldsymbol{Z}(\mathfrak{W})) \subseteq \mathfrak{W}_1$ and let $\mathfrak{W}_{-1} = \langle 1 \rangle$.

Let $\mathfrak{M}_0 = \ker(\boldsymbol{O}_{p',p,p'}(\mathfrak{K}) \rightarrow \operatorname{Aut}\mathfrak{W}_0)$, $\mathfrak{M}_1 = \ker(\boldsymbol{O}_{p',p,p'}(\mathfrak{K}) \rightarrow \operatorname{Aut}(\mathfrak{W}_1/\mathfrak{W}_0))$. By definition of $\mathfrak{M}$, $\mathfrak{M}$ is contained in $\mathfrak{M}_i$ if and only if $\mathfrak{B}$ acts trivially on $\boldsymbol{O}_{p',p,p'}(\mathfrak{K})/\mathfrak{M}_i$, $i = 0$ or 1. Suppose that $\mathfrak{B}$ does not act trivially on $\boldsymbol{O}_{p',p,p'}(\mathfrak{K})/\mathfrak{M}_i$. Let $\mathfrak{B} = \mathfrak{A}^G$ be a conjugate of $\mathfrak{A}$ which lies in $\mathfrak{B}$ and does not centralize $\boldsymbol{O}_{p',p,p'}(\mathfrak{K})/\mathfrak{M}_i$ ($\mathfrak{B}$ depends on i). In accordance with 3.11, we find a subgroup $\mathfrak{N}_i$ of $\boldsymbol{O}_{p',p,p'}(\mathfrak{K})$ such that $\mathfrak{N}_i/\mathfrak{M}_i$ is a special q-group, is $\mathfrak{B}$-admissible, and such that $\mathfrak{B}$ acts trivially on $\mathfrak{D}_i/\mathfrak{M}_i$, irreducibly and non trivially on $\mathfrak{N}_i/\mathfrak{D}_i$, where $\mathfrak{D}_i = \boldsymbol{D}(\mathfrak{N}_i \bmod \mathfrak{M}_i)$. Let $\mathfrak{B}_i = \ker(\mathfrak{B} \rightarrow \operatorname{Aut}(\mathfrak{N}_i/\mathfrak{M}_i))$, so that $\mathfrak{B}_i$ acts trivially on $\mathfrak{N}_i/\mathfrak{M}_i$ and $\mathfrak{B}/\mathfrak{B}_i$ is cyclic.

Let $\mathfrak{X}_i$ be a subgroup of $\mathfrak{W}_i/\mathfrak{W}_{i-1}$ of minimal order subject to being $\mathfrak{B}\mathfrak{N}_i$-admissible and not centralized by $\mathfrak{N}_i$. The minimal nature of $\mathfrak{X}_i$ guarantees that $\mathfrak{B}_i$ acts trivially on $\mathfrak{X}_i$. If $\mathfrak{B}_i B_i$ is a generator for $\mathfrak{B}/\mathfrak{B}_i$, then (B) guarantees that the minimal polynomial of B_i on $\mathfrak{X}_i$ is $(x-1)^r$ where $r = r_i = |\mathfrak{B} : \mathfrak{B}_i|$.

Suppose $i = 0$. Since $\mathfrak{X}_0$ is a p-group, while $\boldsymbol{O}_{p'}(\mathfrak{K})$ is a p'-group, we can find a p-subgroup $\mathfrak{H}_0$ of $\mathfrak{K}$ such that $\mathfrak{H}_0$ and $\mathfrak{X}_0$ are incident, and such that $\mathfrak{H}_0$ is $\mathfrak{B}$-admissible. In particular, $\mathfrak{B}_0$ centralizes $\mathfrak{H}_0$. Let $\mathfrak{P}^*$ be a S_p-subgroup of $\boldsymbol{N}(\mathfrak{B})$, so that $\mathfrak{P}^*$ is a S_p-subgroup of $\mathfrak{G}$. If $\mathfrak{B}_0 \cap \boldsymbol{Z}(\mathfrak{P}^*)^\#$ is non empty, we apply Lemma 16.1 and have a contradiction. Otherwise, Lemma 16.2 gives the contradiction.

We can now suppose that $\boldsymbol{Z}(\mathfrak{W})$ is cyclic. In particular, $\mathfrak{W}_0$ is of order p. Since $\mathfrak{X}_1$ is of the form $\mathfrak{Y}_1/\mathfrak{W}_0$ where $\mathfrak{Y}_1$ is a suitable subgroup of $\mathfrak{W}_1$, we can find a p-subgroup $\mathfrak{H}_1$ of $\mathfrak{K}$ incident with $\mathfrak{Y}_1$ and $\mathfrak{B}$-admissible.

Choose B in $\mathfrak{B}_1$. Since $\mathfrak{B}_1$ centralizes $\mathfrak{Y}_1/\mathfrak{W}_0$ and since $\mathfrak{W}_0$ is of order p, it follows that $\mathfrak{H}_2 = C_{\mathfrak{H}_1}(B)$ is of index 1 or p in $\mathfrak{H}_1$. If $\mathfrak{B}_1 \cap Z(\mathfrak{P}^*)^\#$ is non empty, application of Lemma 16.1 gives $\gamma^3\mathfrak{H}_2\mathfrak{B}^3 = \langle 1 \rangle$, and so $\gamma^4\mathfrak{H}_1\mathfrak{B}^4 = \langle 1 \rangle$, the desired contradiction. Otherwise, we apply Lemma 16.2 and conclude that $\gamma^4\mathfrak{H}_2\mathfrak{B}^4 = \langle 1 \rangle$, and so $\gamma^5\mathfrak{H}_1\mathfrak{B}^5 = \langle 1 \rangle$, from which we conclude that $|\mathfrak{B} : \mathfrak{B}_1| = 5$. In this case, however, setting $\mathfrak{Z} = Z(\mathfrak{P}^*) \cap \mathfrak{B}$, we have $\mathfrak{B} = \langle \mathfrak{B}_1, \mathfrak{Z} \rangle$, and so the extra push comes from Lemma 16.2 which asserts that $\gamma^3\mathfrak{H}_2\mathfrak{Z}^3 = \langle 1 \rangle$, and so $\gamma^4\mathfrak{H}_1\mathfrak{Z}^4 = \langle 1 \rangle$, completing the proof of the lemma.

19. An E-theorem

It is convenient to assume Burnside's theorem that groups of order $p^a q^b$ are solvable. The interested reader can reword certain of the lemmas to yield a proof of the main theorem of this paper without using the theorem of Burnside.

If $p, q \in \pi_3 \cup \pi_4$, we write $p \sim q$ provided $\mathfrak{G}$ contains elementary subgroups $\mathfrak{E}$ and $\mathfrak{F}$ of orders p^3 and q^3 respectively such that $\langle \mathfrak{E}, \mathfrak{F} \rangle \subset \mathfrak{G}$. Clearly, $\sim$ is reflexive and symmetric.

Hypothesis 19.1.

(i) *$p \in \pi_3 \cup \pi_4$, $q \in \pi(\mathfrak{G})$ and $p \neq q$.*

(ii) *A S_p-subgroup $\mathfrak{P}$ of $\mathfrak{G}$ centralizes every element of $\mathsf{M}(\mathfrak{P}; q)$.*

LEMMA 19.1. *Under Hypothesis* 19.1, *if $\mathfrak{B} \in \mathscr{U}(p)$, then $\mathfrak{B}$ centralizes every element of $\mathsf{M}(\mathfrak{B}; q)$.*

Proof. Suppose false, and that $\mathfrak{Q}$ is an element of $\mathsf{M}(\mathfrak{B}; q)$ minimal with respect to $\gamma\mathfrak{B}\mathfrak{Q} \neq \langle 1 \rangle$. From 3.11 we conclude that $\mathfrak{B}$ centralizes $D(\mathfrak{Q})$ and acts irreducibly and non trivially on $\mathfrak{Q}/D(\mathfrak{Q})$, so in particular, $\mathfrak{Q} = \gamma\mathfrak{Q}\mathfrak{B}$ and $\mathfrak{B}_0 = \ker(\mathfrak{B} \to \mathrm{Aut}\,\mathfrak{Q}) \neq \langle 1 \rangle$. Let $\mathfrak{C} = C(\mathfrak{B}_0)$, let $\mathfrak{P}$ be a S_p-subgroup of $N(\mathfrak{B})$, and let $\mathfrak{P}_0 = C(\mathfrak{B}) \cap \mathfrak{P}$. Since $\mathfrak{B} \in \mathscr{U}(p)$, $\mathfrak{P}_0$ is of index at most p in a S_p-subgroup $\mathfrak{P}_1$ of $\mathfrak{C}$, and so $\mathfrak{P}_0 \lhd \mathfrak{P}_1$. Hence $\gamma\mathfrak{P}_1\mathfrak{B} \subseteq \mathfrak{P}_0$. Since $\mathfrak{P}_0$ centralizes $\mathfrak{B}$, we have $\gamma^2\mathfrak{P}_1\mathfrak{B}^2 = \langle 1 \rangle$, so $\mathfrak{B} \subseteq O_{p',p}(\mathfrak{C}) = \mathfrak{K}$. Let $\mathfrak{L} = O_{p'}(\mathfrak{C})$. Since $\mathfrak{B} \subseteq \mathfrak{K} \lhd \mathfrak{C}$, $\gamma\mathfrak{Q}\mathfrak{B} \subseteq \mathfrak{K}$, so $\gamma\mathfrak{Q}\mathfrak{B} \subseteq \mathfrak{K} \cap \mathfrak{Q} \subseteq \mathfrak{L}$. Since $\mathfrak{Q} = \gamma\mathfrak{Q}\mathfrak{B}$, we have $\mathfrak{Q} \subseteq \mathfrak{L}$.

By Lemma 8.9, $\mathfrak{B}$ is contained in an element $\mathfrak{A}$ of $\mathscr{S}\mathscr{C}\mathscr{N}_3(\mathfrak{P})$. Since $\mathfrak{A}$ centralizes $\mathfrak{B}$, we have $\mathfrak{A} \subseteq \mathfrak{P}_0$. Let $\mathfrak{D} = \mathfrak{A}\mathfrak{L}$, and observe that $\mathfrak{L}$ is a normal p-complement for $\mathfrak{A}$ in $\mathfrak{D}$. By Hypothesis 19.1 (ii), Theorem 17.1, Corollary 17.2, and $D_{p,q}$ in $\mathfrak{D}$, $\mathfrak{A}$ centralizes a S_q-subgroup of $\mathfrak{D}$, so $\mathfrak{D}$ satisfies $E_{p,q}^n$ and every p, q-subgroup of $\mathfrak{D}$

is nilpotent. But $\mathfrak{Q}\mathfrak{B} \subseteq \mathfrak{D}$, and $\mathfrak{Q} = \gamma\mathfrak{Q}\mathfrak{B} \neq \langle 1 \rangle$, so $\mathfrak{Q}\mathfrak{B}$ is not nilpotent. This contradiction completes the proof of this lemma.

Hypothesis 19.2.
(i) *$p, q \in \pi_3 \cup \pi_4$ and $p \neq q$.*
(ii) $p \sim q$.
(iii) *A S_p-subgroup $\mathfrak{P}$ of $\mathfrak{G}$ centralizes every element of $И(\mathfrak{P}; q)$ and a S_q-subgroup $\mathfrak{Q}$ of $\mathfrak{G}$ centralizes every element of $И(\mathfrak{Q}; p)$.*

THEOREM 19.1. *Under Hypothesis* 19.2, *$\mathfrak{G}$ satisfies $E^n_{p,q}$.*

We proceed by way of contradiction, proving the theorem by a sequence of lemmas. Lemmas 19.2 through 19.14 all assume Hypothesis 19.2. We remark that Hypothesis 19.2 is symmetric in p and q.

LEMMA 19.2. $\langle \mathfrak{A}, \mathfrak{B} \rangle \quad \mathfrak{G}$, *whenever $\mathfrak{A} \in \mathscr{U}(p)$ and $\mathfrak{B} \in \mathscr{U}(q)$.*

Proof. Suppose $\langle \mathfrak{A}, \mathfrak{B} \rangle = \mathfrak{K} \subset \mathfrak{G}$, where $\mathfrak{A} \in \mathscr{U}(p)$, $\mathfrak{B} \in \mathscr{U}(q)$, and $\mathfrak{K}$ is minimal. By $D_{p,q}$ in $\mathfrak{K}$, it follows that $\mathfrak{K}$ is a p, q-group.

By the previous lemma $\mathfrak{A}^{\mathfrak{K}}$ centralizes $O_q(\mathfrak{K})$ and $\mathfrak{B}^{\mathfrak{K}}$ centralizes $O_p(\mathfrak{K})$. Since $\mathfrak{B}$ and $\mathfrak{A}$ are abelian, $\mathfrak{K}/\mathfrak{A}^{\mathfrak{K}}$ and $\mathfrak{K}/\mathfrak{B}^{\mathfrak{K}}$ are abelian, so $\mathfrak{K}'$ centralizes $O_p(\mathfrak{K}) \times O_q(\mathfrak{K}) = F(\mathfrak{K})$. Hence $\mathfrak{K}' \subseteq Z(F(\mathfrak{K}))$ by 3.3.

Let $\mathscr{C}$ be a chief series for $\mathfrak{K}$, one of whose terms is $\mathfrak{K}'$, and let $\mathfrak{C}/\mathfrak{D}$ be a chief factor of $\mathscr{C}$. If $\mathfrak{K}' \subseteq \mathfrak{D}$, then $\mathfrak{C}/\mathfrak{D}$ is obviously a central factor. If $\mathfrak{C} \subseteq \mathfrak{K}'$, and $\mathfrak{C}/\mathfrak{D}$ is a p-group, then $\mathfrak{B}^{\mathfrak{K}}$ centralizes $\mathfrak{C}/\mathfrak{D}$, and since $\mathfrak{C}/\mathfrak{D}$ is a chief factor, $\mathfrak{A}$ must also centralize $\mathfrak{C}/\mathfrak{D}$, so $\mathfrak{C}/\mathfrak{D}$ is a central factor. The situation being symmetric in p and q, every chief factor of $\mathscr{C}$ is central, and so $\mathfrak{K}$ is nilpotent, and $\mathfrak{K} = \mathfrak{A} \times \mathfrak{B}$.

Let $\mathfrak{N} = N(\mathfrak{A})$, let $\mathfrak{M}$ be a $S_{p,q}$-subgroup of $\mathfrak{N}$ with Sylow system $\mathfrak{P}, \mathfrak{Q}$, $\mathfrak{P}$ being a S_p-subgroup of $\mathfrak{G}$, since $\mathfrak{A} \in \mathscr{U}(p)$. By $D_{p,q}$ in $\mathfrak{N}$, $\mathfrak{B}_1 = \mathfrak{B}^N \subseteq \mathfrak{Q}$ for suitable N in $\mathfrak{N}$. Let $\mathfrak{M}_1$ be a maximal p, q-subgroup of $\mathfrak{G}$ containing $\mathfrak{M}$, with Sylow system $\mathfrak{P}$, $\mathfrak{Q}_1$ where $\mathfrak{Q} \subseteq \mathfrak{Q}_1$. Let $\mathfrak{Q}$ be a S_q-subgroup of $\mathfrak{G}$ containing $\mathfrak{Q}_1$. Finally, let $\mathfrak{B} = V(ccl_{\mathfrak{G}}(\mathfrak{B}); \mathfrak{Q}_1)$ and observe that $\mathfrak{B}_1 \subseteq \mathfrak{B}$. By Hypothesis 19.2, $\mathfrak{P}$ centralizes $O_q(\mathfrak{M}_1)$ By the previous lemma, $\mathfrak{B}$ centralizes $O_p(\mathfrak{M}_1)$.

We next show that $\mathfrak{B} \subseteq F(\mathfrak{M}_1)$. Consider $O_{q,p}(\mathfrak{M}_1)$, and let $\mathfrak{P}_1 = \mathfrak{P} \cap O_{q,p}(\mathfrak{M}_1)$. Since $\mathfrak{P}$ centralizes $O_q(\mathfrak{M}_1)$, so does $\mathfrak{P}_1$, so $O_{q,p}(\mathfrak{M}_1) = \mathfrak{P}_1 \times O_q(\mathfrak{M}_1)$ is nilpotent. But now $\mathfrak{B}$ centralizes $\mathfrak{P}_1$, and so Lemma 1.2.3 of [21] implies that $\mathfrak{B} \subseteq O_q(\mathfrak{M}_1)$. It follows that $\mathfrak{B} \lhd \mathfrak{M}_1$. Since $\mathfrak{B}$ is weakly closed in a S_q-subgroup of $\mathfrak{M}_1$, it follows that $\mathfrak{M}_1$ is a $S_{p,q}$-subgroup of $\mathfrak{G}$.

Again, $\mathfrak{P}$ centralizes $O_q(\mathfrak{M}_1)$, and now $\mathfrak{Q}_1$ centralizes $O_p(\mathfrak{M}_1)$ both assertions being a consequence of Hypothesis 19.2 (iii). It follows readily that every chief factor of $\mathfrak{M}_1$ is central, and so $\mathfrak{M}_1$ is nilpotent. Since we are advancing by way of contradiction, we accept this lemma.

LEMMA 19.3. *If $\mathfrak{A} \in \mathscr{U}(p)$, then either $C(\mathfrak{A})$ is a q'-group or a S_q-subgroup $\mathfrak{E}$ of $C(\mathfrak{A})$ is of order q, and $\mathfrak{E}$ has the property that it does not centralize any $\mathfrak{B} \in \mathscr{U}(q)$.*

Proof. Let $\mathfrak{E}$ be a S_q-subgroup of $C(\mathfrak{A})$, and suppose $\mathfrak{E} \neq \langle 1 \rangle$. By Lemma 19.2, no element of $\mathfrak{E}^\#$ centralizes any $\mathfrak{B} \in \mathscr{U}(q)$. Let $\mathfrak{Q}$ be a S_q-subgroup of $\mathfrak{G}$ containing $\mathfrak{E}$ and let $\mathfrak{B} \in \mathscr{U}(\mathfrak{Q})$. Then $C_{\mathfrak{Q}}(\mathfrak{B})$ is of index 1 or q in $\mathfrak{Q}$ and is disjoint from $\mathfrak{E}$. $|\mathfrak{E}| = q$ follows.

Lemmas 19.2 and 19.3 remain valid if p and q are interchanged throughout. In Lemmas 19.4 through 19.14 this symmetry is destroyed by the assumption that $p > q$ (which is not an assumption but a choice of notation).

We now define a family of subgroups of $\mathfrak{G}$, $\mathscr{F} = \mathscr{F}(p)$. First, $\mathscr{F}$ is the set theoretic union of the subfamilies $\mathscr{F}(\mathfrak{P})$, where $\mathfrak{P}$ ranges over the S_p-subgroups of $\mathfrak{G}$. Next, $\mathscr{F}(\mathfrak{P})$ is the set theoretic union of the subfamilies $\mathscr{F}(\mathfrak{A}; \mathfrak{P})$, where $\mathfrak{A}$ ranges through the elements of $\mathscr{S}\mathscr{C}\mathscr{N}_3(\mathfrak{P})$. We proceed to build up $\mathscr{F}(\mathfrak{A}; \mathfrak{P})$. Form $V(\mathfrak{A}) = V(ccl_{\mathfrak{G}}(\mathfrak{A}); \mathfrak{P})$. Consider the collection $\mathscr{K} = \mathscr{K}(\mathfrak{A}) = \mathscr{K}(\mathfrak{A}, q)$ of all p, q-subgroups $\mathfrak{K}$ of $\mathfrak{G}$ which have the following properties:

1. $\mathfrak{P} \subseteq \mathfrak{K}$.

(K) 2. $V(\mathfrak{A}) \subseteq O_{q,p}(\mathfrak{K})$.

3. Every characteristic abelian subgroup of $\mathfrak{P} \cap O_{q,p}(\mathfrak{K})$ is cyclic.

If $\mathscr{K}(\mathfrak{A}, q)$ is empty, we define $\mathscr{F}(\mathfrak{A}; \mathfrak{P})$ to consist of all subgroups of $\mathfrak{A}$ of type (p, p). If $\mathscr{K}(\mathfrak{A}, q)$ is non empty, we define $\mathscr{F}(\mathfrak{A}; \mathfrak{P})$ to consist of all subgroups of $\mathfrak{A}$ of type (p, p) together with all subgroups of $\mathfrak{P} \cap O_{q,p}(\mathfrak{K})$ of type (p, p) which contain $\Omega_1(Z(\mathfrak{P} \cap O_{q,p}(\mathfrak{K})))$, and $\mathfrak{K}$ ranges over $\mathscr{K}(\mathfrak{A}, q)$.

Notice that $\mathscr{F}(p)$ depends on q, too, but we write $\mathscr{F}(p)$ to emphasize that its elements are p-subgroups of $\mathfrak{G}$. The nature of $\mathscr{F}$ is somewhat limited by

LEMMA 19.4. *If $\mathfrak{A}_1, \mathfrak{A}_2 \in \mathscr{S}\mathscr{C}\mathscr{N}_3(\mathfrak{P})$, $\mathfrak{P}$ is a S_p-subgroup of $\mathfrak{G}$, $\mathscr{K}(\mathfrak{A}_1)$ and $\mathscr{K}(\mathfrak{A}_2)$ are non empty, and if $\mathfrak{K}_i \in \mathscr{K}(\mathfrak{A}_i)$, $i = 1, 2$, then $\mathfrak{P} \cap O_{q,p}(\mathfrak{K}_1) = \mathfrak{P} \cap O_{q,p}(\mathfrak{K}_2)$.*

Proof. Let $\mathfrak{P}_i = \mathfrak{P} \cap O_{q,p}(\mathfrak{K}_i)$, $i = 1, 2$. Then $\mathfrak{P}_i \lhd \mathfrak{P}$, $i = 1, 2$. From 3.5 and the definition of $\mathscr{F}(p)$, we have $cl(\mathfrak{P}_i) = 2$, $i = 1, 2$. Hence $\gamma^3\mathfrak{P}_1\mathfrak{P}_2^3 = \langle 1 \rangle$ and $\gamma^3\mathfrak{P}_2\mathfrak{P}_1^3 = \langle 1 \rangle$. From (B), we conclude that $\mathfrak{P}_2 \subseteq \mathfrak{P}_1$ and $\mathfrak{P}_1 \subseteq \mathfrak{P}_2$, as required.

Using Lemma 8.9 and Lemma 19.4, we arrive at an alternative definition of $\mathscr{F}(\mathfrak{P})$, $\mathfrak{P}$ being a S_p-subgroup of $\mathfrak{G}$. If $\mathscr{K}(\mathfrak{A})$ is empty for all $\mathfrak{A} \in \mathscr{SCN}_3(\mathfrak{P})$, $\mathscr{T}(\mathfrak{P})$ is the set of all subgroups $\mathfrak{B}$ of $\mathfrak{P}$ of type (p, p) such that $\mathfrak{B}^{\mathfrak{P}}$ is abelian. If $\mathscr{K}(\mathfrak{A})$ is non empty for some $\mathfrak{A} \in \mathscr{SCN}_3(\mathfrak{P})$ and $\mathfrak{K} \in \mathscr{K}(\mathfrak{A})$, then $\mathscr{T}(\mathfrak{P})$ consists of all subgroups of type (p, p) in $O_{q,p}(\mathfrak{K}) \cap \mathfrak{P}$ which contain $\Omega_1(Z(O_{q,p}(\mathfrak{K}) \cap \mathfrak{P}))$, together with all subgroups $\mathfrak{B}$ of $O_{q,p}(\mathfrak{K}) \cap \mathfrak{P}$ of type (p, p) such that $\mathfrak{B}^{\mathfrak{P}}$ is abelian. Here we are also using (B) to conclude that $O_{q,p}(\mathfrak{K}) \cap \mathfrak{P}$ contains every element of $\mathscr{SCN}(\mathfrak{P})$.

LEMMA 19.5. *Let $\mathfrak{K} \in \mathscr{K}(\mathfrak{A})$, where $\mathfrak{A} \in \mathscr{SCN}_3(\mathfrak{P})$ and $\mathfrak{P}$ is a S_p-subgroup of $\mathfrak{G}$. Let $\mathfrak{P}_0 = \mathfrak{P} \cap O_{q,p}(\mathfrak{K})$. If $\mathfrak{M}$ is any proper subgroup of $\mathfrak{G}$ containing $\mathfrak{P}$, then $\mathfrak{P}_0 O_{p'}(\mathfrak{M}) \lhd \mathfrak{M}$.*

Proof. Since $\gamma^3\mathfrak{P}\mathfrak{P}_0^3 = 1$, it follows from (B) that $\mathfrak{P}_0 \subseteq \mathfrak{P} \cap O_{p',p}(\mathfrak{M}) = \mathfrak{P}_1$, say. By Sylow's theorem, $\mathfrak{M} = O_{p'}(\mathfrak{M}) N_{\mathfrak{M}}(\mathfrak{P}_1)$, so it suffices to show that $\mathfrak{P}_0 \lhd N_{\mathfrak{M}}(\mathfrak{P}_1) = \mathfrak{N}$. Choose N in $\mathfrak{N}$. Then $[\mathfrak{P}_0^N, \mathfrak{P}_0, \mathfrak{P}_0, \mathfrak{P}_0] = 1$. Since $\mathfrak{P}_0 \subseteq \mathfrak{P}_1 \subseteq \mathfrak{P}^N \subseteq \mathfrak{K}^N$, it follows from (B) applied to $\mathfrak{K}^N$ that $\mathfrak{P}_0 \subseteq \mathfrak{P}_0^N$, so that $\mathfrak{P}_0 = \mathfrak{P}_0^N$, as required.

LEMMA 19.6. *Let $\mathfrak{K} \in \mathscr{K}(\mathfrak{A})$, $\mathfrak{A} \in \mathscr{SCN}_3(\mathfrak{P})$, $\mathfrak{P}$ being a S_p-subgroup of $\mathfrak{G}$, and let $\mathfrak{L}$ be a subgroup of index p in $\mathfrak{P}_0 = O_{q,p}(\mathfrak{K}) \cap \mathfrak{P}$. Then $\mathfrak{B} = V(ccl_{\mathfrak{G}}(\mathfrak{L}); \mathfrak{P}) \subseteq \mathfrak{P} \cap O_{q,p}(\mathfrak{K})$.*

Proof. Since $\mathscr{SCN}_3(\mathfrak{P})$ is non empty, (B) implies that $\mathfrak{L}$ is non abelian. Now $\Omega_1(Z(\mathfrak{P}))$ is of order p and is contained in $\mathfrak{L}$. By 3.5 $\mathfrak{L}/\Omega_1(Z(\mathfrak{P}))$ is abelian.

Let $\mathfrak{L}^G = \mathfrak{L}_1$ be a conjugate of $\mathfrak{L}$ contained in $\mathfrak{P}$, $G \in \mathfrak{G}$. First, suppose that $(\Omega_1(Z(\mathfrak{P})))^G = \mathfrak{Z}$ is contained in $\mathfrak{P}_0$. Then $C_{\mathfrak{P}_0}(\mathfrak{Z}) = \mathfrak{C}_1$ is of index 1 or p in $\mathfrak{P}_0$. Set $\mathfrak{C}_2 = C(\mathfrak{Z})$. By Lemma 19.5, with $\mathfrak{C}_2$ in the role of $\mathfrak{M}$, $\mathfrak{P}^G$ in the role of $\mathfrak{P}$, $\mathfrak{P}_0^G$ in the role of $\mathfrak{P}_0$, we see that $\gamma^3\mathfrak{C}_1\mathfrak{L}_1^3 = \langle 1 \rangle$, and it follows that $\gamma^4\mathfrak{P}_0\mathfrak{L}_1^4 = \langle 1 \rangle$, so by (B), $\mathfrak{L}_1 \subseteq \mathfrak{P}_0$. (Recall that $p \geqq 5$.)

Thus, if $\mathfrak{L}_1 \not\subseteq \mathfrak{P}_0$, but $\mathfrak{L}_1 \subseteq \mathfrak{P}$, then $\mathfrak{Z} \not\subseteq \mathfrak{P}_0$. But $\mathfrak{L}_1$ normalizes $\mathfrak{P}_0$, so $\mathfrak{P}_0 \cap \mathfrak{L}_1 \lhd \mathfrak{L}_1$. Since $\mathfrak{L}_1$ is of index p in $\mathfrak{P}_0^G$, any non cyclic normal subgroup of $\mathfrak{L}_1$ contains $\mathfrak{Z}$. Hence, $\mathfrak{P}_0 \cap \mathfrak{L}_1$ is cyclic and disjoint from $\mathfrak{Z}$. If now $\Omega_1(\mathfrak{P}_0)$ is extra special of order p^{2r+1}, we see that $\Omega_1(\mathfrak{L}_1)$ contains an extra special subgroup $\mathfrak{T}$ of order p^{2r-1} which is disjoint from $\mathfrak{P}_0$.

Consider now the configuration $(\mathfrak{K}, \mathfrak{P}, \mathfrak{L}; \mathfrak{B}, \mathfrak{M}, \mathfrak{W})$, and observe that $\mathfrak{W} \cong \mathfrak{P}_0$. $\mathfrak{T}$ is disjoint from $\mathfrak{P}_0$, so is faithfully represented on $\mathfrak{F} = O_{q,p,q}(\mathfrak{K})/O_{q,p}(\mathfrak{K})$, a q-group. Furthermore, $\mathfrak{F}$ is faithfully represented on $\Omega_1(\mathfrak{W})/\Omega_1(Z(\mathfrak{W}))$, which makes sense, since $O_{q,p}(\mathfrak{K})$ acts trivially on $\Omega_1(\mathfrak{W})/\Omega_1(Z(\mathfrak{W}))$. Let $\mathfrak{F}_1$ be the subgroup of $\mathfrak{F}$ which acts trivially on $\Omega_1(Z(\mathfrak{W}))$, which also makes sense, since $O_{q,p}(\mathfrak{K})$ acts trivially on $\Omega_1(Z(\mathfrak{W}))$. Then $\mathfrak{F}/\mathfrak{F}_1$ is cyclic and $\mathfrak{T}$ acts trivially on $\mathfrak{F}/\mathfrak{F}_1$ since $p > q$.

Since $\mathfrak{T}$ is a p-group, $\mathfrak{T}$ acts faithfully on $\mathfrak{F}_1$, so acts faithfully on $\mathfrak{F}_1/D(\mathfrak{F}_1)$. If $|\mathfrak{F}_1 : D(\mathfrak{F}_1)| = q^n$, then $|\mathfrak{T}|$ divides $(q^n-1)(q^{n-1}-1)\cdots(q-1)$, and so $|\mathfrak{T}| < q^n$, by Lemma 5.2.

On the other hand, $\mathfrak{F}_1$ acts faithfully on $\Omega_1(\mathfrak{W})/\Omega_1(Z(\mathfrak{W}))$, and trivially on $\Omega_1(Z(\mathfrak{W}))$, so $\mathfrak{F}_1$ is isomorphic to a subgroup of the symplectic group $Sp(2r, p)$. Hence, $|\mathfrak{F}_1|$ divides $|Sp(2r, p)|_{p'} = (p^{2r}-1)\cdots(p^2-1)$ [6], so by Lemma 5.2 (ii), $|\mathfrak{F}_1| < p^{2r-1}$. Combining this with the previous paragraph, we have $|\mathfrak{T}| = p^{2r-1} < q^n \leqq |\mathfrak{F}_1| < p^{2r-1}$, a contradiction, completing the proof of the lemma.

We can now translate this information about $\mathfrak{L}$ to the general p, q-subgroup of $\mathfrak{G}$. To do this, we let $\mathscr{L}(p)$ be the set theoretic union of sets $\mathscr{L}(\mathfrak{P})$, $\mathfrak{P}$ ranging over the S_p-subgroups of $\mathfrak{G}$. $\mathscr{L}(\mathfrak{P})$ is the set of all subgroups $\mathfrak{L}$ which can occur in the previous lemma. Formally, $\mathscr{L}(\mathfrak{P})$ is the set of all subgroups of index p in $\mathfrak{P} \cap O_{q,p}(\mathfrak{K})$, where $\mathfrak{K} \in \mathscr{K}(\mathfrak{A})$, and $\mathfrak{A} \in \mathscr{SCN}_3(\mathfrak{P})$.

LEMMA 19.7. *If $\mathfrak{L} \in \mathscr{L}(p)$ and $\mathfrak{H}$ is a p, q-subgroup of $\mathfrak{G}$, then* $\mathfrak{V}_1 = V(ccl_{\mathfrak{G}}(\mathfrak{L}); \mathfrak{H}) \subseteq O_{q,p}(\mathfrak{H})$.

Proof. Let $(\mathfrak{H}, \mathfrak{P}_1, \mathfrak{L}; \mathfrak{V}, \mathfrak{M}, \mathfrak{W})$ be a configuration. The lemma is clearly equivalent to the statement that $\mathfrak{V} \subseteq O_{q,p}(\mathfrak{H})$. Let $\mathfrak{P}_2$ be a S_p-subgroup of $\mathfrak{G}$ containing $\mathfrak{P}_1$ and let $\mathfrak{L}_1 = \mathfrak{L}^G$ be a conjugate of $\mathfrak{L}$ contained in $\mathfrak{P}_1$. Since $\mathfrak{L}_1 \in \mathscr{L}(p)$, we have $\mathfrak{L}_1 \in \mathscr{L}(\mathfrak{P}_3)$ for some S_p-subgroup $\mathfrak{P}_3$ of $\mathfrak{G}$. Now $\mathfrak{P}_3 = \mathfrak{P}_2^X$ for some X in $\mathfrak{G}$, and so $\mathfrak{L}_1^X \subseteq \mathfrak{P}_3$. By Lemma 19.6, we have $\gamma^3\mathfrak{P}_3(\mathfrak{L}_1^X)^3 = \langle 1\rangle$, and so $\gamma^3\mathfrak{P}_2\mathfrak{L}_1^3 = \langle 1\rangle$; in particular, $\gamma^3\mathfrak{P}_1\mathfrak{L}_1^3 = \langle 1\rangle$, so (B) and $p \geqq 5$ imply this lemma.

LEMMA 19.8. *If $\mathfrak{A} \in \mathscr{SCN}_3(p)$, then $\mathfrak{V} \subseteq O_{q,p}(\mathfrak{K})$ for every configuration $(\mathfrak{K}, \mathfrak{P}, \mathfrak{A}; \mathfrak{V}, \mathfrak{M}, \mathfrak{W})$ for which $\mathfrak{K}$ is a p, q-group.*

Proof. Suppose false, and that $\mathfrak{K}$ is chosen to maximize $\mathfrak{P}$, and, with this restriction to minimize $|\mathfrak{K}|_q$. It follows readily that $O_p(\mathfrak{K})$ is a S_p-subgroup of $O_{q,p}(\mathfrak{K})$ and that $\mathfrak{P}$ is a S_p-subgroup of every p, q-subgroup of $\mathfrak{G}$ which contains $\mathfrak{K}$.

By Lemma 18.1 and the isomorphism $O_p(\mathfrak{K}) \cong O_{q,p}(\mathfrak{K})/O_q(\mathfrak{K}) = \mathfrak{W}$, we conclude that $\mathfrak{M}$ centralizes $Z(O_p(\mathfrak{K}))$. By minimality of $|\mathfrak{K}|_q$, we also have $\mathfrak{K} = \mathfrak{P}\mathfrak{M}$.

If $\mathfrak{P}^*$ is a S_p-subgroup of $\mathfrak{G}$ containing $\mathfrak{P}$, we see that $Z(\mathfrak{P}^*)$ centralizes $O_p(\mathfrak{K})$, and so $Z(\mathfrak{P}^*) \subseteq Z(O_p(\mathfrak{K}))$, by maximality of $\mathfrak{P}$. It now follows that $\mathfrak{K}$ centralizes $Z(\mathfrak{P}^*)$, and maximality of $\mathfrak{P}$ yields $\mathfrak{P} = \mathfrak{P}^*$.

Since $\mathfrak{V}$ does not act trivially on $O_{q,p,q}(\mathfrak{K})/O_{q,p}(\mathfrak{K})$, and since $p > q$, it follows that $\mathfrak{M}$ contains an elementary subgroup of order q^3. But

$\mathfrak{M}$ centralizes $Z(O_p(\mathfrak{K})) = \mathfrak{Z}$ and if $\mathfrak{Z}$ is non cyclic, then $\mathfrak{Z}$ contains an element of $\mathscr{U}(\mathfrak{P})$, in violation of Lemma 19.3. Hence, $\mathfrak{Z}$ is cyclic. In this case, we conclude from Lemma 18.1 that a S_q-subgroup of $\mathfrak{M}$ centralizes $Z_2(O_p(\mathfrak{K})) = \mathfrak{Z}_2$. But $\mathfrak{Z}_2$ contains an element of $\mathscr{U}(\mathfrak{P})$, so once again Lemma 19.3 is violated. This contradiction completes the proof of this lemma.

LEMMA 19.9. *If $\mathfrak{L} \in \mathscr{L}(p) \cup \mathscr{SCN}_3(p)$, then $\mathfrak{L} \subseteq O_p(\mathfrak{K})$ for every p, q-subgroup $\mathfrak{K}$ of $\mathfrak{G}$ which contains $\mathfrak{L}$.*

Proof. By Lemmas 19.7 and 19.8, it suffices to show that $\mathfrak{L}$ centralizes $O_q(\mathfrak{K})$. If $\mathfrak{L} \in \mathscr{SCN}_3(p)$, Theorem 17.1, Corollary 17.2 and Hypothesis 19.2 imply that $\mathfrak{L}$ centralizes $O_q(\mathfrak{K})$. If $\mathfrak{L} \in \mathscr{L}(p)$, then $\mathfrak{L} \in \mathscr{L}(\mathfrak{P})$ for some S_p-subgroup $\mathfrak{P}$ of $\mathfrak{G}$. In this case, if $\mathfrak{A} \in \mathscr{SCN}_3(\mathfrak{P})$, the definition of $\mathscr{L}(\mathfrak{P})$ implies that $\mathfrak{A} \cap \mathfrak{L} = \mathfrak{A}_0$ is non cyclic. Hence, $O_q(\mathfrak{K})$ is generated by its subgroups $C(A) \cap O_q(\mathfrak{K})$ as A ranges over $\mathfrak{A}_0^\#$. By the preceding argument, $\mathfrak{A}$ is contained in $O_p(\mathfrak{K}_0)$ for every p, q-subgroup $\mathfrak{K}_0$ of $C(A)$ which contains $\mathfrak{A}$. Lemma 7.5 implies that $\mathfrak{A}_0$ centralizes $O_q(\mathfrak{K})$. In particular, $\Omega_1(Z(\mathfrak{P}))$ centralizes $O_q(\mathfrak{K})$.

Consider $C(\Omega_1(Z(\mathfrak{P}))) \supseteq \langle \mathfrak{P}, O_q(\mathfrak{K})\rangle$. Since $\mathfrak{L} \subseteq O_p(\mathfrak{K}_1)$ for every p, q-subgroup $\mathfrak{K}_1$ of $\langle \mathfrak{P}, O_q(\mathfrak{K})\rangle$ which contains $\mathfrak{L}$ by (B) and Hypothesis 19.2, a second application of Lemma 7.5 shows that $\mathfrak{L}$ centralizes $O_q(\mathfrak{K})$, as required.

LEMMA 19.10. *If $\mathfrak{B} \in \mathscr{T}(p)$, then $\mathfrak{B}$ centralizes every element of $\mathsf{M}(\mathfrak{B}; q)$.*

Proof. Suppose false, and $\mathfrak{Q}$ is chosen minimal subject to $\mathfrak{Q} \in \mathsf{M}(\mathfrak{B}; q)$ and $\gamma\mathfrak{Q}\mathfrak{B} \neq \langle 1\rangle$, so that we have $\mathfrak{Q} = \gamma\mathfrak{Q}\mathfrak{B}$ and $\mathfrak{B}_0 = \ker(\mathfrak{B} \rightarrow \operatorname{Aut}\mathfrak{Q}) \neq \langle 1\rangle$. Let $\mathfrak{C} = C(\mathfrak{B}_0)$. Since $\mathfrak{B} \in \mathscr{T}(p)$, we have $\mathfrak{B} \in \mathscr{T}(\mathfrak{P})$ for a suitable S_p-subgroup $\mathfrak{P}$ of $\mathfrak{G}$. By definition of $\mathscr{T}(\mathfrak{P})$, either $C(\mathfrak{B})$ contains an element $\mathfrak{A}_1$ of $\mathscr{SCN}_3(\mathfrak{P})$ or else $C(\mathfrak{B})$ contains a subgroup $\mathfrak{P}_1$ of index p in $\mathfrak{P} \cap O_{q,p}(\mathfrak{K})$, $\mathfrak{K} \in \mathscr{K}(\mathfrak{A})$ and $\mathfrak{A} \in \mathscr{SCN}_3(\mathfrak{P})$. Let $\mathfrak{H}$ be a $S_{p,q}$-subgroup of $\mathfrak{C}$ containing $\mathfrak{A}_1$ in the first case, and $\mathfrak{P}_1$ in the second case. Lemma 19.9 implies that $\mathfrak{A}_1 \subseteq O_p(\mathfrak{H})$ in the first case and $\mathfrak{P}_1 \subseteq O_p(\mathfrak{H})$ in the second case. In both cases, we have $\mathfrak{B} \subseteq O_p(\mathfrak{H})$. Now let $\mathfrak{H}_1$ be a $S_{p,q}$-subgroup of $\mathfrak{C}$ containing $\mathfrak{B}\mathfrak{Q}$. By Lemma 7.5, we have $\mathfrak{B} \subseteq O_p(\mathfrak{H}_1)$ and so $\gamma\mathfrak{Q}\mathfrak{B} \subseteq O_p(\mathfrak{H}_1) \cap \mathfrak{Q} = \langle 1\rangle$, contrary to assumption.

LEMMA 19.11. *If $\mathfrak{B} \in \mathscr{T}(p)$, $\mathfrak{A} \in \mathscr{U}(q)$, then $\mathfrak{G} = \langle \mathfrak{A}, \mathfrak{B}\rangle$.*

Proof. Suppose $\langle \mathfrak{A}, \mathfrak{B}\rangle = \mathfrak{K} \subset \mathfrak{G}$, and $\mathfrak{A}$ and $\mathfrak{B}$ are chosen to minimize $\mathfrak{K}$. By the minimal nature of $\mathfrak{K}$, $\mathfrak{K}$ is a p, q-group. By the previous lemmas, $\mathfrak{A}^{\mathfrak{K}}$ centralizes $O_p(\mathfrak{K})$, and $\mathfrak{B}^{\mathfrak{K}}$ centralizes $O_q(\mathfrak{K})$. It follows readily that $\mathfrak{K}$ is nilpotent, so $\mathfrak{K} = \mathfrak{A} \times \mathfrak{B}$. But now $C(\mathfrak{A})$ contains $\mathfrak{B}$ in violation of Lemma 19.3, with p and q interchanged. This interchange is permissible since Lemma 19.3 was proved before we discarded the symmetry in p and q.

LEMMA 19.12. *If $\mathfrak{D}$ is a p, q-subgroup of $\mathfrak{G}$ and if $\mathfrak{D}$ possesses an elementary subgroup of order p^3, then a S_p-subgroup of $\mathfrak{D}$ is normal in $\mathfrak{D}$.*

Proof. Case 1. $\mathfrak{D}$ contains a S_p-subgroup $\mathfrak{P}$ of $\mathfrak{G}$. Let $\mathfrak{Q}$ be a S_q-subgroup of $\mathfrak{D}$, let $\mathfrak{Q}_1 = \mathfrak{Q} \cap O_{p,q}(\mathfrak{D})$, let $\tilde{\mathfrak{Q}}$ be a S_q-subgroup of $\mathfrak{G}$ containing $\mathfrak{Q}$, let $\mathfrak{B} \in \mathscr{U}(\tilde{\mathfrak{Q}})$, and $\mathfrak{Q}_2 = C_{\mathfrak{Q}_1}(\mathfrak{B})$. Then $\mathfrak{Q}_2$ is of index 1 or q in $\mathfrak{Q}_1$.

Next, let $\mathfrak{K} = O_p(\mathfrak{D})$, and assume by way of contradiction that $\mathfrak{K} \subset \mathfrak{P}$. By the preceding lemmas, $\mathfrak{K}$ contains $V(ccl_{\mathfrak{G}}(\mathfrak{A}); \mathfrak{P})$ for every $\mathfrak{A} \in \mathscr{SCN}_3(\mathfrak{P})$. By the preceding lemma, no element of $\mathfrak{Q}_2^\#$ centralizes any element of $\mathscr{F}(p)$.

If $\mathfrak{K}$ contains a non cyclic characteristic subgroup $\mathfrak{C}$, then every subgroup of $\mathfrak{C}$ of type (p, p) belongs to $\mathscr{F}(\mathfrak{P})$, and so $C_{\mathfrak{C}}(Q)$ is cyclic for $Q \in \mathfrak{Q}_2$. This implies that $\mathscr{SCN}_3(\mathfrak{Q}_2)$ is empty, and if $\mathfrak{Q}_2$ possesses a subgroup of type (q, q), then $p \equiv 1 \pmod q$. However, if $\mathfrak{K}$ does not contain any non cyclic characteristic abelian subgroup, then every subgroup of $\mathfrak{K}$ of type (p, p) which contains $\Omega_1(Z(\mathfrak{K}))$ lies in $\mathscr{F}(\mathfrak{P})$, and we again conclude that $\mathscr{SCN}_3(\mathfrak{Q}_2)$ is empty, and if $\mathfrak{Q}_2$ is non cyclic, then $p \equiv 1 \pmod q$.

Now $\mathfrak{Q}_1 \cong O_{p,q}(\mathfrak{D})/\mathfrak{K}$ admits a non trivial p-automorphism since $\mathfrak{K} \subset \mathfrak{P}$, so $\mathscr{SCN}_3(\mathfrak{Q}_1)$ is non empty, by Lemma 8.4 (ii) and $p > q$. Hence, $\mathfrak{Q}_2$ is non cyclic, being of index at most q in $\mathfrak{Q}_1$, and this yields $p \equiv 1 \pmod q$. We apply Lemma 8.8 and conclude that $p = 1 + q + q^2$, and $\mathfrak{Q}_1$ is elementary of order q^3. This implies that any two subgroups of $\mathfrak{Q}_1$ of the same order are conjugate in $\mathfrak{D}$. Since at least one subgroup of $\mathfrak{Q}_1$ of order q centralizes $\mathfrak{B}$, every subgroup of $\mathfrak{Q}_1$ of order q centralizes some element of $\mathscr{U}(q)$. Since at least one subgroup of $\mathfrak{Q}_1$ of order q centralizes some element of $\mathscr{F}(\mathfrak{P})$, every subgroup of $\mathfrak{Q}_1$ of order q centralizes some element of $\mathscr{F}(p)$. This conflicts with Lemma 19.11.

Case 2. $\mathfrak{D}$ does not contain a S_p-subgroup of $\mathfrak{G}$. Among all $\mathfrak{D}$ which satisfy the hypotheses but not the conclusion of this lemma, choose $\mathfrak{D}$ so that $|\mathfrak{D} \cap \Omega_1(\mathfrak{A})|$ is a maximum, where $\mathfrak{A}$ ranges over all elements of $\mathscr{SCN}_3(p)$, and with this restriction, maximize $|\mathfrak{D}|_p$.

Let $\mathfrak{D}_1$ be a S_p-subgroup of $\mathfrak{D}$, and let $\mathfrak{P}$ be a S_p-subgroup of $\mathfrak{G}$ containing $\mathfrak{D}_1$.

First, assume $\mathfrak{D}_1$ centralizes $O_q(\mathfrak{D})$. In this case, $O_p(\mathfrak{D})$ is a S_p-subgroup of $O_{q,p}(\mathfrak{D})$. By maximality of $|\mathfrak{D}|_p$, $\mathfrak{D}_1$ is a S_p-subgroup of $N(O_p(\mathfrak{D}))$. This implies that $\mathfrak{D}_1$ contains every element of $\mathscr{SCN}_3(\mathfrak{P})$. To see this, let $\mathfrak{A} \in \mathscr{SCN}_3(\mathfrak{P})$, and let $\mathfrak{A}_1 = \mathfrak{A} \cap \mathfrak{D}_1$. Since $O_p(\mathfrak{D})$ is a S_p-subgroup of $O_{q,p}(\mathfrak{D})$, it follows that $\mathfrak{A} \cap \mathfrak{D}_1 \subseteq O_p(\mathfrak{D})$. If $\mathfrak{A}_1$ were a proper subgroup of $\mathfrak{A}$, then $\mathfrak{D}_1$ would be a proper subgroup of $N_{\mathfrak{A}\mathfrak{D}_1}(O_p(\mathfrak{D}))$. Since this is not possible, we have $\mathfrak{A} = \mathfrak{A}_1$. But now, $V(ccl_{\mathfrak{G}}(\mathfrak{A}); \mathfrak{D}_1) \lhd \mathfrak{D}$, and by maximality of $|\mathfrak{D}|_p$, $\mathfrak{D}_1 = \mathfrak{P}$ follows, and we are in the preceding case.

We can now assume that $\mathfrak{D}_1$ does not centralize $O_q(\mathfrak{D})$. Suppose $\mathfrak{D}_1$ contains some element $\mathfrak{B}$ of $\mathscr{F}(p)$. By Lemma 19.10, $\mathfrak{B}$ centralizes $O_q(\mathfrak{D})$. Since $\mathfrak{D}_1$ does not centralize $O_q(\mathfrak{D})$, $|O_q(\mathfrak{D})| > q$, and so Lemma 19.11 is violated in $C(Q)$, Q being a suitable element of $O_q(\mathfrak{D})$. Thus, we can suppose that $\mathfrak{D}_1$ does not contain any element of $\mathscr{F}(p)$. In particular, $\mathfrak{D} \cap \Omega_1(\mathfrak{A})$ is of order 1 or p for all $\mathfrak{A} \in \mathscr{SCN}_3(p)$. Let $\mathfrak{B} \in \mathscr{U}(\mathfrak{P})$, and $\mathfrak{D}_2 = C_{\mathfrak{D}_1}(\mathfrak{B})$. Since $\mathscr{SCN}_3(\mathfrak{D}_1)$ is non empty by hypothesis, $\mathfrak{D}_2$ is non cyclic. Let $\mathfrak{E}$ be a subgroup of $\mathfrak{D}_2$ of type (p, p). Since $\mathfrak{B} \not\subseteq \mathfrak{D}_1$, $\langle \mathfrak{E}, \mathfrak{B} \rangle$ is elementary of order at least p^3. If $\mathfrak{E}$ does not centralize $O_q(\mathfrak{D})$, then there is an element E in $\mathfrak{E}^\#$ such that $\mathfrak{E}$ does not centralize $C(E) \cap O_q(\mathfrak{D})$. But in this case, a $S_{p,q}$-subgroup of $C(E)$ is larger than $\mathfrak{D}$ in our ordering since $\mathfrak{B} \subseteq C(E)$, $C(E)$ possesses an elementary subgroup of order p^3, and a S_p-subgroup of a $S_{p,q}$-subgroup $\mathfrak{F}$ of $C(E)$ is not normal in $\mathfrak{F}$. This conflict forces every subgroup of $\mathfrak{D}_2$ of type (p, p) to centralize $O_q(\mathfrak{D})$. Thus, $\Omega_1(\mathfrak{D}_2) = \mathfrak{D}^*$ centralizes $O_q(\mathfrak{D})$, since $\mathfrak{D}^*$ is generated by its subgroups of type (p, p). However, we now have $N(\mathfrak{D}^*) \supseteq \langle \mathfrak{D}_1, \mathfrak{B}, O_q(\mathfrak{D}) \rangle$ and a $S_{p,q}$-subgroup $\mathfrak{F}_1$ of $N(\mathfrak{D}^*)$ is larger than $\mathfrak{D}$ in our ordering, possesses an elementary subgroup of order p^3, and has the additional property that its S_p-subgroups are not normal in $\mathfrak{F}_1$. This conflict completes the proof of this lemma.

Lemma 19.12 gives us a fairly good idea of the structure of the p, q-subgroups of $\mathfrak{G}$. The remaining analysis is still somewhat detailed, but the moves are more obvious.

For the remainder of this section, $\mathfrak{P}$ denotes a S_p-subgroup of $\mathfrak{G}$, $\mathfrak{Q}$ a S_q-subgroup of $N(\mathfrak{P})$, and $\tilde{\mathfrak{Q}}$ a S_q-subgroup of $\mathfrak{G}$ which contains $\mathfrak{Q}$.

LEMMA 19.13. *$\mathscr{SCN}_3(\mathfrak{Q})$ is non empty.*

Proof. We apply Hypothesis 19.2 (ii) and let $\mathfrak{D}$ be a maximal p, q-subgroup of $\mathfrak{G}$ which contains elementary subgroups of order p^3 and q^3. By Lemma 19.12, $\mathfrak{D}_p \lhd \mathfrak{D}$, $\mathfrak{D}_p$ being a S_p-subgroup of $\mathfrak{D}$. Since $\mathfrak{D}$ is a maximal p, q-subgroup of $\mathfrak{G}$, $\mathfrak{D}_p$ is a S_p-subgroup of $\mathfrak{G}$, so $\mathfrak{D}_p = \mathfrak{P}^G$ and the lemma follows.

We now choose $\mathfrak{B}$ in $\mathscr{U}(\tilde{\mathfrak{Q}})$ and set $\mathfrak{Q}_1 = C_{\mathfrak{Q}}(\mathfrak{B})$.

LEMMA 19.14.

(i) *$\mathscr{S}\mathscr{C}\mathscr{N}_3(\mathfrak{Q}_1)$ is empty.*

(ii) *$\mathfrak{Q}$ contains $\Omega_1(Z(\tilde{\mathfrak{Q}}))$.*

(iii) $p \equiv 1 \pmod q$.

(iv) *$\mathfrak{Q}^\#$ contains an element Y which centralizes an element of $\mathscr{F}(\mathfrak{P})$, and has the additional property that $C_{\mathfrak{Q}}(Y)$ contains an elementary subgroup of order q^3.*

(v) *If $X \in \mathfrak{Q}^\#$ and X centralizes an element of $\mathscr{F}(\mathfrak{P})$, then X does not centralize any element of $\mathscr{U}(\tilde{\mathfrak{Q}})$, and $C(X)$ does not contain an elementary subgroup of order q^4.*

Proof. Let $\mathfrak{E}$ be an elementary subgroup of $\mathfrak{Q}$ of order q^3, and choose $\mathfrak{E}$ in $\mathfrak{Q}_1$ if possible. If $\mathfrak{P}$ possesses a non cyclic characteristic abelian subgroup $\mathfrak{C}$, then some element of $\mathfrak{E}$ has a non cyclic fixed point set on $\mathfrak{C}$. Since every subgroup of $\mathfrak{C}$ of type (p, p) lies in $\mathscr{F}(\mathfrak{P})$, (iv) is established in this case.

If every characteristic abelian subgroup of $\mathfrak{P}$ is cyclic, then some non cyclic subgroup $\mathfrak{E}_1$ of $\mathfrak{E}$ centralizes $Z(\mathfrak{P})$. Since any non cyclic subgroup of $\mathfrak{P}$ which contains $\Omega_1(Z(\mathfrak{P}))$ is normal in $\mathfrak{P}$, by 3.5, some element of $\mathfrak{E}_1$ centralizes an element of $\mathscr{F}(\mathfrak{P})$, so (iv) is proved.

If $\mathfrak{E} \subseteq \mathfrak{Q}_1$, then Lemma 19.11 is violated in $C(E)$, $E \in \mathfrak{E}^\#$, E centralizing an element of $\mathscr{F}(\mathfrak{P})$. Hence, (i) is proved.

On the other hand, $\mathscr{S}\mathscr{C}\mathscr{N}_3(\mathfrak{Q})$ is non empty, so $\mathfrak{Q}_1$ possesses a subgroup $\mathfrak{F}_1$ of type (q, q). If $p \not\equiv 1 \pmod q$, then some element of $\mathfrak{F}_1$ is seen to centralize an element of $\mathscr{F}(\mathfrak{P})$. Since this is forbidden by Lemma 19.11, (iii) follows.

We now turn attention to (v). In view of Lemma 19.11, we only need to show that if X in $\mathfrak{Q}^\#$ centralizes an element of $\mathscr{F}(\mathfrak{P})$, then $C(X)$ does not contain an elementary subgroup of order q^4.

Let $\mathfrak{A}$ be an element of $\mathscr{F}(\mathfrak{P})$ centralized by X, let $\mathfrak{H}$ be a $S_{p,q}$-subgroup of $C(X)$ and let $\mathfrak{K}$ be a maximal p, q-subgroup of $\mathfrak{G}$ containing $\mathfrak{H}$. By $D_{p,q}$ in $C(X)$, $\mathfrak{A}_1 = \mathfrak{A}^G \subseteq \mathfrak{H} \subseteq \mathfrak{K}$, for some G in $C(X)$. Suppose by way of contradiction that $C(X)$ contains an elementary subgroup of order q^4. By $D_{p,q}$ in $C(X)$, $\mathfrak{H}$ contains an elementary subgroup of order q^4; thus, $\mathfrak{K}$ contains such a subgroup.

We first show that a S_p-subgroup of $\mathfrak{K}$ is not normal in $\mathfrak{K}$. Suppose false. In this case, since $\mathfrak{K}$ is a maximal p, q-subgroup of $\mathfrak{G}$, a S_p-subgroup of $\mathfrak{K}$ is conjugate to $\mathfrak{P}$, and so $\mathfrak{K}$ is conjugate to $\mathfrak{P}\mathfrak{Q}$. However, (i) implies that $\mathfrak{Q}$ does not contain an elementary subgroup of order q^4, since $|\mathfrak{Q} : \mathfrak{Q}_1| = q$, so $\mathfrak{K}$ does not contain one either.

We now apply Lemma 19.12 and conclude that $\mathfrak{K}$ does not possess an elementary subgroup of order p^3. It follows directly from Lemma 8.13 that $\mathfrak{K}$ has p-length one. Let $\mathfrak{K}_p$ be a S_p-subgroup of $\mathfrak{K}$ containing $\mathfrak{A}_1$, and let $\mathfrak{B}_1 = V(ccl_{\mathfrak{G}}(\mathfrak{A}_1); \mathfrak{K}_p)$. By Lemma 19.10, $\mathfrak{B}_1$ centralizes $O_q(\mathfrak{K})$. Since $\mathfrak{K}$ has p-length one, $\mathfrak{B}_1 \lhd \mathfrak{K}$. But then $N(\mathfrak{B}_1) = \mathfrak{N}$ contains S_p-subgroups of larger order than $|\mathfrak{K}_p|$, and $\mathfrak{N}$ also contains $\mathfrak{K}$, contrary to the assumption that $\mathfrak{K}$ is a maximal p, q-subgroup of $\mathfrak{G}$. This contradiction proves (v).

We now turn to (ii). Choose Y to satisfy (iv) and let $\mathfrak{E}$ be an elementary subgroup of $C_{\mathfrak{Q}}(Y)$ of order q^3. If $\Omega_1(Z(\tilde{\mathfrak{Q}})) = \Omega_1$ were not contained in $\mathfrak{E}$, then $\langle \mathfrak{E}, \Omega_1 \rangle$ would contain an elementary subgroup of order q^4, and (v) would be violated. This completes the proof of this lemma.

We remark that Lemma 19.2 and Lemma 19.14 (ii) imply that $Z(\tilde{\mathfrak{Q}})$ is cyclic.

Theorem 19.1 can now be proved fairly easily. We again denote by $\mathfrak{E}$ an elementary subgroup of $\mathfrak{Q}$ of order q^3, and we let Y be an element of $\mathfrak{E}^\#$ which centralizes an element of $\mathscr{T}(\mathfrak{P})$. Let $\mathfrak{E}_1 = C_{\mathfrak{E}}(\mathfrak{B})$. Since $\Omega_1 = \Omega_1(Z(\tilde{\mathfrak{Q}}))$ centralizes $\mathfrak{B}$, Ω_1 does not centralize any element of $\mathscr{U}(\mathfrak{P})$, by Lemma 19.2, and so does not centralize $\mathfrak{P}$. Thus, we can find an element E in $\mathfrak{E}_1^\#$ with the property that Ω_1 does not centralize $C_{\mathfrak{P}}(E)$. Consider $\mathfrak{C} = C(E)$. We see that $\mathfrak{C}$ contains both Y and $\mathfrak{B}$. Since Y does not centralize $\mathfrak{B}$, $\langle Y, \mathfrak{B} \rangle$ is a non abelian group of order q^3, with center Ω_1. Let $\mathfrak{L}$ be a $S_{p,q}$-subgroup of $\mathfrak{C}$ which contains $\langle Y, \mathfrak{B} \rangle$; since $\mathfrak{L}$ contains $\mathfrak{B}$, $\mathfrak{L}$ does not contain an elementary subgroup of order p^3. Since Ω_1 is contained in the derived group of $\langle Y, \mathfrak{B} \rangle$, Ω_1 is contained in $\mathfrak{L}'$. We apply Lemma 8.13 and conclude that Ω_1 centralizes every chief p-factor of $\mathfrak{L}$. It follows that $\gamma^n \mathfrak{L} \Omega_1^n = \langle 1 \rangle$ for suitably large n, and so $\Omega_1 \subseteq O_q(\mathfrak{L})$. But now if $\mathfrak{H}$ is any $S_{p,q}$-subgroup of $\mathfrak{C}$ which contains Ω_1, we have $\Omega_1 \subseteq O_q(\mathfrak{H})$, by Lemma 7.5, and so $[\Omega_1, C_{\mathfrak{P}}(E)]$ is both a p-group and q-group, so is $\langle 1 \rangle$, contrary to construction. This completes the proof of Theorem 19.1.

COROLLARY 19.1. *If $p, q \in \pi_3 \cup \pi_4$, $p \neq q$, and $p \sim q$, then either $p \in \pi_3$ or $q \in \pi_3$.*

Proof. If $\mathfrak{G}$ satisfies $E_{p,q}^n$, then both p and q are in π_3. Otherwise, Hypothesis 19.2 is violated and the corollary follows.

20. An E-theorem for π_3

Hypothesis 20.1 $p, q \in \pi_3$, $p \neq q$, *and* $p \sim q$.

THEOREM 20.1. *Under Hypothesis 20.1, $\mathfrak{G}$ satisfies $E_{p,q}$.*

The proof of this theorem is by contradiction. The following lemmas assume that Hypothesis 20.1 is satisfied but $\mathfrak{G}$ does not satisfy $E_{p,q}$.

LEMMA 20.1. *If $\mathfrak{P}$ is a S_p-subgroup of $\mathfrak{G}$ and $\mathfrak{Q}$ is a S_q-subgroup of $\mathfrak{G}$, then either $\mathfrak{P}$ normalizes but does not centralize some q-subgroup of $\mathfrak{G}$, or $\mathfrak{Q}$ normalizes but does not centralize some p-subgroup of $\mathfrak{G}$.*

Proof. This lemma is an immediate consequence of Hypothesis 20.1, Theorem 19.1, and the assumption that $\mathfrak{G}$ does not satisfy $E_{p,q}$.

We assume now that notation is chosen so that $\mathfrak{P}$, a S_p-subgroup of $\mathfrak{G}$, does not centralize $\mathfrak{Q}_1$, a maximal element of $\mathsf{N}(\mathfrak{P}; q)$. Let $\mathfrak{Q}^*$ be a S_q-subgroup of $N(\mathfrak{Q}_1)$ permutable with $\mathfrak{P}$, and let $\mathfrak{Q}$ be a S_q-subgroup of $\mathfrak{G}$ containing $\mathfrak{Q}^*$.

LEMMA 20.2. $O_p(\mathfrak{P}\mathfrak{Q}^*) \neq \langle 1 \rangle$.

Proof. Suppose false. Let $\mathfrak{A}$ be an element of $\mathscr{SCN}_3(\mathfrak{Q})$. By Lemma 7.9, we have $\mathfrak{A} \subseteq O_q(\mathfrak{P}\mathfrak{Q}^*)$. We apply Lemma 17.4 and conclude that $N(\mathfrak{Q}_1) \subseteq N(\mathfrak{Z})$, where $\mathfrak{Z} = Z(\mathfrak{B})$, $\mathfrak{B} = V(ccl_{\mathfrak{G}}(\mathfrak{A}); \mathfrak{Q})$, and so $\mathfrak{G}$ satisfies $E_{p,q}$, contrary to assumption.

Let $\mathfrak{P}_1 = O_p(\mathfrak{P}\mathfrak{Q}^*)$.

LEMMA 20.3. *$\mathfrak{Q}^*$ is a S_q-subgroup of every proper subgroup $\mathfrak{K}$ of $\mathfrak{G}$ which contains $\mathfrak{P}_1\mathfrak{Q}^*$.*

Proof. Let $\mathfrak{T}$ be a $S_{p,q}$-subgroup of $\mathfrak{K}$ with Sylow system $\mathfrak{Q}_2$, $\mathfrak{P}_2$ where $\mathfrak{Q}^* \subseteq \mathfrak{Q}_2$ and $\mathfrak{P}_1 \subseteq \mathfrak{P}_2$, and let $F(\mathfrak{T}) = \mathfrak{T}_1 \times \mathfrak{T}_2$, where $\mathfrak{T}_1 = O_p(\mathfrak{T})$, $\mathfrak{T}_2 = O_q(\mathfrak{T})$.

We first show that $\mathfrak{T}_1 \subseteq \mathfrak{P}_1$. Suppose by way of contradiction that $\mathfrak{T}_1 \cap \mathfrak{P}_1 \subset \mathfrak{T}_1$. Since $\mathfrak{Q}^*$ and $\mathfrak{P}_1$ both normalize $\mathfrak{T}_1 \cap \mathfrak{P}_1$ and both normalize $\mathfrak{T}_1$, setting $\mathfrak{T}_1^* = N_{\mathfrak{T}_1}(\mathfrak{T}_1 \cap \mathfrak{P}_1)$, we see that $\mathfrak{T}_1^*\mathfrak{Q}^*\mathfrak{P}_1$ is a group, and that $\mathfrak{Q}^*\mathfrak{P}_1$ normalizes $\mathfrak{T}_1^*$. Let $\mathfrak{T}^*/\mathfrak{T}_1 \cap \mathfrak{P}_1$ be a chief factor of $\mathfrak{T}_1^*\mathfrak{Q}^*\mathfrak{P}_1$ with $\mathfrak{T}^* \subseteq \mathfrak{T}_1^*$. Since $\mathfrak{P}_1 \lhd \mathfrak{P}_1\mathfrak{Q}^*$, it follows that $\mathfrak{P}_1$ centralizes $\mathfrak{T}^*/\mathfrak{T}_1 \cap \mathfrak{P}_1$, that is $\gamma\mathfrak{T}^*\mathfrak{P}_1 \subseteq \mathfrak{T}_1 \cap \mathfrak{P}_1$. In particular $\mathfrak{T}^*$ normalizes $\mathfrak{P}_1$. Now $\mathfrak{P}\mathfrak{Q}^*$ is a maximal p, q-subgroup of $\mathfrak{G}$ by Lemma 7.3, so $\mathfrak{Q}^*$ is a S_q-subgroup of $N(\mathfrak{P}_1)$. A second application of Lemma 7.3 yields that $\mathfrak{P}_1$ is a S_p-subgroup of $O_{q'}(N(\mathfrak{P}_1))$. But $\mathfrak{P}_1\mathfrak{T}^*$ is normalized by $\mathfrak{Q}^*$, so a third application of Lemma 7.3 yields $\mathfrak{P}_1\mathfrak{T}^* \subseteq O_{q'}(N(\mathfrak{P}_1))$, so $\mathfrak{T}^* \subseteq \mathfrak{P}_1$, contrary to our choice of $\mathfrak{T}^*$. Thus, $\mathfrak{T}_1 \subseteq \mathfrak{P}_1$.

We next show that $\mathfrak{T}_2 \subseteq \mathfrak{Q}^*$. To do this, it suffices to show that

$\mathfrak{P}_1$ centralizes $\mathfrak{T}_2$, for if this is the case, then $\mathfrak{T}_2 \subseteq C(\mathfrak{P}_1) \subseteq N(\mathfrak{P}_1)$, and so $\mathfrak{T}_2\mathfrak{Q}^*$ is a q-subgroup of $N(\mathfrak{P}_1)$. Since $\mathfrak{Q}^*$ is a S_q-subgroup of $N(\mathfrak{P}_1)$, $\mathfrak{T}_2 \subseteq \mathfrak{Q}^*$ follows.

To show that $\mathfrak{P}_1$ centralizes $\mathfrak{T}_2$, we first show that $\mathfrak{P}_1$ centralizes $C_{\mathfrak{T}_2}(\mathfrak{Q}_1)$. By definition, $\mathfrak{Q}^*$ is a S_q-subgroup of $N(\mathfrak{Q}_1)$, and since $\mathfrak{Q}^*C_{\mathfrak{T}_2}(\mathfrak{Q}_1)$ is a q-subgroup of $N(\mathfrak{Q}_1)$, we have $C_{\mathfrak{T}_2}(\mathfrak{Q}_1) \subseteq \mathfrak{Q}^*$. Hence, $[C_{\mathfrak{T}_2}(\mathfrak{Q}_1), \mathfrak{P}_1] \subseteq \mathfrak{T}_2 \cap [\mathfrak{Q}^*, \mathfrak{P}_1] \subseteq \mathfrak{T}_2 \cap \mathfrak{P}_1 = \langle 1 \rangle$. Suppose that $\mathfrak{P}_1$ does not centralize $\mathfrak{T}_2$ and that $\mathfrak{T}_3$ is a $\mathfrak{P}_1\mathfrak{Q}_1$-invariant subgroup of $\mathfrak{T}_2$, minimal subject to the condition $\gamma\mathfrak{T}_3\mathfrak{P}_1 \neq \langle 1 \rangle$. By minimality of $\mathfrak{T}_3$, we have $\mathfrak{T}_3 = \gamma\mathfrak{T}_3\mathfrak{P}_1$. Since $\mathfrak{T}_3$ is a q-group, $\gamma\mathfrak{T}_3\mathfrak{Q}_1 \subset \mathfrak{T}_3$, and so $\gamma^2\mathfrak{T}_3\mathfrak{Q}_1\mathfrak{P}_1 = \langle 1 \rangle$. Since $\gamma\mathfrak{Q}_1\mathfrak{P}_1 = \langle 1 \rangle$, we also have $\gamma^2\mathfrak{Q}_1\mathfrak{P}_1\mathfrak{T}_3 = \langle 1 \rangle$. The three subgroups lemma now yields $\gamma^2\mathfrak{P}_1\mathfrak{T}_3\mathfrak{Q}_1 = \langle 1 \rangle$, so $\mathfrak{Q}_1$ centralizes $\gamma\mathfrak{P}_1\mathfrak{T}_3 = \mathfrak{T}_3$. By what we have already shown this implies that $\mathfrak{P}_1$ centralizes $\mathfrak{T}_3$. This conflict forces $\gamma\mathfrak{P}_1\mathfrak{T}_2 = \langle 1 \rangle$.

We next show that $\mathfrak{Q}_1 \subseteq \mathfrak{T}_2$. To do this, consider $C_{\mathfrak{T}}(\mathfrak{T}_1) = \mathfrak{C} \lhd \mathfrak{T}$. Since $\mathfrak{T}_1 \subseteq \mathfrak{P}_1$, we see that $\mathfrak{Q}_1 \subseteq \mathfrak{C}$. On the other hand, $Z(\mathfrak{P}_1)$ centralizes both $\mathfrak{T}_1$ and $\mathfrak{T}_2$, so $Z(\mathfrak{P}_1) \subseteq \mathfrak{T}_1$, by 3.3. Hence, $\mathfrak{C} \subseteq C_{\mathfrak{T}}(Z(\mathfrak{P}_1)) \subseteq C(Z(\mathfrak{P}_1)) \subseteq N(Z(\mathfrak{P}_1))$. Since $\mathfrak{Q}_1 = O_q(\mathfrak{P}\mathfrak{Q}^*)$, Lemma 7.5 implies that $\mathfrak{Q}_1 \subseteq O_q(\mathfrak{C})$ char $\mathfrak{C} \lhd \mathfrak{T}$, and so $\mathfrak{Q}_1 \subseteq \mathfrak{T}_2$.

Consider finally $C_{\mathfrak{T}}(\mathfrak{T}_2)$. Since $\mathfrak{Q}_1 \subseteq \mathfrak{T}_2$, we have $C_{\mathfrak{T}}(\mathfrak{T}_2) \subseteq C_{\mathfrak{T}}(\mathfrak{Q}_1) \subseteq C(\mathfrak{Q}_1) \subseteq N(\mathfrak{Q}_1)$. Since $\mathfrak{P}_1 = O_p(\mathfrak{P}\mathfrak{Q}^*)$, Lemma 7.5 implies that $\mathfrak{P}_1 \subseteq O_p(C_{\mathfrak{T}}(\mathfrak{T}_2))$ char $C_{\mathfrak{T}}(\mathfrak{T}_2) \lhd \mathfrak{T}$, and so $\mathfrak{P}_1 \subseteq \mathfrak{T}_1$. Since we have already shown that $\mathfrak{T}_1 \subseteq \mathfrak{P}_1$, we have $\mathfrak{T}_1 = \mathfrak{P}_1 \lhd \mathfrak{T}$, and so $\mathfrak{Q}^*$ is a S_q-subgroup of $\mathfrak{T}$, as required.

To prove Theorem 20.1 recall that $\mathfrak{Q}$ is a S_q-subgroup of $\mathfrak{G}$ containing $\mathfrak{Q}^*$. Choose $\mathfrak{A}$ in $\mathscr{S}\mathscr{C}\mathscr{N}_3(\mathfrak{Q})$, and let $\mathfrak{A}^* = \mathfrak{A} \cap \mathfrak{Q}^*$. We first show that $\mathfrak{A}^* \subset \mathfrak{A}$. Suppose by way of contradiction that $\mathfrak{A}^* = \mathfrak{A}$. Then $\mathfrak{A}$ normalizes $\mathfrak{P}_1$. Lemma 7.3 and the previous lemma imply that $\mathfrak{P}_1$ is a maximal element of $\mathsf{M}(\mathfrak{A}; p)$. By Corollary 17.1, $N(\mathfrak{P}_1)$ contains a S_q-subgroup of $\mathfrak{G}$, and $\mathfrak{G}$ satisfies $E_{p,q}$. Since we are advancing by contradiction, we have $\mathfrak{A}^* \subset \mathfrak{A}$.

We next show that $\mathfrak{A}^* \cap \mathfrak{Q}_1 = \langle 1 \rangle$. To do this, we observe that $\mathfrak{A}^* \cap \mathfrak{Q}_1 \lhd \mathfrak{Q}^*$, so if $\mathfrak{A}^* \cap \mathfrak{Q}_1 \neq \langle 1 \rangle$, then $\mathfrak{A}^* \cap \mathfrak{Q}_1 \cap Z(\mathfrak{Q}^*) \neq \langle 1 \rangle$. In this case, however, $C(\mathfrak{A}^* \cap \mathfrak{Q}_1 \cap Z(\mathfrak{Q}^*))$ contains $\mathfrak{P}_1$ and also contains $\mathfrak{Q}^*\mathfrak{A}$, contrary to the previous lemma. Thus, $\mathfrak{A}^* \cap \mathfrak{Q}_1 = \langle 1 \rangle$. Since $\mathfrak{A}^*$ and $\mathfrak{Q}_1$ are both normal in $\mathfrak{Q}^*$, we have $\gamma\mathfrak{A}^*\mathfrak{Q}_1 = \langle 1 \rangle$.

Let $\mathfrak{A}_1 = N_{\mathfrak{A}}(\mathfrak{Q}^*)$, so that $\mathfrak{A}^* \subset \mathfrak{A}_1 \subseteq \mathfrak{A}$. Observe that $\gamma\mathfrak{A}_1\mathfrak{Q}^* \subseteq \mathfrak{Q}^* \cap \mathfrak{A} \subset \mathfrak{A}_1$ and so $\mathfrak{Q}^*$ normalizes $\mathfrak{A}_1$. Let $\mathfrak{B}$ be any subgroup of $\mathfrak{A}_1$ which contains $\mathfrak{A}^*$ properly. Since $[\mathfrak{B}, \mathfrak{Q}_1\mathfrak{A}^*] \subseteq \mathfrak{A}^*$, we see that $\mathfrak{B}$ normalizes $\mathfrak{Q}_1\mathfrak{A}^* = \mathfrak{Q}_1 \times \mathfrak{A}^*$. Since $\mathfrak{Q}^*$ normalizes $\mathfrak{B}$, $\mathfrak{Q}^*$ also normalizes $C_{\mathfrak{Q}_1}(\mathfrak{B}) = \mathfrak{D}$, say. If $\mathfrak{D} \neq \langle 1 \rangle$, then $\mathfrak{D} \cap Z(\mathfrak{Q}^*) \neq \langle 1 \rangle$. But then the previous lemma is violated in $C(\mathfrak{D} \cap Z(\mathfrak{Q}^*))$. Hence, $\mathfrak{D} = \langle 1 \rangle$. Since $C(\mathfrak{A}_1) \cap \mathfrak{Q}_1\mathfrak{A}^* \supseteq \mathfrak{A}^*$, we have $C(\mathfrak{B}) \cap \mathfrak{Q}_1\mathfrak{A}^* = \mathfrak{A}^*$.

Since $\mathfrak{B}$ normalizes $\mathfrak{Q}_1 \times \mathfrak{A}^*$, $\mathfrak{B}$ also normalizes $(\mathfrak{Q}_1 \times \mathfrak{A}^*)' = \mathfrak{Q}_1'$. Since $\mathfrak{B}$ has no fixed points on $\mathfrak{Q}_1^\#$ by the above argument, $\mathfrak{Q}_1$ is abelian. But now $\mathfrak{Q}_1\mathfrak{A}^*$ and $\mathfrak{B}$ are normal abelian subgroups of $\langle\mathfrak{Q}_1, \mathfrak{B}\rangle$, so $\langle\mathfrak{Q}_1, \mathfrak{B}\rangle$ is of class two, so is regular. It follows that if $B \in \mathfrak{B}$, $Q \in \mathfrak{Q}_1$, then $[B^q, Q] = [B, Q^q] = [B, Q]^q$. But $\mathfrak{B}$ is an arbitrary subgroup of $\mathfrak{A}_1$ which contains $\mathfrak{A}^*$ properly, so we can choose $\mathfrak{B}$ such that $\mho^1(\mathfrak{B}) \subseteq \mathfrak{A}^*$. For such a $\mathfrak{B}$, the element B centralizes $\mho^1(\mathfrak{Q}_1)$. It now follows that $\mathfrak{Q}_1$ is elementary.

We take a different approach for an instant. $\mathfrak{P}$ does not centralize the elementary abelian group $\mathfrak{Q}_1$, and $N(\mathfrak{Q}_1)$ has no normal subgroup of index p, by Lemma 17.3. It follows that $\mathfrak{Q}_1$ is not of order q.

Returning to the groups $\mathfrak{A}^*$ and $\mathfrak{B}$, since $\mathfrak{B}$ has no fixed points on $\mathfrak{Q}_1$, if $B \in \mathfrak{B}$, $B \notin \mathfrak{A}^*$, then the mapping $\phi_B : \mathfrak{Q}_1 \rightarrow \mathfrak{A}^*$ defined by $\phi_B(Q) = [B, Q]$, Q in $\mathfrak{Q}_1$, is an isomorphism of $\mathfrak{Q}_1$ onto a subgroup of $\mathfrak{A}^*$. Hence, $\mathfrak{A}^*$ is not cyclic.

From the definition of $\mathfrak{A}^*$, we see that $\mathfrak{A}^*$ contains $Z(\mathfrak{Q})$. We wish to show that $\mathfrak{A}^*$ contains an element of $\mathscr{U}(\mathfrak{Q})$. This is immediate if $Z(\mathfrak{Q})$ is non cyclic, so suppose $Z(\mathfrak{Q})$ is cyclic. If $\mathfrak{A}^*$ does not contain any element of $\mathscr{U}(\mathfrak{Q})$, then the element B above can be taken to lie in some element of $\mathscr{U}(\mathfrak{Q})$. However, $[Q, B] \in \Omega_1(Z(\mathfrak{Q}))$, so ϕ_B could not map $\mathfrak{Q}_1$ onto a subgroup of order exceeding q. We conclude that $\mathfrak{A}^*$ contains $Z(\mathfrak{Q})$ and also some element of $\mathscr{U}(\mathfrak{Q})$.

We will now show that for each element Z of $Z(\mathfrak{Q})^\#$, we can find a p-subgroup $\mathfrak{H}(Z)$ in $\mathsf{M}(\mathfrak{A}; p)$ which is not centralized by Z. Namely, $\mathfrak{A}^*$ is faithfully represented on $\mathfrak{P}_1$, since $\mathfrak{A}^* \cap \mathfrak{Q}_1 = \langle 1\rangle$ and $\mathfrak{A}^*$ is a normal abelian subgroup of $\mathfrak{Q}^*$. We first consider the case in which $Z(\mathfrak{Q})$ is non cyclic. Let $\mathfrak{E}$ be a subgroup of $Z(\mathfrak{Q})$ of type (q, q) which has non trivial intersection with $\langle Z\rangle$, that is let $\mathfrak{E}$ contain $\mathfrak{Z}_1 = \Omega_1(\langle Z\rangle)$. Since $\mathfrak{Z}_1$ acts non trivially on $\mathfrak{P}_1$, $\mathfrak{Z}_1$ acts non trivially on $C_{\mathfrak{P}_1}(E)$ for suitable E in $\mathfrak{E}^\#$. Let $\mathfrak{C} = C(E)$, and let $\mathfrak{R}$ be a S_p-subgroup of $\mathfrak{C}$ permutable with $\mathfrak{Q}$. It is easy to see that $\mathfrak{Z}_1$ does not centralize $O_p(\mathfrak{Q}\mathfrak{R}) \in \mathsf{M}(\mathfrak{A}; p)$.

If $Z(\mathfrak{Q})$ is cyclic, we use the fact that $\mathfrak{A}^*$ contains an element $\mathfrak{U}$ of $\mathscr{U}(\mathfrak{Q})$. We can find an element U in $\mathfrak{U}^\#$ such that $\mathfrak{Z}_1 = \Omega_1(Z(\mathfrak{Q}))$ does not centralize $C_{\mathfrak{P}_1}(U)$. Let $\mathfrak{C} = C(U)$. By (B), it follows that $\mathfrak{U} \subseteq O_{q',q}(\mathfrak{C})$, and so $[\mathfrak{Z}_1, C_{\mathfrak{P}_1}(U)] \subseteq O_{q'}(\mathfrak{C})$. Thus, $\mathfrak{C}$ contains an element of $\mathsf{M}(\mathfrak{A}; p)$ which $\mathfrak{Z}_1$ does not centralize.

It now follows from Theorem 17.1 and the preceding argument that if $\tilde{\mathfrak{P}}$ is a maximal element of $\mathsf{M}(\mathfrak{Q}; p)$, then $Z(\mathfrak{Q})$ is faithfully represented on $\tilde{\mathfrak{P}}$. If $\hat{\mathfrak{P}}$ is a S_p-subgroup of $N(\tilde{\mathfrak{P}})$ permutable with $\mathfrak{Q}$, then Lemma 20.2 is violated with p and q interchanged. This completes the proof of Theorem 20.1.

21. A C^*-theorem for π_3 and a C-theorem for π_3

It is convenient to introduce another proposition which is "between" C_π and D_π.

C_π^*: $\mathfrak{X}$ satisfies C_π, and if $\mathfrak{T}$ is a π-subgroup of $\mathfrak{X}$ with the property that $|\mathfrak{T}|_p = |\mathfrak{X}|_p$ for at least one prime p in π, then $\mathfrak{T}$ is contained in a S_π-subgroup of $\mathfrak{X}$.

THEOREM 21.1 *If $p, q \in \pi_3$ and $p \sim q$, then $\mathfrak{G}$ satisfies $C_{p,q}^*$*

Proof. We can suppose $p \neq q$. We first show that $\mathfrak{G}$ satisfies $C_{p,q}$. By Theorem 20.1, $\mathfrak{G}$ satisfies $E_{p,q}$. Let $\mathfrak{H}$ be a $S_{p,q}$-subgroup of $\mathfrak{G}$ with Sylow system $\mathfrak{P}$, $\mathfrak{Q}$, where $\mathfrak{P}$ is a S_p-subgroup of $\mathfrak{G}$. We assume notation is chosen so that $|\mathfrak{P}| > |\mathfrak{Q}|$. Then $O_p(\mathfrak{H}) \neq \langle 1 \rangle$ by Lemma 5.2. Lemma 7.3 implies that $O_p(\mathfrak{H})$ is a maximal element of $\mathsf{V}(\mathfrak{Q}; p)$. If $\mathfrak{H}_1$ is another $S_{p,q}$-subgroup of $\mathfrak{G}$ containing $\mathfrak{Q}$, then $O_p(\mathfrak{H}_1)$ is also a maximal element of $\mathsf{V}(\mathfrak{Q}; p)$. From Section 17 we conclude that $O_p(\mathfrak{H}_1) = G^{-1}O_p(\mathfrak{H})G$ for suitable G in $\mathfrak{G}$. Hence, $G\mathfrak{H}_1G^{-1}$ and $\mathfrak{H}$ both normalize $O_p(\mathfrak{H})$ so are conjugate in $N(O_p(\mathfrak{H}))$.

Turning to $C_{p,q}^*$, we drop the hypothesis $|\mathfrak{P}| > |\mathfrak{Q}|$, and let $\mathfrak{T}$ be a maximal p, q-subgroup of $\mathfrak{G}$ containing $\mathfrak{P}$. Let $\mathfrak{H}$ be a $S_{p,q}$-subgroup of $\mathfrak{G}$ containing $\mathfrak{P}$.

First, assume that $O_q(\mathfrak{T}) \neq 1$. In this case, $O_q(\mathfrak{T})$ is a maximal element of $\mathsf{V}(\mathfrak{P}; q)$. If $O_q(\mathfrak{H}) \neq 1$, then $O_q(\mathfrak{H})$ is also a maximal element of $\mathsf{V}(\mathfrak{P}; q)$. Thus, Theorem 17.1 yields that $\mathfrak{H}$ is conjugate to $\mathfrak{T}$. (Here, as elsewhere, we are using the fact that every maximal element of $\mathsf{V}(\mathfrak{P}; q)$ is also a maximal element of $\mathsf{V}(\mathfrak{A}; q)$ for all $\mathfrak{A}$ in $\mathscr{SCN}_3(\mathfrak{P})$.) Thus, suppose $O_q(\mathfrak{H}) = 1$. In this case, if $\mathfrak{A} \in \mathscr{SCN}_3(\mathfrak{P})$, then $\mathfrak{B} \lhd \mathfrak{H}$, $\mathfrak{B} = V(ccl_{\mathfrak{G}}(\mathfrak{A}); \mathfrak{P})$, by Lemma 17.5, so $|\mathfrak{G}|_q = |N(\mathfrak{B}) : C(\mathfrak{B})|_q$. But $N(O_q(\mathfrak{T}))$ dominates $\mathfrak{B}$, so $|N(O_q(\mathfrak{T}))|_q > |\mathfrak{G}|_q$, which is absurd.

We can now suppose that $O_q(\mathfrak{T}) = 1$. We apply Lemma 17.5 and conclude that $\mathfrak{B} \lhd \mathfrak{T}$, where $\mathfrak{B} = V(ccl_{\mathfrak{G}}(\mathfrak{A}); \mathfrak{P})$, and $\mathfrak{A} \in \mathscr{SCN}_3(\mathfrak{P})$. Let $\mathfrak{Q}_0$ be a S_q-subgroup of $\mathfrak{T}$. Since $\mathfrak{T}$ is a maximal p, q-subgroup of $\mathfrak{G}$, $\mathfrak{Q}_0$ is a S_q-subgroup of $N(\mathfrak{B})$.

Let $\mathfrak{H}$ be a $S_{p,q}$-subgroup of $\mathfrak{G}$ containing $\mathfrak{P}$ and let $\mathfrak{Q}$ be a S_q-subgroup of $\mathfrak{H}$. Let $\mathfrak{Q}_1 = O_q(\mathfrak{H})$. If $\mathfrak{Q}_1 = \langle 1 \rangle$, then $\mathfrak{H} \subseteq N(\mathfrak{B})$, by Lemma 17.5, and we are done. Otherwise, $\mathfrak{H} = \mathfrak{Q}_1 N_{\mathfrak{H}}(\mathfrak{B})$, again by Lemma 17.5, and we assume without loss of generality that $N_{\mathfrak{H}}(\mathfrak{B}) \subseteq \mathfrak{T}$.

Assume that $N_{\mathfrak{H}}(\mathfrak{B}) \cap \mathfrak{Q}_1 \neq \langle 1 \rangle$. Then in particular, $\mathfrak{T} \cap \mathfrak{Q}_1 \neq \langle 1 \rangle$, contrary to $O_q(\mathfrak{T}) = \langle 1 \rangle$. Hence, $N_{\mathfrak{H}}(\mathfrak{B}) \cap \mathfrak{Q}_1 = \langle 1 \rangle$.

We will now show directly that $N_{\mathfrak{H}}(\mathfrak{B}) = \mathfrak{T}$. Since $N_{\mathfrak{H}}(\mathfrak{B}) \subseteq \mathfrak{T}$, it suffices to show that $|N_{\mathfrak{H}}(\mathfrak{B})|_q \geq |\mathfrak{T}|_q$. Now $N(\mathfrak{Q}_1) = O_{p'}(N(\mathfrak{Q}_1)) \cdot (N(\mathfrak{Q}_1) \cap N(\mathfrak{B}))$, by Lemma 17.1, and since $N_{\mathfrak{H}}(\mathfrak{B}) \cap \mathfrak{Q}_1 = \langle 1 \rangle$, it follows easily that $|N_{\mathfrak{H}}(\mathfrak{B})|_q = |N(\mathfrak{Q}_1) \cap N(\mathfrak{B})|_q$.

Let $\mathfrak{N}_1 = N(Z(\mathfrak{B}))$. By Lemma 17.3 we have $\mathfrak{N}_1 = O_{p'}(\mathfrak{N}_1)\cdot(N(\mathfrak{Q}_1)\cap\mathfrak{N}_1)$. Let $\mathfrak{M} = N(\mathfrak{Q}_1)\cap\mathfrak{N}_1$. Since $\mathfrak{M}$ contains $\mathfrak{P}$, $O_{p'}(\mathfrak{M}) = O_{p'}(\mathfrak{N}_1)\cap\mathfrak{M}$. By Lemma 17.5, we now have $\mathfrak{M} = (O_{p'}(\mathfrak{N}_1)\cap\mathfrak{M})\cdot(N(\mathfrak{B})\cap\mathfrak{M})$, which yields $\mathfrak{N}_1 = O_{p'}(\mathfrak{N}_1)\cdot(N(\mathfrak{Q}_1)\cap N(\mathfrak{B}))$. Now $\mathfrak{N}_1$ contains $\mathfrak{T}$ and $\mathfrak{T}\cap O_{p'}(\mathfrak{N}_1) = \langle 1\rangle$, since $O_q(\mathfrak{T}) = \langle 1\rangle$. Thus, $\mathfrak{Q}_0$ is mapped isomorphically into $\mathfrak{N}_1/O_{p'}(\mathfrak{N}_1) \cong (N(\mathfrak{Q}_1)\cap N(\mathfrak{B}))/(O_{p'}(\mathfrak{N}_1)\cap N(\mathfrak{Q}_1)\cap N(\mathfrak{B}))$, and it follows that $|N(\mathfrak{Q}_1)\cap N(\mathfrak{B})|_q \geqq |\mathfrak{Q}_0| = |\mathfrak{T}|_q$, as required.

Since $N_{\mathfrak{H}}(\mathfrak{B}) = \mathfrak{T}$, it follows that $\mathfrak{T}\subseteqq\mathfrak{H}$, proving the theorem.

THEOREM 21.2. *Let σ be a subset of π_3. Assume that $\mathfrak{G}$ satisfies $E_{p\,q}$ for all p, q in σ. Then $\mathfrak{G}$ satisfies C_σ.*

Proof. By the preceding theorem, we can assume that σ contains at least three elements. By induction on $|\sigma|$, we assume that $\mathfrak{G}$ satisfies C_τ for every proper subset τ of σ.

Let $\sigma = \{p_1, \cdots, p_n\}$, $n \geqq 3$, and let $\sigma_i = \sigma - p_i$, $\sigma_{ij} = \sigma - p_i - p_j$, $1\leqq i,\ j\leqq n$, $i\neq j$. Let $\mathfrak{S}_i$ be a S_{σ_i}-subgroup of $\mathfrak{G}$, $1\leqq i\leqq n$. Then the $S_{\sigma_{ij}}$-subgroups of $\mathfrak{S}_i$ are conjugate to the $S_{\sigma_{ij}}$-subgroups of $\mathfrak{S}_j$.

For $i\neq j$, let $m_{ij} = |O_{p_i}(\mathfrak{S}_j)|$. Note that by C_{σ_j}, m_{ij} depends only on i and j and not on the particular S_{σ_j}-subgroup of $\mathfrak{G}$ we choose.

Fix i, j, k, $i\neq j\neq k\neq i$, let $\mathfrak{P}_i$ be a S_{p_i}-subgroup of $\mathfrak{G}$, let $\mathfrak{S}_j^*$ be a S_{σ_j}-subgroup of $\mathfrak{G}$ containing $\mathfrak{P}_i$ and $\mathfrak{S}_k^*$ be a S_{σ_k}-subgroup of $\mathfrak{G}$ containing $\mathfrak{P}_i$, chosen so that $\mathfrak{S}_j^*\cap\mathfrak{S}_k^*$ is a $S_{\sigma_{j,k}}$-subgroup of $\mathfrak{G}$ which is possible by $C_{\sigma_{j,k}}$, C_{σ_j} and C_{σ_k}.

Let $\mathfrak{P}_{ij} = O_{p_i}(\mathfrak{S}_j^*)$, $\mathfrak{P}_{ik} = O_{p_i}(\mathfrak{S}_k^*)$. Suppose that $\mathfrak{P}_{ij}\cap\mathfrak{P}_{ik} = \langle 1\rangle$. With this assumption, we will show that $m_{ij}\leqq m_{jk}$. We can assume that $i=1$, $j=2$, $k=3$, that $\mathfrak{P}_{12}\cap\mathfrak{P}_{13} = \langle 1\rangle$, and try to show that $m_{12}\leqq m_{23}$.

Let $\mathfrak{P}_1, \mathfrak{R}_4, \mathfrak{R}_5, \cdots, \mathfrak{R}_n$ be a Sylow system for $\mathfrak{S}_2^*\cap\mathfrak{S}_3^*$, and let $\mathfrak{P}_1, \mathfrak{R}_3, \cdots, \mathfrak{R}_n$ and $\mathfrak{P}_1, \mathfrak{R}_2, \mathfrak{R}_4, \cdots, \mathfrak{R}_n$ be Sylow systems for $\mathfrak{S}_2^*$ and $\mathfrak{S}_3^*$ respectively. Here $\mathfrak{R}_i$ is a S_{p_i}-subgroup of $\mathfrak{G}$, $i = 2, \cdots, n$.

Since $\mathfrak{P}_{13}$ is the S_{p_1}-subgroup of $F(\mathfrak{S}_3^*)$, the condition $\mathfrak{P}_{12}\cap\mathfrak{P}_{13} = \langle 1\rangle$ says that $\mathfrak{P}_{12}$ is faithfully represented as automorphisms of $F(\mathfrak{S}_3^*)$. Now

$$F(\mathfrak{S}_3^*) = F(\mathfrak{S}_3^*)\cap\mathfrak{P}_1\times F(\mathfrak{S}_3^*)\cap\mathfrak{R}_2\times F(\mathfrak{S}_3^*)\cap\mathfrak{R}_4\times\cdots\times F(\mathfrak{S}_3^*)\cap\mathfrak{R}_n\,,$$

where $\mathfrak{P}_{13} = F(\mathfrak{S}_3^*)\cap\mathfrak{P}_1$. Since $\mathfrak{P}_{12}$ and $\mathfrak{P}_{13}$ are disjoint normal subgroups of $\mathfrak{P}_1$, $\mathfrak{P}_{12}$ centralizes $\mathfrak{P}_{13}$. If $4\leqq s\leqq n$, then $\langle\mathfrak{P}_{12}, F(\mathfrak{S}_3^*)\cap\mathfrak{R}_s\rangle = \mathfrak{H}_s$ is clearly contained in $\mathfrak{S}_2^*\cap\mathfrak{S}_3^*$ and so $\mathfrak{P}_{12}$ and $F(\mathfrak{S}_3^*)\cap\mathfrak{R}_s$ are disjoint normal subgroups of $\mathfrak{H}_s$, and so commute elementwise. But $\mathfrak{P}_{12}$ is faithfully represented as automorphisms of $F(\mathfrak{S}_3^*)$, so is faithfully represented as automorphisms of $F(\mathfrak{S}_3^*)\cap\mathfrak{R}_2$. It follows from Lemma 5.2 that $m_{12}\leqq m_{23}$.

Returning to the general situation, if $O_{p_i}(\mathfrak{S}_j)\cap O_{p_i}(\mathfrak{S}_k) = \langle 1\rangle$, whenever $i\neq j\neq k\neq i$, and $\mathfrak{S}_j\cap\mathfrak{S}_k$ is a $S_{\sigma_{j,k}}$-subgroup of $\mathfrak{G}$, then

$m_{ij} \leqq m_{jk}$. Permuting i, j, k cyclically, we would have $m_{ij} \leqq m_{jk} \leqq m_{ki} \leqq m_{ij}$. The integers m_{ij}, m_{jk}, m_{ki} being pairwise relatively prime, we would find $m_{ij} = 1$ for all $i \neq j$. This is not possible since a S_{σ_i}-subgroup of $\mathfrak{G}$ is solvable.

Returning to the groups $\mathfrak{S}_2^*$ and $\mathfrak{S}_3^*$, we suppose without loss of generality that $\mathfrak{P}_{12} \cap \mathfrak{P}_{13} = \mathfrak{D}_{123} \neq \langle 1 \rangle$. Since $\mathfrak{D}_{123} \subseteqq \mathfrak{P}_{12} \lhd \mathfrak{S}_2^*$, $\mathfrak{D}_{123}$ commutes elementwise with $O_{p_1'}(\mathfrak{S}_2^*)$. Similarly, $\mathfrak{D}_{123}$ commutes elementwise with $O_{p_1'}(\mathfrak{S}_3^*)$. Hence $\langle \mathfrak{P}_1, O_{p_1'}(\mathfrak{S}_2^*), O_{p_1'}(\mathfrak{S}_3^*) \rangle = \mathfrak{L}$ is a proper subgroup of $\mathfrak{G}$ normalizing $\mathfrak{D}_{123}$. By Lemma 7.5, both $O_{p_1'}(\mathfrak{S}_2^*)$ and $O_{p_1'}(\mathfrak{S}_3^*)$ are S-subgroups of $O_{p_1'}(\mathfrak{L})$; in particular, $\mathfrak{L}$ has a normal p_1-complement. Since $\mathfrak{L}$ has a normal p_1-complement, we can find an element C in $C_{\mathfrak{L}}(\mathfrak{P})$ such that $O_{p_1'}(\mathfrak{S}_2^*)$ is permutable with $C^{-1}O_{p_1'}(\mathfrak{S}_3^*)C$. For such an element C, let $\mathfrak{M} = \langle O_{p_1'}(\mathfrak{S}_2^*), C^{-1}O_{p_1'}(\mathfrak{S}_3^*)C \rangle$.

We will now show directly that for each q in σ, $N(\mathfrak{M})$ contains a S_q-subgroup of $\mathfrak{G}$. This is trivially true if $\mathfrak{M} = \langle 1 \rangle$, so suppose that $\mathfrak{M} \neq \langle 1 \rangle$. Let $\mathfrak{M}_2, \cdots, \mathfrak{M}_n$ be a Sylow system for $\mathfrak{M}$ which is normalized by $\mathfrak{P}_1$, where $\mathfrak{M}_i$ is an S_{p_i}-subgroup of $\mathfrak{M}$, $i = 2, \cdots, n$. We remark that by $C^*_{p_1, p_i}$, each $\mathfrak{M}_i$ is a maximal element of $\mathsf{M}(\mathfrak{P}_1; p_i)$.

Let $|\mathfrak{M}_i| = p_i^{e_i}$ and let $|\mathfrak{G}|_{p_i} = p_i^{f_i}$. By Lemma 17.5 and $C^*_{p_1, p_i}$ we see that $p_i^{f_i - e_i} = |N(\mathfrak{B}) : C(\mathfrak{B})|_{p_i}$, where $\mathfrak{B} = V(ccl_{\mathfrak{G}}(\mathfrak{A}); \mathfrak{P})$, $\mathfrak{P} = \mathfrak{P}_1$, and $\mathfrak{A} \in \mathscr{SCN}_3(\mathfrak{P})$.

Let $\mathfrak{N}_1 = N(Z(\mathfrak{B}))$. Let $\mathfrak{C}$ be a coset of $O_{p'}(\mathfrak{N}_1)$ in $\mathfrak{N}_1$. Then $\mathfrak{C}$ contains an element N of $N(\mathfrak{B})$ by Lemma 17.5. Hence, $\mathfrak{M}_i^{N^{-1}} = \mathfrak{M}_i^{C_i}$, $i = 2, \cdots, n$ where $C_2, \cdots, C_n$ all lie in $C(\mathfrak{A})$. Let $\mathfrak{K} = \langle \mathfrak{P}, \mathfrak{M}_2, \cdots, \mathfrak{M}_n, C_2, \cdots, C_n \rangle$. Since $\mathfrak{D}_{123} \cap Z(\mathfrak{P}_1) \neq 1$, and since $\mathfrak{K}$ centralizes $\mathfrak{D}_{123} \cap Z(\mathfrak{P}_1)$, we have $\mathfrak{K} \subset \mathfrak{G}$. Let $\mathfrak{L} = O_{p'}(\mathfrak{K})$ $(p = p_1)$ so that $\mathfrak{L}\mathfrak{P} = \mathfrak{K}$ by Lemmas 7.3 and 7.4. Hence, $\mathfrak{L}$ contains both $\mathfrak{M}$ and $\mathfrak{M}^{N^{-1}}$, and since $\mathfrak{B}$ normalizes $\mathfrak{M}$, $\mathfrak{A}$ normalizes both $\mathfrak{M}$ and $\mathfrak{M}^{N^{-1}}$. By C^*_{p, p_i}, $i = 2, \cdots, n$, $\mathfrak{M}$ is a S-subgroup of $\mathfrak{L}$. By the conjugacy of Sylow systems in $\mathfrak{A}\mathfrak{L}$, there is an element C in $\mathfrak{L}\mathfrak{A}$ such that $\mathfrak{A}^C = \mathfrak{A}$, $\mathfrak{M}^C = \mathfrak{M}^{N^{-1}}$. Since $\mathfrak{L}\mathfrak{A}$ has a normal p-complement, $C \in C(\mathfrak{A}) \subseteqq O_{p'}(\mathfrak{N}_1)$, so $\mathfrak{C}$ contains $CN \in N(\mathfrak{M})$.

Thus, if $\mathfrak{T} = \mathfrak{N}_1 \cap N(\mathfrak{M})$, we have $\mathfrak{N}_1 = O_{p'}(\mathfrak{N}_1)\mathfrak{T}$. Since $\mathfrak{P} \subseteqq \mathfrak{T}$, we have $O_{p'}(\mathfrak{T}) = \mathfrak{T} \cap O_{p'}(\mathfrak{N}_1)$. Hence $\mathfrak{T} = O_{p'}(\mathfrak{T})N_{\mathfrak{T}}(\mathfrak{B})$ by Lemma 17.5, so that $\mathfrak{N}_1 = O_{p'}(\mathfrak{N}_1)N_{\mathfrak{T}}(\mathfrak{B})$. Thus $N_{\mathfrak{T}}(\mathfrak{B})$ maps onto $N(\mathfrak{B})/C(\mathfrak{B})$. Since $N_{\mathfrak{T}}(\mathfrak{B}) \cap \mathfrak{M}$ centralizes $\mathfrak{B}$, it follows that $|\mathfrak{T} : \mathfrak{T} \cap \mathfrak{M}|_{p_i} = p_i^{f_i - e_i}$, $i = 2, \cdots, n$. Hence $|\mathfrak{T}\mathfrak{M}|_{p_i} = |\mathfrak{G}|_{p_i}$, as required.

If now $\mathfrak{M} \neq \langle 1 \rangle$, then $N(\mathfrak{M}) \subset \mathfrak{G}$ and so $\mathfrak{G}$ satisfies E_σ.

We now treat the possibility that $\mathfrak{M} = \langle 1 \rangle$. In this case, both $F(\mathfrak{P}_2^*)$ and $F(\mathfrak{P}_3^*)$ are p_1-groups. By (B), both groups contain $\mathfrak{A}$. By Lemma 17.4, both $\mathfrak{P}_2^*$ and $\mathfrak{P}_3^*$ are contained in $N(Z(\mathfrak{B}))$, so once again $\mathfrak{G}$ satisfies E_σ.

It remains to prove C_σ, given E_σ and C_τ for every proper subset

τ of σ.

Let $\mathfrak{H}$ and $\mathfrak{H}_1$ be two S_σ-subgroups of $\mathfrak{G}$ with Sylow systems $\mathfrak{P}_1, \cdots, \mathfrak{P}_n$ and $\mathfrak{Q}_1, \cdots, \mathfrak{Q}_n$ respectively, $\mathfrak{P}_i$ and $\mathfrak{Q}_i$ being S_{p_i}-subgroups of $\mathfrak{G}$, $1 \leqq i \leqq n$.

If $F(\mathfrak{H})$ and $F(\mathfrak{H}_1)$ are p_1-groups, we apply Lemma 17.4 and conclude that $\mathfrak{H}$ and $\mathfrak{H}_1$ are conjugate in $N(Z(\mathfrak{B}))$, where $\mathfrak{B} = V(ccl_{\mathfrak{G}}(\mathfrak{A}); \mathfrak{P}_1)$, $\mathfrak{A} \in \mathscr{SCN}_3(\mathfrak{P}_1)$ and we have normalized by taking $\mathfrak{P}_1 = \mathfrak{Q}_1$.

If $F(\mathfrak{H})$ is a p_1-group, then C_{p_1, p_i} for $i = 2, \cdots, n$ imply that $F(\mathfrak{H}_1)$ is a p_1-group. Thus, we can assume that neither $F(\mathfrak{H})$ nor $F(\mathfrak{H}_1)$ is a p-group for any prime p.

Let $m_i = |O_{p_i}(\mathfrak{H})|$, $m'_i = |O_{p_i}(\mathfrak{H}_1)|$, $1 \leqq i \leqq n$. For each i, we can choose G_i in $\mathfrak{G}$ so that $\mathfrak{Q}_j^{G_i} = \mathfrak{P}_j$, $1 \leqq j \leqq n$, $i \neq j$. Let $\mathfrak{K}_i = \mathfrak{H}_1^{G_i}$, $i = 1, \cdots, n$, so that $\mathfrak{H} \cap \mathfrak{K}_i$ contains a S_{σ_i}-subgroup of $\mathfrak{G}$.

Suppose $O_{p_j}(\mathfrak{K}_i) \cap O_{p_j}(\mathfrak{H}) = \langle 1 \rangle$ for some i, j, $i \neq j$. Then $O_{p_j}(\mathfrak{K}_i)$ is faithfully represented on $F(\mathfrak{H})$, since $O_{p_j}(\mathfrak{K}_i) \subseteqq \mathfrak{H}$. But in this case, $O_{p_j}(\mathfrak{K}_i)$ centralizes $O_{p_j}(\mathfrak{H})$ and also centralizes $O_{p_k}(\mathfrak{H})$ for $k \neq i$. Hence, $O_{p_j}(\mathfrak{K}_i)$ is faithfully represented on $O_{p_i}(\mathfrak{H})$, and so $m'_j \leqq m_i$ by Lemma 5.2. For the same reasons, $m_j \leqq m'_i$, since $O_{p_j}(\mathfrak{H})$ is faithfully represented on $F(\mathfrak{K}_i)$. If for all i, j, $1 \leqq i$, $j \leqq n$, $i \neq j$, $O_{p_j}(\mathfrak{K}_i) \cap O_{p_j}(\mathfrak{H}) = \langle 1 \rangle$, we find $m'_j \leqq m_i \leqq m'_j$, and so $m'_j = m_i = 1$. This is not possible since $\mathfrak{H}$ and $\mathfrak{H}_1$ are solvable.

Hence, we assume without loss of generality that $\mathfrak{D}_{12} = O_{p_1}(\mathfrak{K}_2) \cap O_{p_1}(\mathfrak{H}) \neq \langle 1 \rangle$. We will now show that $O_{p'_1}(\mathfrak{K}_2)$ is conjugate to $O_{p'_1}(\mathfrak{H})$. To see this, we first apply Lemma 7.4 and C_{p_1, p_i} to conclude that $O_{p'_1}(\mathfrak{K}_2)$ and $O_{p'_1}(\mathfrak{H})$ have the same order. Since $\mathfrak{D}_{12}$ centralizes both $O_{p'_1}(\mathfrak{K}_2)$ and $O_{p'_1}(\mathfrak{H})$, it follows that $\mathfrak{L} = \langle \mathfrak{P}_1, O_{p'_1}(\mathfrak{K}_2), O_{p'_1}(\mathfrak{H}) \rangle \subset \mathfrak{G}$. By Lemma 7.4, it follows that $\langle O_{p'_1}(\mathfrak{K}_2), O_{p'_1}(\mathfrak{H}) \rangle \subseteqq O_{p'_1}(\mathfrak{L})$. By Theorem 17.1 and $C^*_{p_1, p_i}$, $O_{p'_1}(\mathfrak{K}_2)$ and $O_{p'_1}(\mathfrak{H})$ are S-subgroups of $O_{p'_1}(\mathfrak{L})$, so are conjugate in $\mathfrak{L}$, being of the same order. Since $O_{p'_1}(\mathfrak{H}) \neq \langle 1 \rangle$, C_σ follows immediately.

22. Linking Theorems

One of the purposes of this section is to clarify the relationship between π_3 and π_4.

Hypothesis 22.1.

(i) $p \in \pi_3$, $q \in \pi(\mathfrak{G})$.

(ii) *A S_p-subgroup $\mathfrak{P}$ of $\mathfrak{G}$ does not centralize every element of $\mathsf{N}(\mathfrak{P}; q)$.*

THEOREM 22.1. *Under Hypothesis 22.1, if $\mathfrak{Q}_1$ is a maximal element of $\mathsf{N}(\mathfrak{P}; q)$ and Q is an element of $\mathfrak{Q}_1$ of order q, then $C_{\mathfrak{Q}_1}(Q)$ contains an elementary subgroup of order q^3. In particular, $q \in \pi_3 \cup \pi_4$.*

Proof. Choose $\mathfrak{C}$ char $\mathfrak{Q}_1$ in accordance with Lemma 8.2, and set $\mathfrak{C}_1 = \Omega_1(\mathfrak{C})$. From 3.6 and Lemma 8.2, it follows that $\mathfrak{P}$ does not centralize $\mathfrak{C}_1$. Since $cl(\mathfrak{C}) \leqq 2$, $\mathfrak{C}_1$ is of exponent q.

Since $N(\mathfrak{C}_1) \supseteqq N(\mathfrak{Q}_1)$, Lemma 17.3 implies that $O^p(N(\mathfrak{C}_1)) = N(\mathfrak{C}_1)$. Since $N(\mathfrak{C}_1)$ has odd order, this in turn implies that $\mathfrak{C}_1$ is not generated by two elements. Consider the chain $\mathscr{C}: \mathfrak{C}_1 \supseteqq \gamma\mathfrak{C}_1\mathfrak{Q}_1 \supseteqq \gamma^2\mathfrak{C}_1\mathfrak{Q}_1^2 \supseteqq \cdots$. Since $\mathfrak{P}$ does not centralize $\mathfrak{C}_1$, $\mathfrak{P}$ does not stabilize $\mathscr{C}$, so we can find an integer n and subgroups $\mathfrak{A}_1$, $\mathfrak{A}_2$ such that $\gamma^{n+1}\mathfrak{C}_1\mathfrak{Q}_1^{n+1} \subseteqq \mathfrak{A}_1 \subseteqq \mathfrak{A}_2 \subseteqq \gamma^n\mathfrak{C}_1\mathfrak{Q}_1^n$ and such that $\mathfrak{B} = \mathfrak{A}_2/\mathfrak{A}_1$ is a chief factor of $N(\mathfrak{Q}_1)$ and with the additional property that $\mathfrak{P}$ does not centralize $\mathfrak{B}$. Since $N(\mathfrak{Q}_1) = O^p(N(\mathfrak{Q}_1))$, we also have $\bar{\mathfrak{N}} = O^p(\bar{\mathfrak{N}})$, where $\bar{\mathfrak{N}} = (N(\mathfrak{B}) \cap N(\mathfrak{Q}_1))/(C(\mathfrak{B}) \cap N(\mathfrak{Q}_1))$. Since $|N(\mathfrak{Q}_1)|$ is odd it follows that $|\mathfrak{B}| \geqq q^3$. Since $\gamma\mathfrak{A}_2\mathfrak{Q}_1 \subseteqq \mathfrak{A}_1$, it follows that $|C_{\mathfrak{A}_2}(Q)| \geqq q^3$. If $C_{\mathfrak{Q}_1}(Q)$ did not contain an elementary subgroup of order q^3, then we would necessarily have $Q \in \mathfrak{A}_2$ since $\mathfrak{A}_2$ is of exponent q. Since $|C_{\mathfrak{A}_2}(Q)| \geqq q^3$, the only possibility is that $C_{\mathfrak{A}_2}(Q)$ is the non abelian group of order q^3 and exponent q. But in this case $Q \in C_{\mathfrak{A}_2}(Q)' \subseteqq Z(\mathfrak{C}_1)$, and $C_{\mathfrak{Q}_1}(Q)$ contains an elementary subgroup of order q^3 since $\mathfrak{Q}_1$ does, by Lemma 8.13, Lemma 8.1, and the equation $N(\mathfrak{Q}_1) = O^p(N(\mathfrak{Q}_1))$.

Hypothesis 22.2.

(i) *$\mathfrak{P}$ is a S_p-subgroup of $\mathfrak{G}$ and $p \in \pi_3$.*

(ii) *$q, r \in \pi_3 \cup \pi_4$; $\mathfrak{P}$ does not centralize every element of $\mathcal{N}(\mathfrak{P}; q)$ and $\mathfrak{P}$ does not centralize every element of $\mathcal{N}(\mathfrak{P}; r)$.*

THEOREM 22.2. *Under Hypothesis* 22.2, $q \sim r$.

The proof of this theorem is by contradiction. The following lemmas assume that $q \not\sim r$.

Since Hypothesis 22.2 is symmetric in q and r we can assume that $q > r$, thereby destroying the symmetry.

Let $\mathfrak{A} \in \mathscr{S}\mathscr{C}\mathscr{N}_3(\mathfrak{P})$. Let $\mathfrak{Q}_1$, $\mathfrak{R}_1$ be maximal elements in $\mathcal{N}(\mathfrak{P}; q)$, $\mathcal{N}(\mathfrak{P}; r)$ respectively.

LEMMA 22.1. *If $\mathfrak{H}$ is an $\mathfrak{A}$-invariant q, r-subgroup of $\mathfrak{G}$, and if a S_q-subgroup $\mathfrak{H}_q$ of $\mathfrak{H}$ is non cyclic, then $\mathfrak{H}_q \lhd \mathfrak{H}$.*

Proof. Let $\mathfrak{H}_r$ be a S_r-subgroup of $\mathfrak{H}$ normalized by $\mathfrak{A}$. Since $q \not\sim r$, either $\mathscr{S}\mathscr{C}\mathscr{N}_3(\mathfrak{H}_r)$ or $\mathscr{S}\mathscr{C}\mathscr{N}_3(\mathfrak{H}_q)$ is empty. If $\mathscr{S}\mathscr{C}\mathscr{N}_3(\mathfrak{H}_r)$ is empty, application of Lemma 8.5 to $\mathfrak{H}$ yields this lemma.

Suppose $\mathscr{S}\mathscr{C}\mathscr{N}_3(\mathfrak{H}_r)$ is non empty. Then $\mathscr{S}\mathscr{C}\mathscr{N}_3(\mathfrak{H}_q)$ is empty, so $\mathfrak{H}$ has q-length one. Thus, it suffices to show that $\mathfrak{H}_q$ centralizes $O_r(\mathfrak{H})$. We suppose without loss of generality that $\mathfrak{A}$ normalizes $\mathfrak{H}_q$.

Then by Corollary 17.2 $\mathfrak{H}_q$ is contained in a conjugate of $\mathfrak{Q}_1$, so $C(H)$ possesses an elementary subgroup of order q^3 for H in $\mathfrak{H}_q$, H of order q, by Theorem 22.1. We will show that $\Omega_1(\mathfrak{H}_q)$ centralizes $O_r(\mathfrak{H})$. Since $\mathfrak{H}_q$ is assumed non cyclic, $\Omega_1(\mathfrak{H}_q)$ is generated by its subgroups $\mathfrak{E}$ which are elementary of order q^2, so it suffices to show that each such $\mathfrak{E}$ centralizes $O_r(\mathfrak{H})$. If $\mathfrak{E}$ does not centralize $O_r(\mathfrak{H})$, then $\mathfrak{E}$ does not centralize $O_r(\mathfrak{H}) \cap C(E)$ for suitable E in $\mathfrak{E}^\#$. By Lemma 8.4, $\mathscr{SCN}_3(O_r(\mathfrak{H}) \cap C(E))$ is non empty for such an E, so $q \not\sim r$ is violated in $C(E)$.

Since $\Omega_1(\mathfrak{H}_q)$ centralizes $O_r(\mathfrak{H})$, it follows that $\mathscr{SCN}_3(O_r(\mathfrak{H}))$ is empty, since $q \not\sim r$. Hence, $\mathfrak{H}_q$ centralizes $O_r(\mathfrak{H})$ by Lemma 8.4, as required.

We define $\mathscr{K}$ as the set of q, r-subgroups of $\mathsf{M}(\mathfrak{A})$ which have the additional property that no S_q- or S_r-subgroup is centralized by $\mathfrak{A}$.

LEMMA 22.2. *$\mathscr{K}$ is non empty.*

Proof. Suppose that $\gamma\mathfrak{Q}_1\mathfrak{A} = \langle 1 \rangle$. If we also had $\gamma\mathfrak{R}_1\mathfrak{A} = \langle 1 \rangle$, then $q \not\sim r$ would be violated in $C(\mathfrak{A})$. Hence, $\gamma\mathfrak{R}_1\mathfrak{A} \neq \langle 1 \rangle$, and we can find $\mathfrak{R}_2 \subseteq \mathfrak{R}_1$, $\mathfrak{R}_2 \neq \langle 1 \rangle$, such that $\mathfrak{R}_2 = \gamma\mathfrak{R}_2\mathfrak{A}$ and such that $\mathfrak{A}_1 = C_{\mathfrak{A}}(\mathfrak{R}_2) \neq \langle 1 \rangle$. Consider $C(\mathfrak{A}_1) \supseteq \langle \mathfrak{A}, \mathfrak{Q}_1, \mathfrak{R}_2 \rangle = \mathfrak{L}$. By Lemma 17.6, $\mathfrak{A} \supseteq O_{p',p}(\mathfrak{L})$ and it follows readily that $\mathfrak{L}$ possesses a normal complement $\mathfrak{H}_0$ to $\mathfrak{A}$. We can then find C in $C_{\mathfrak{L}}(\mathfrak{A})$ such that $\mathfrak{H} = \langle \mathfrak{Q}_1, \mathfrak{R}_2^C \rangle$ is a q, r-group. By Lemma 22.1 and the fact that $\mathfrak{Q}_1$ is a maximal element of $\mathsf{M}(\mathfrak{A}; q)$, we have $\mathfrak{Q}_1 \lhd \mathfrak{H}$. But now $\mathfrak{R}_2^C \subseteq N(\mathfrak{Q}_1) = O^p(N(\mathfrak{Q}_1))$. Since $q \not\sim r$, if $\mathfrak{S}_r$ is a S_r-subgroup of $N(\mathfrak{Q}_1)$, then $\mathscr{SCN}_3(\mathfrak{S}_r)$ is empty. By Lemma 8.13 $N(\mathfrak{Q}_1)'$ centralizes every chief r-factor of $N(\mathfrak{Q}_1)$. It follows that $\mathfrak{A}$ centralizes $\mathfrak{R}_2^C$, contrary to construction, so we can assume that $\gamma\mathfrak{Q}_1\mathfrak{A} \neq \langle 1 \rangle$.

Suppose $\gamma\mathfrak{R}_1\mathfrak{A} = \langle 1 \rangle$. Since $\mathfrak{A}$ possesses an elementary subgroup of order p^3, we can find A in $\mathfrak{A}$ such that $C_{\mathfrak{Q}_1}(A)$ is non cyclic. Consider $C(A) \supseteq \langle \mathfrak{A}, C_{\mathfrak{Q}_1}(A), \mathfrak{R}_1 \rangle$. By Lemma 17.6 we can assume that $\mathfrak{H} = \langle C_{\mathfrak{Q}_1}(A), \mathfrak{R}_1 \rangle$ is a q, r-group. Then Lemma 22.1 implies that $\mathfrak{S}_q \lhd \mathfrak{H}$, $\mathfrak{S}_q$ being a S_q-subgroup of $\mathfrak{H}$. Enlarge $\mathfrak{H}$ to $\mathfrak{K}$, a maximal $\mathfrak{A}$-invariant q, r-subgroup with Sylow system $\mathfrak{K}_q, \mathfrak{R}_1$. Lemma 17.6, Lemma 22.1 and maximality of $\mathfrak{K}$ imply that $\mathfrak{K}_q$ is a maximal element of $\mathsf{M}(\mathfrak{A}; q)$, contrary to $q \not\sim r$.

We can now assume that $\gamma\mathfrak{Q}_1\mathfrak{A} \neq \langle 1 \rangle$ and $\gamma\mathfrak{R}_1\mathfrak{A} \neq \langle 1 \rangle$.

Let $\mathfrak{Q}_2$ be an $\mathfrak{A}$-invariant subgroup of $\mathfrak{Q}_1$ of minimal order subject to $\gamma\mathfrak{Q}_2\mathfrak{A} \neq \langle 1 \rangle$. Let $\mathfrak{R}_2$ be an $\mathfrak{A}$-invariant subgroup of $\mathfrak{R}_1$ of minimal order subject to $\gamma\mathfrak{R}_2\mathfrak{A} \neq \langle 1 \rangle$. Let $\mathfrak{A}_1 = \ker(\mathfrak{A} \rightarrow \operatorname{Aut} \mathfrak{Q}_2)$, $\mathfrak{A}_2 = \ker(\mathfrak{A} \rightarrow \operatorname{Aut} \mathfrak{R}_2)$. Since $\mathfrak{A}$ acts irreducibly on $\mathfrak{Q}_2/D(\mathfrak{Q}_2)$ and on $\mathfrak{R}_2/D(\mathfrak{R}_2)$, it follows that $\mathfrak{A}/\mathfrak{A}_i$ is cyclic, $i = 1, 2$. Since $\mathfrak{A} \in \mathscr{SCN}_3(\mathfrak{P})$, $\mathfrak{A}_1 \cap \mathfrak{A}_2 =$

$\mathfrak{A}_3 \neq \langle 1 \rangle$. An $\mathfrak{A}$-invariant $S_{q,r}$-subgroup of $\langle \mathfrak{A}, \mathfrak{Q}_2, \mathfrak{R}_2 \rangle \subseteq C(\mathfrak{A}_3)$ satisfies the conditions defining $\mathscr{K}$, by Lemma 17.6 and $D_{p,q,r}$ in $\langle \mathfrak{A}, \mathfrak{Q}_2, \mathfrak{R}_2 \rangle$.

Let $\mathfrak{K}$ be a maximal element of $\mathscr{K}$, with Sylow system $\mathfrak{K}_q$, $\mathfrak{K}_r$, chosen so that $\mathfrak{A}$ normalizes both $\mathfrak{K}_q$ and $\mathfrak{K}_r$, $\mathfrak{K}_q$ being a S_q-subgroup of $\mathfrak{K}$.

LEMMA 22.3. *$\mathfrak{K}_q$ is cyclic and $O_q(\mathfrak{K}) = \langle 1 \rangle$.*

Proof. Suppose $\mathfrak{K}_q$ is non cyclic. Then Lemma 22.1 yields $\mathfrak{K}_q \lhd \mathfrak{K}$. The maximal nature of $\mathfrak{K}$, together with Lemma 17.6, imply that $\mathfrak{K}_q$ is a maximal element of $\mathsf{M}(\mathfrak{A}; q)$, so is conjugate to $\mathfrak{Q}_1$.

By Lemma 17.3, $N(\mathfrak{K}_q) = \mathfrak{N} = O^p(\mathfrak{N})$. Since $q \not\sim r$, $N(\mathfrak{K}_q)$ does not possess an elementary subgroup of order r^3. Now $\mathfrak{N} = O^p(\mathfrak{N})$ and Lemma 8.13 imply that $\gamma\mathfrak{K}_r\mathfrak{A} = \langle 1 \rangle$, contrary to construction. Hence, $\mathfrak{K}_q$ is cyclic.

If $O_q(\mathfrak{K}) \neq \langle 1 \rangle$, then $\Omega_1(O_q(\mathfrak{K})) = \Omega_1(\mathfrak{K}_q) \lhd \mathfrak{K}$. The maximal nature of $\mathfrak{K}$ now conflicts with Lemma 17.6 and Theorem 22.1 proving this lemma.

We choose C in $C(\mathfrak{A})$ so that $\mathfrak{K}_r^C \subseteq \mathfrak{R}_1$; since $\mathfrak{K}^C$ is also a maximal element of $\mathscr{K}$, we assume without loss of generality that $\mathfrak{K}_r \subseteq \mathfrak{R}_1$.

LEMMA 22.4.

(i) *$\mathfrak{K}_r$ is non abelian.*

(ii) *No non identity weakly closed subgroup of $\mathfrak{K}_r$ is contained in $O_r(\mathfrak{K})$.*

(iii) *$O_r(\mathfrak{K})$ contains an element of $\mathscr{U}(\mathfrak{R})$, $\mathfrak{R}$ being any S_r-subgroup of $\mathfrak{G}$ containing a S_r-subgroup $\mathfrak{R}^*$ of $N(\mathfrak{R}_1)$.*

Proof. We first prove (ii). Suppose $\mathfrak{T} \neq \langle 1 \rangle$, $\mathfrak{T}$ is weakly closed in $\mathfrak{K}_r$, and $\mathfrak{T} \subseteq O_r(\mathfrak{K})$. Then $\mathfrak{T} \lhd \mathfrak{K}\mathfrak{A}$, so the maximal nature of $\mathfrak{K}$ together with Lemma 17.6 imply that $\mathfrak{K}_r = \mathfrak{R}_1$.

Since $N(\mathfrak{R}_1) = O^p(N(\mathfrak{R}_1))$, so also $N(\mathfrak{T}) = O^p(N(\mathfrak{T}))$. Since $q \not\sim r$, Lemma 8.13 implies $\gamma\mathfrak{A}\mathfrak{K}_q = \langle 1 \rangle$, contrary to construction, proving (ii).

If $\mathfrak{K}_r$ were abelian, then $O_q(\mathfrak{K}) = \langle 1 \rangle$ and Lemma 1.2.3 of [21] imply that $\mathfrak{K}_r = O_r(\mathfrak{K})$, in violation of (ii). This proves (i).

Suppose $r \in \pi_3$. In this case, $C^*_{p,r}$ implies $\mathfrak{R}^* = \mathfrak{R}$, and since $\mathfrak{R}_1' \neq \langle 1 \rangle$, it is clear that $\mathfrak{R}_1$ contains an element $\mathfrak{U}$ of $\mathscr{U}(\mathfrak{R})$. Since $\mathfrak{K}_r = N(O_r(\mathfrak{K})) \cap \mathfrak{R}_1$, it follows that $\mathfrak{U} \cap Z(\mathfrak{R}) \subseteq \mathfrak{K}_r$ and so by (B), $\mathfrak{U} \cap Z(\mathfrak{R}) \subseteq O_r(\mathfrak{K})$. It now follows that $\mathfrak{U} \subseteq \mathfrak{K}_r$, and so $\mathfrak{U} \subseteq O_r(\mathfrak{K})$, again by (B). Next, suppose that $r \in \pi_4$. In this case, since $\mathfrak{R}_1' \neq \langle 1 \rangle$, $\mathfrak{R}^*$ contains an element $\mathfrak{B}$ of $\mathscr{U}(\mathfrak{R})$, $\mathfrak{R}^*$ being a S_r-subgroup of $N(\mathfrak{R}_1')$. Since $\mathfrak{B}$ centralizes $O_p(\mathfrak{P}\mathfrak{R}^*)$, by Lemma 19.1, we have $\mathfrak{B} \subseteq \mathfrak{R}_1$. Since $\mathfrak{B} \subseteq \mathfrak{R}_1$, $\mathfrak{B} \cap Z(\mathfrak{R}) \subseteq \mathfrak{K}_r$ and so by (B), $\mathfrak{B} \cap Z(\mathfrak{R}) \subseteq O_r(\mathfrak{K})$. It follows that $\mathfrak{B} \subseteq O_r(\mathfrak{K})$. This proves (iii).

To prove Theorem 22.2 we will now show that $\mathfrak{K}_q$ centralizes $Z(O_r(\mathfrak{K})) = \mathfrak{Z}$. Suppose by way of contradiction that this is not the case. We can choose $\mathfrak{C} \in \mathscr{U}(r)$ such that $\mathfrak{C} \subseteq \mathfrak{K}_r$ but $\mathfrak{C} \not\subseteq O_r(\mathfrak{K})$. Since $\mathfrak{K}_q$ is cyclic $\mathfrak{C}_1 = \mathfrak{C} \cap O_r(\mathfrak{K})$ is of order r. From (B), we then have $\gamma^{r-1}\mathfrak{Z}\mathfrak{C}^{r-1} \neq \langle 1 \rangle$.

If $r \geqq 5$, we apply Lemma 16.2 and conclude that $\gamma^4\mathfrak{Z}\mathfrak{C}^4 = \langle 1 \rangle$, contrary to the above statement. Hence $r = 3$, and by Lemma 16.3 we have $\gamma^2\mathfrak{Z}\mathfrak{C}^2 = \mathfrak{C}_1$; in particular, $\mathfrak{C}_1 \subseteq \mathfrak{Z}$. Now apply Lemma 16.3 again, this time with $O_3(\mathfrak{K})$ in the role of $\mathfrak{F}$, and conclude that $\gamma^2 O_3(\mathfrak{K})\mathfrak{C}^2 = \mathfrak{C}_1$.

Let $\mathfrak{T} = \gamma O_3(\mathfrak{K})\mathfrak{K}_q$. By Lemma 8.11, we have $\mathfrak{T} = \gamma\mathfrak{T}\mathfrak{K}_q$, and so $\mathfrak{C}_1 \subseteq \Omega_1(Z(\mathfrak{T}))$. Hence by (B), $\mathfrak{K}_q$ acts trivially on $\mathfrak{T}/\Omega_1(Z(\mathfrak{T}))$, and this implies that $\mathfrak{T} = \Omega_1(Z(\mathfrak{T}))$, so that $\mathfrak{T}$ is elementary.

The equality $\gamma^2\mathfrak{T}\mathfrak{C}^2 = \mathfrak{C}_1$ and (B) imply that an element of $\mathfrak{C} - \mathfrak{C}_1$ induces an automorphism of $\mathfrak{T}$ with matrix J_3. Since $|\mathfrak{K}_q|$ divides $3^3 - 1$, we have $|\mathfrak{K}_q| = 13$.

By definition of $\mathscr{K}$ we have $p/12 = |\operatorname{Aut} \mathfrak{K}_{13}|$. Since $p \neq r = 3$, we have a contradiction, completing the proof that $\mathfrak{K}_q$ centralizes $\mathfrak{Z}$ in all cases.

Now $Z(\mathfrak{N}_1)$ centralizes $O_r(\mathfrak{K})$, so by maximality of $\mathfrak{K}$, we have $Z(\mathfrak{N}_1) \subseteq \mathfrak{K}_r$ and (B) implies that $Z(\mathfrak{N}_1) \subseteq Z(O_r(\mathfrak{K}))$. Hence, $\mathfrak{K} \subseteq N(Z(\mathfrak{N}_1)) = \mathfrak{N}_1$. But $\mathfrak{N}_1 = O^p(\mathfrak{N}_1)$ and since $q \not\sim r$, $\mathfrak{N}_1$ does not possess an elementary subgroup of order q^3. Lemma 8.13 implies that $\gamma\mathfrak{K}_q\mathfrak{A} = \langle 1 \rangle$, contrary to construction, completing the proof of Theorem 22.2.

For p in $\pi_3 \cup \pi_4$, let $\mathscr{W}(p)$ be the set of all subgroups $\mathfrak{W}$ of $\mathfrak{G}$ of type (p, p) such that every element W of $\mathfrak{W}$ centralizes an element $\mathfrak{B}$ of $\mathscr{U}(p)$. We allow $\mathfrak{B}$ to depend on W.

Hypothesis 22.3.

(i) $p \in \pi_3$, $q \in \pi(\mathfrak{G})$.

(ii) $p \not\sim q$.

THEOREM 22.3. *Under Hypothesis* 22.3, *if* $\mathfrak{K}$ *is a* p, q*-subgroup of* $\mathfrak{G}$ *and if* $\mathfrak{K}$ *contains an element of* $\mathscr{W}(p)$, *then a* S_p*-subgroup of* $\mathfrak{K}$ *is normal in* $\mathfrak{K}$.

Proof. Let $\mathscr{K}$ be the set of subgroups of $\mathfrak{G}$ satisfying the hypotheses but not the conclusion of this theorem and let $\mathscr{K}_1$ be the subset of all $\mathfrak{K}$ in $\mathscr{K}$ which contain at least one element of $\mathscr{U}(p)$.

We first show that $\mathscr{K}_1$ is empty. Suppose false and $\mathfrak{K}$ in $\mathscr{K}_1$ is chosen to maximize $|\mathfrak{K}|_p$. Let $\mathfrak{K}_p$ be a S_p-subgroup of $\mathfrak{K}$, and let $\mathfrak{V} = V(ccl_{\mathfrak{G}}(\mathfrak{B}); \mathfrak{K}_p)$ where $\mathfrak{B} \in \mathscr{U}(p)$ and $\mathfrak{B} \subseteq \mathfrak{K}_p$.

Since $p \not\sim q$, Hypothesis 22.1 does not hold. Hence, Hypothesis

19.1 holds. Apply Lemma 19.1 and conclude that $\mathfrak{B}$ centralizes $O_q(\mathfrak{K})$.

Suppose $\mathfrak{K}_p$ is a S_p-subgroup of $\mathfrak{G}$. Then $\mathfrak{K}_p$ centralizes $O_q(\mathfrak{K})$. By Lemma 17.5 and Hypothesis 22.3 (i), if $\mathfrak{A} \in \mathscr{S}\mathscr{C}\mathscr{N}_3(\mathfrak{K}_p)$, and $\mathfrak{B}_1 = V(ccl_{\mathfrak{G}}(\mathfrak{A}); \mathfrak{K}_p)$, then $\mathfrak{B}_1 \subseteq O_{q,p}(\mathfrak{K})$. Since $\mathfrak{K}_p$ centralizes $O_q(\mathfrak{K})$, it follows that $\mathfrak{B}_1 \subseteq O_p(\mathfrak{K})$, and so $\mathfrak{B}_1 \lhd \mathfrak{K}$. By Lemma 17.2, $N(Z(\mathfrak{B}_1)) = O^p(N(Z(\mathfrak{B}_1)))$. Since $p \not\sim q$, $N(Z(\mathfrak{B}_1))$ does not possess an elementary subgroup of order q^3, so Lemma 8.13 implies that $\mathfrak{K}_p \lhd \mathfrak{K}$, contrary to the definition of $\mathscr{K}_1$. Hence, in showing that $\mathscr{K}_1$ is empty, we can suppose that $\mathfrak{K}_p$ is not a S_p-subgroup of $\mathfrak{G}$.

Since $\mathfrak{B}$ centralizes $O_q(\mathfrak{K})$, we have $\mathfrak{K}_p \cdot O_q(\mathfrak{K}) \subseteq N(\mathfrak{B})$. Since $\mathfrak{B}$ is weakly closed in $\mathfrak{K}_p$ and $\mathfrak{K}_p$ is not a S_p-subgroup of $\mathfrak{G}$, $\mathfrak{K}_p$ is not a S_p-subgoup of $N(\mathfrak{B})$. Maximality of $|\mathfrak{K}|_p$ implies that $\mathfrak{K}_p \lhd \mathfrak{K}_p \cdot O_q(\mathfrak{K})$, and so $O_p(\mathfrak{K})$ is a S_p-subgroup of $O_{q,p}(\mathfrak{K})$.

Let $\mathfrak{P}$ be a S_p-subgroup of $\mathfrak{G}$ containing $\mathfrak{K}_p$, and let $\mathfrak{A} \in \mathscr{S}\mathscr{C}\mathscr{N}_3(\mathfrak{P})$. Since $O_p(\mathfrak{K})$ is a S_p-subgroup of $O_{q,p}(\mathfrak{K})$, it follows from (B) that $\mathfrak{A} \cap \mathfrak{K}_p = \mathfrak{A} \cap O_p(\mathfrak{K})$. By maximality of $|\mathfrak{K}|_p$, $\mathfrak{K}_p$ is a S_p-subgroup of $N(O_p(\mathfrak{K}))$ and it follows readily that $\mathfrak{A} \subseteq O_p(\mathfrak{K})$. But in this case, $\mathfrak{B}_2 = V(ccl_{\mathfrak{G}}(\mathfrak{A}); \mathfrak{K}_p) \lhd \mathfrak{K}$, by Lemma 17.5. Since $\mathfrak{K}_p$ is not a S_p-subgroup of $\mathfrak{G}$, it is not a S_p-subgroup of $N(\mathfrak{B}_2)$, and the maximality of $\mathfrak{K}_p$ is violated in a $S_{p,q}$-subgroup of $N(\mathfrak{B}_2)$. This contradiction shows that $\mathscr{K}_1$ is empty.

Now let $\mathfrak{K}$ be in $\mathscr{K}$ with $|\mathfrak{K}|_p$ maximal. Let $\mathfrak{W} \subseteq \mathfrak{K}_p$, $\mathfrak{W} \in \mathscr{W}(p)$. If $\gamma\mathfrak{W}O_q(\mathfrak{K}) \neq \langle 1 \rangle$, then $\mathfrak{W}$ does not centralize $C(W) \cap O_q(\mathfrak{K})$ for suitable W in $\mathfrak{W}^{\#}$. But in this case a $S_{p,q}$-subgroup of $C(W)$ contains an element of $\mathscr{U}(p)$ and also contains non normal S_p-subgroups, and $\mathscr{K}_1$ is non empty. Since this is not the case, $\mathfrak{W}$ centralizes $O_q(\mathfrak{K})$, and so $\mathfrak{W}_1 = V(ccl_{\mathfrak{G}}(\mathfrak{W}); \mathfrak{K}_p)$ centralizes $O_q(\mathfrak{K})$, $\mathfrak{W}$ being an arbitrary element of $\mathscr{W}(p)$ contained in $\mathfrak{K}_p$. Since $\mathfrak{K}_p$ is not a S_p-subgroup of $\mathfrak{G}$, it is not a S_p-subgroup of $N(\mathfrak{W}_1)$, so maximality of $|\mathfrak{K}|_p$ implies that $\mathfrak{K}_p$ centralizes $O_q(\mathfrak{K})$. Hence, $O_p(\mathfrak{K})$ is a S_p-subgroup of $O_{q,p}(\mathfrak{K})$. Since $\mathfrak{K}_p$ is a S_p-subgroup of $N(O_p(\mathfrak{K}))$ in this case, $Z(\mathfrak{P}) \subseteq O_p(\mathfrak{K})$ for every S_p-subgroup $\mathfrak{P}$ of $\mathfrak{G}$ which contains $\mathfrak{K}_p$. It follows that $\mathfrak{K}_p$ contains an element of $\mathscr{U}(p)$. This contradiction completes the proof of this theorem.

If $p \in \pi_3 \cup \pi_4$, we define $\pi(p)$ to be the set of primes q such that $p \sim q$, and we set $\pi_3(p) = \pi(p) \cap \pi_3$.

THEOREM 22.4. *If $p, q \in \pi_3$ and $p \sim q$, then $\pi_3(p) = \pi_3(q)$.*

Proof. We only need to show that if $r \in \pi_3$ and $p \sim r$, then $r \sim q$.

Apply Theorem 21.1, let $\mathfrak{K}$ be a $S_{p,q}$-subgroup of $\mathfrak{G}$ with Sylow system $\mathfrak{P}$, $\mathfrak{Q}$, and let $\mathfrak{L}$ be a $S_{p,r}$-subgroup of $\mathfrak{G}$ with Sylow system $\mathfrak{P}$, $\mathfrak{R}$.

If Hypothesis 22.2 is satisfied, Theorem 22.2 applies and yields this theorem. Hence, we suppose without loss of generality that $\mathfrak{P}$ centralizes $O_q(\mathfrak{K})$.

Let $\mathfrak{A} \in \mathscr{SCN}_3(\mathfrak{P})$, $\mathfrak{B} = V(ccl_{\mathfrak{G}}(\mathfrak{A}); \mathfrak{P})$. Apply Lemma 17.5 and conclude that $\mathfrak{B} \lhd \mathfrak{K}$.

If $\mathfrak{P}$ also centralizes $O_r(\mathfrak{L})$, then we also have $\mathfrak{B} \lhd \mathfrak{L}$, and $q \sim r$ follows from consideration of $N(\mathfrak{B})$. We can suppose that $\mathfrak{P}$ does not centralize $O_r(\mathfrak{L})$.

Suppose we are able to show that $N(O_r(\mathfrak{L}))$ contains a S_q-subgroup of $C(\mathfrak{P})$. Apply Lemma 17.3 and conclude that $N(Z(\mathfrak{B})) = \mathfrak{N}_1 = O_{p'}(\mathfrak{N}_1) \cdot \mathfrak{N}_1 \cap \mathfrak{N}$, where $\mathfrak{N} = N(O_r(\mathfrak{L}))$. Let $\mathfrak{Q}_1$ be a S_q-subgroup of $C(\mathfrak{P})$ which is contained in $\mathfrak{N}$. Since $\mathfrak{P}$ centralizes $O_q(\mathfrak{K})$, it follows that $\mathfrak{Q}_1$ is a S_q-subgroup of $O_{p'}(\mathfrak{N}_1)$. Let $\mathfrak{N}_1^*$ be a $S_{q'}$-subgroup of $O_{p'}(\mathfrak{N}_1)$, so that $O_{p'}(\mathfrak{N}_1) = \mathfrak{N}_1^* \mathfrak{Q}_1$. Hence,

$$\mathfrak{N}_1 = O_{p'}(\mathfrak{N}_1) \cdot \mathfrak{N}_1 \cap \mathfrak{N} = \mathfrak{N}_1^* \mathfrak{Q}_1 \cdot \mathfrak{N}_1 \cap \mathfrak{N} = \mathfrak{N}_1^* \cdot \mathfrak{N}_1 \cap \mathfrak{N} ,$$

since $\mathfrak{Q}_1 \subseteq \mathfrak{N}_1 \cap \mathfrak{N}$. Since $\mathfrak{N}_1$ contains a S_q-subgroup of $\mathfrak{G}$, so does $\mathfrak{N}_1 \cap \mathfrak{N}$. But $\mathfrak{N}$ contains a S_r-subgroup of $\mathfrak{G}$ as well, and so $q \sim r$.

Thus, in proving this theorem, it suffices to show that $N(O_r(\mathfrak{L}))$ contains a S_q-subgroup of $C(\mathfrak{P})$.

We wish to show first that some element A of $\mathfrak{A}^\#$ centralizes a subgroup $\mathfrak{W}$ of $\mathscr{W}(r)$ with $\mathfrak{W} \subseteq O_r(\mathfrak{L})$. If $D(O_r(\mathfrak{L})) = \mathfrak{D}$ is non cyclic, then every subgroup of $\mathfrak{D}$ of type (r, r) is in $\mathscr{W}(r)$ and since $\mathfrak{A}$ possesses an elementary subgroup of order p^3, an element A is available. Suppose then that $\mathfrak{D}$ is cyclic. If $\mathfrak{D} = \langle 1 \rangle$, then of course $\mathfrak{P}$ centralizes $\mathfrak{D}$. If $\mathfrak{D} \neq \langle 1 \rangle$, then $N(\mathfrak{D}) = O^p(N(\mathfrak{D}))$ and once again $\mathfrak{P}$ centralizes $\mathfrak{D}$. It now follows that $\mathfrak{A}^\#$ contains an element A whose fixed-point set on $\Omega_1(O_r(\mathfrak{L}))/\Omega_1(\mathfrak{D})$ is non cyclic, and this implies that $C(\mathfrak{A}) \cap O_r(\mathfrak{L})$ contains an element of $\mathscr{W}(r)$.

For such an element A, let $\mathfrak{H}$ be a $S_{q,r}$-subgroup of $O_{p'}(C(A))$ which is $\mathfrak{A}$-invariant and contains $O_q(\mathfrak{K})$. Then Lemma 17.5 implies that $\mathfrak{H}$ contains an element of $\mathscr{W}(r)$. Apply Theorem 22.3 and conclude that $\mathfrak{H}_r \lhd \mathfrak{H}$, $\mathfrak{H}_r$ being a S_r-subgroup of $\mathfrak{H}$. If $\mathfrak{H}^*$ is a maximal element of $\mathsf{M}(\mathfrak{A}; q, r)$ containing $\mathfrak{H}$, then Theorem 22.3 implies that $\mathfrak{H}_r^* \lhd \mathfrak{H}^*$, $\mathfrak{H}_r^*$ being a S_r-subgroup of $\mathfrak{H}^*$. By maximality of $\mathfrak{H}^*$, $\mathfrak{H}_r^*$ is a maximal element of $\mathsf{M}(\mathfrak{A}; r)$. Since $\mathfrak{H}$ contains a maximal element of $\mathsf{M}(\mathfrak{A}; q)$, namely, $O_q(\mathfrak{K})$, so does $\mathfrak{H}^*$. It follows that $N(O_r(\mathfrak{L}))$ contains a maximal element $\mathfrak{Q}^*$ of $\mathsf{M}(\mathfrak{P}^*; q)$ where $\mathfrak{P}^*$ is a suitable S_p-subgroup of $N(O_r(\mathfrak{L}))$. But $\mathfrak{P} \subseteq N(O_r(\mathfrak{L}))$, and so $\mathfrak{P} = \mathfrak{P}^{*N}$ for some N in $N(O_r(\mathfrak{L}))$, and so $\mathfrak{Q}^{*N} = \mathfrak{Q}_2$ is a maximal element of $\mathsf{M}(\mathfrak{P}; q)$ normalizing $O_r(\mathfrak{L})$. By Lemma 17.4, $\mathfrak{Q}_2$ is a maximal element of $\mathsf{M}(\mathfrak{A}; q)$.

Now $\mathfrak{P}$ centralizes $O_q(\mathfrak{K})$, and $O_q(\mathfrak{K})$ is a maximal element of $\mathsf{M}(\mathfrak{P}; q)$. It follows that $N(O_q(\mathfrak{K}))/C(O_q(\mathfrak{K}))$ is a p'-group. Since $\mathfrak{Q}_2$ and $O_q(\mathfrak{K})$

are conjugate by Theorem 17.1, it follows that $N(\mathfrak{Q}_2)/C(\mathfrak{Q}_2)$ is a p'-group, and so $\mathfrak{P}$ centralizes $\mathfrak{Q}_2$. By $C^*_{p,q}$, it follows that $\mathfrak{Q}_2$ is a S_q-subgroup of $C(\mathfrak{P})$, completing the proof of this theorem.

THEOREM 22.5. *If $p \in \pi_3$, then $\mathfrak{G}$ satisfies $C_{\pi_3(p)}$.*

Proof. By Theorem 22.4, if $q, r \in \pi_3(p)$, then $q \sim r$. By Theorem 20.1, $\mathfrak{G}$ satisfies $E_{q,r}$ for $q, r \in \pi_3(p)$. By Theorem 21.2, $\mathfrak{G}$ satisfies $C_{\pi_3(p)}$.

Hypothesis 22.4.

(i) $p \in \pi_3$, $q \in \pi_3 \cup \pi_4$.

(ii) *If $\mathfrak{P}$ is a S_p-subgroup of $\mathfrak{G}$, then $\mathfrak{P}$ contains a normal subgroup $\mathfrak{E}$ of type (p, p) which centralizes at least one maximal element of $\mathsf{M}(\mathfrak{P}; q)$.*

LEMMA 22.5. *Under Hypothesis 22.4, $\mathfrak{E}$ centralizes every element of $\mathsf{M}(\mathfrak{E}; q)$.*

Proof. Suppose false and $\mathfrak{Q}$ is an element of $\mathsf{M}(\mathfrak{E}; q)$ minimal with respect to $\gamma\mathfrak{Q}\mathfrak{E} \neq \langle 1 \rangle$. Then $\mathfrak{Q} = \gamma\mathfrak{Q}\mathfrak{E}$ and $\mathfrak{E}_0 = C_{\mathfrak{E}}(\mathfrak{Q}) \neq \langle 1 \rangle$. Let $\mathfrak{H} = C(\mathfrak{E}_0)$. Then $\mathfrak{H}$ contains an element $\mathfrak{A}$ of $\mathscr{SCN}_3(\mathfrak{P})$ with $\mathfrak{E} \subseteq \mathfrak{A}$. By Lemma 17.5, $\mathfrak{A} \subseteq O_{p',p}(\mathfrak{H})$, and so $\mathfrak{Q} = \gamma\mathfrak{Q}\mathfrak{E}$ is contained in $O_{p'}(\mathfrak{H})$. If $\mathfrak{Q}^*$ is an $\mathfrak{A}$-invariant S_q-subgroup of $O_{p'}(\mathfrak{H})$, it follows readily that $\gamma\mathfrak{Q}^*\mathfrak{E} \neq \langle 1 \rangle$. If $\tilde{\mathfrak{Q}}$ is a maximal element of $\mathsf{M}(\mathfrak{A}; q)$ containing $\mathfrak{Q}^*$, then $\mathfrak{E}$ does not centralize $\tilde{\mathfrak{Q}}$. Let $\mathfrak{Q}_0$ be a maximal element of $\mathsf{M}(\mathfrak{P}; q)$ centralizing $\mathfrak{E}$. Since $\mathfrak{Q}_0$ is also a maximal element of $\mathsf{M}(\mathfrak{A}; q)$, we have $\mathfrak{Q}_0 = \tilde{\mathfrak{Q}}^C$ for suitable C in $C(\mathfrak{A}) \subseteq C(\mathfrak{E})$. Since $\mathfrak{E}$ does not centralize $\tilde{\mathfrak{Q}}$, $\mathfrak{E}^C = \mathfrak{E}$ does not centralize $\mathfrak{Q}_0$. This contradiction completes the proof of this lemma.

The next theorem is fairly delicate and brings π_4 into play explicitly for the first time.

Hypothesis 22.5.

(i) $p \in \pi_3$, $q \in \pi_4$.

(ii) $p \sim q$.

THEOREM 22.6. *Under Hypothesis 22.5, if $\mathfrak{P}$ is a S_p-subgroup of $\mathfrak{G}$ and $\mathfrak{Q}_1$ is a maximal element of $\mathsf{M}(\mathfrak{P}; q)$, then $\mathfrak{Q}_1 \neq \langle 1 \rangle$. If $\mathfrak{Q}_2$ is a S_q-subgroup of $N(\mathfrak{Q}_1)$ permutable with $\mathfrak{P}$ and $\mathfrak{Q}_3$ is a S_q-subgroup of $\mathfrak{G}$ containing $\mathfrak{Q}_2$, then $\mathfrak{Q}_1$ contains every element of $\mathscr{SCN}(\mathfrak{Q}_3)$. Furthermore, $O_p(\mathfrak{P}\mathfrak{Q}_2) = \langle 1 \rangle$.*

Proof. By Theorem 19.1, $\mathfrak{P}$ does not centralize $\mathfrak{Q}_1$, so in particular $\mathfrak{Q}_1 \neq \langle 1 \rangle$.

Suppose that $\mathfrak{Q}_2$ contains an element $\mathfrak{B}$ of $\mathscr{U}(\mathfrak{Q}_3)$. By Lemma 19.1, $\mathfrak{B}$ centralizes $O_p(\mathfrak{P}\mathfrak{Q}_2)$ and since $\mathfrak{B}$ is a normal abelian subgroup of $\mathfrak{Q}_2$, (B) implies that $\mathfrak{B} \subseteq O_q(\mathfrak{P}\mathfrak{Q}_2)$. Let $\mathfrak{A}$ be an element of $\mathscr{SCN}_3(\mathfrak{Q}_3)$ containing $\mathfrak{B}$. Let $\mathfrak{H} = N(\mathfrak{B}) \supseteq \langle \mathfrak{Q}_3, O_p(\mathfrak{P}\mathfrak{Q}_2)\rangle$. Since $q \in \pi_4$, $O_{q'}(\mathfrak{H}) = \langle 1 \rangle$, and (B) implies that $\mathfrak{A} \subseteq O_q(\mathfrak{H})$. Hence, $[\mathfrak{A} \cap \mathfrak{Q}_2, O_p(\mathfrak{P}\mathfrak{Q}_2)] \subseteq O_q(\mathfrak{H}) \cap O_p(\mathfrak{P}\mathfrak{Q}_2) = \langle 1 \rangle$, and by (B), $\mathfrak{A} \cap \mathfrak{Q}_2 \subseteq O_{p,q}(\mathfrak{P}\mathfrak{Q}_2)$, and so $\mathfrak{A} \cap \mathfrak{Q}_2 \subseteq O_q(\mathfrak{P}\mathfrak{Q}_2)$, that is, $\mathfrak{A} \cap \mathfrak{Q}_2 = \mathfrak{A} \cap \mathfrak{Q}_1$. If $\mathfrak{A} \cap \mathfrak{Q}_1 \subset \mathfrak{A}$, then $\mathfrak{A} \cap \mathfrak{Q}_1 \subset N_{\mathfrak{A}}(\mathfrak{Q}_1)$, contrary to $N_{\mathfrak{A}}(\mathfrak{Q}_1) \subseteq \mathfrak{A} \cap \mathfrak{Q}_2 = \mathfrak{A} \cap \mathfrak{Q}_1$. Hence, $\mathfrak{A} \subseteq \mathfrak{Q}_1$. Since $q \in \pi_4$, Corollary 17.3 implies that $\mathsf{M}(\mathfrak{A}; p)$ is trivial, so $O_p(\mathfrak{P}\mathfrak{Q}_2) = \langle 1 \rangle$. By Lemma 7.9, it follows that $\mathfrak{Q}_1$ contains every element of $\mathscr{SCN}(\mathfrak{Q}_3)$, and not merely $\mathfrak{A}$. This proves the theorem in this case.

We can now assume that $\mathfrak{Q}_2$ does not contain any element of $\mathscr{U}(\mathfrak{Q}_3)$, and try to derive a contradiction.

Since $\mathfrak{Q}_2$ is a S_q-subgroup of the normalizer of every non identity normal subgroup of $\mathfrak{P}\mathfrak{Q}_2$, if $D(\mathfrak{Q}_1) \neq \langle 1 \rangle$, then $N_{\mathfrak{Q}_3}(D(\mathfrak{Q}_1))$ contains an element of $\mathscr{U}(\mathfrak{Q}_3)$, and since $N_{\mathfrak{Q}_3}(D(\mathfrak{Q}_1)) = \mathfrak{Q}_2$ in this case, $\mathfrak{Q}_2$ contains an element of $\mathscr{U}(\mathfrak{Q}_3)$, contrary to assumption. Hence, $D(\mathfrak{Q}_1) = \langle 1 \rangle$.

Let $\mathfrak{Q}_1^* = O_{p,q}(\mathfrak{P}\mathfrak{Q}_2) \cap \mathfrak{Q}_2$. Since $[\mathfrak{Q}_1^*, \mathfrak{Q}_1] = [O_{p,q}(\mathfrak{P}\mathfrak{Q}_2), \mathfrak{Q}_1] \lhd \mathfrak{P}\mathfrak{Q}_2$, and since every element of $\mathscr{U}(\mathfrak{Q}_3)$ normalizes $[\mathfrak{Q}_1^*, \mathfrak{Q}_1]$, we conclude that $\mathfrak{Q}_1 \subseteq Z(\mathfrak{Q}_1^*)$. Since $D(\mathfrak{Q}_1^*) \cap \mathfrak{Q}_1$ is normalized by every element of $\mathscr{U}(\mathfrak{Q}_3)$ and also by $\langle O_p(\mathfrak{P}\mathfrak{Q}_2), \mathfrak{P}\mathfrak{Q}_2 \cap N(\mathfrak{Q}_1^*)\rangle = \mathfrak{P}\mathfrak{Q}_2$, we have $D(\mathfrak{Q}_1^*) \cap \mathfrak{Q}_1 = \langle 1 \rangle$. This implies that $\mathfrak{Q}_1^* = \mathfrak{Q}_1 \times \mathfrak{F}$ for a suitable subgroup $\mathfrak{F}$ of $\mathfrak{Q}_1^*$.

Since $Z(\mathfrak{Q}_3) \subseteq \mathfrak{Q}_2$, we have $Z(\mathfrak{Q}_3) \subseteq \mathfrak{Q}_1^*$, by (B). Since $\mathfrak{Q}_2$ contains no element of $\mathscr{U}(\mathfrak{Q}_3)$, $Z(\mathfrak{Q}_3)$ is cyclic. For the same reason, $Z(\mathfrak{Q}_3) \cap \mathfrak{Q}_1 = \langle 1 \rangle$, since otherwise, $\Omega_1(Z(\mathfrak{Q}_3)) \subseteq \mathfrak{Q}_1$ and every element of $\mathscr{U}(\mathfrak{Q}_3)$ normalizes $\mathfrak{Q}_1$. In particular, $\mathfrak{Q}_1$ is a proper subgroup of $\mathfrak{Q}_1^*$. This implies that $O_p(\mathfrak{P}\mathfrak{Q}_2) \neq \langle 1 \rangle$. More exactly, $\mathfrak{Q}_1 = C_{\mathfrak{Q}_1^*}(O_p(\mathfrak{P}\mathfrak{Q}_2))$.

Let $\mathfrak{B} \in \mathscr{U}(\mathfrak{Q}_3)$ and let $\tilde{\mathfrak{Q}}_1 = C_{\mathfrak{Q}_1}(\mathfrak{B})$, so that $|\mathfrak{Q}_1 : \tilde{\mathfrak{Q}}_1| = q$.

Suppose $O_p(\mathfrak{P}\mathfrak{Q}_2)$ is non cyclic. In this case, a basic property of p-groups implies that $O_p(\mathfrak{P}\mathfrak{Q}_2)$ contains a subgroup $\mathfrak{E}$ of type (p, p) which is normal in $\mathfrak{P}$. Since $\mathfrak{Q}_1$ is a maximal element of $\mathsf{M}(\mathfrak{P}; q)$, Hypothesis 22.4 is satisfied. Since $\tilde{\mathfrak{Q}}_1$ is of index q in $\mathfrak{Q}_1$, Theorem 22.1 implies that $\tilde{\mathfrak{Q}}_1 \neq \langle 1 \rangle$. Hence, $\langle \mathfrak{B}, \mathfrak{E} \rangle$ is a proper subgroup of $\mathfrak{G}$ centralizing $\tilde{\mathfrak{Q}}_1$. Choose $\mathfrak{B}_1 \in ccl_{\mathfrak{G}}(\mathfrak{B})$ and $\mathfrak{E}_1 \in ccl_{\mathfrak{G}}(\mathfrak{E})$ so that $\mathfrak{K} = \langle \mathfrak{B}_1, \mathfrak{E}_1 \rangle$ is minimal. By $D_{p,q}$ in $\mathfrak{K}$, it follows that $\mathfrak{K}$ is a p, q-group. By Lemma 19.1, $\mathfrak{B}_1^{\mathfrak{K}}$ centralizes $O_p(\mathfrak{K})$ and by Lemma 22.5, $\mathfrak{E}_1^{\mathfrak{K}}$ centralizes $O_q(\mathfrak{K})$. It follows that $\mathfrak{K} = \mathfrak{B}_1 \times \mathfrak{E}_1$. Let $\mathfrak{N} = N(\mathfrak{B}_1)$. Since $q \in \pi_4$, $F(\mathfrak{N})$ is a q-group. By Lemma 22.5, $\mathfrak{E}_1$ centralizes $F(\mathfrak{N})$ so 3.3 is violated. This contradiction shows that $O_p(\mathfrak{P}\mathfrak{Q}_2)$ is cyclic.

Since $\mathfrak{Q}_1 = C_{\mathfrak{Q}_1^*}(O_p(\mathfrak{P}\mathfrak{Q}_2))$, it follows that $\mathfrak{F}$ is cyclic of an order dividing $p - 1$.

Let $\mathfrak{B} = \Omega_1(\mathfrak{Q}_1^*) = \mathfrak{Q}_1 \times \Omega_1(\mathfrak{F})$, and let $\mathfrak{H}_1 = N_{\mathfrak{P}\mathfrak{Q}_2}(\mathfrak{B})$. We see that $\mathfrak{P}\mathfrak{Q}_2 = \mathfrak{H}_1 O_p(\mathfrak{P}\mathfrak{Q}_2)$, $\mathfrak{H}_1 \cap O_p(\mathfrak{P}\mathfrak{Q}_2) = \langle 1 \rangle$. Let $\mathfrak{M} = N(\mathfrak{B})$, $\mathfrak{M}_1 = C(\mathfrak{B})$. It is clear that $\mathfrak{M}_1 \cap \mathfrak{P}\mathfrak{Q}_2 = \mathfrak{Q}_1^*$, since $\mathfrak{M}_1 \cap O_p(\mathfrak{P}\mathfrak{Q}_2) = \langle 1 \rangle$, and since $\mathfrak{Q}_1^*$ is a S_q-subgroup of $O_{p,q}(\mathfrak{P}\mathfrak{Q}_2)$.

Let $\mathfrak{L} = O_q(\mathfrak{M} \bmod \mathfrak{M}_1)$. We see that $\mathfrak{L} \cap \mathfrak{P}\mathfrak{Q}_2 = \mathfrak{Q}_1^*$, again since $\mathfrak{Q}_1^*$ is a S_q-subgroup of $O_{p,q}(\mathfrak{P}\mathfrak{Q}_2)$. We observe that since $\mathfrak{Q}_1^*$ contains $Z(\mathfrak{Q}_3)$, $\mathfrak{M}$ contains every element of $\mathscr{U}(\mathfrak{Q}_3)$, and so contains $\mathfrak{B}$. By Lemma 7.1, $\mathfrak{L}$ contains $\mathfrak{B}$. Hence, $\mathfrak{L} \supset \mathfrak{M}_1$.

We next show that $\mathfrak{L}/\mathfrak{M}_1 = \bar{\mathfrak{L}}$ is elementary. If $D(\bar{\mathfrak{L}}) \neq \langle 1 \rangle$, then by a basic property of q-groups, $C_{\mathfrak{B}}(D(\bar{\mathfrak{L}}))$ is of order at least q^2. Hence, $\hat{\mathfrak{Q}}_1 = C_{\mathfrak{B}}(D(\bar{\mathfrak{L}})) \cap \mathfrak{Q}_1 \neq \langle 1 \rangle$. But in this case, $\hat{\mathfrak{Q}}_1$ is normalized by $\langle O_p(\mathfrak{P}\mathfrak{Q}_2), \mathfrak{H}_1 \rangle = \mathfrak{P}\mathfrak{Q}_2$, and is centralized by $D(\mathfrak{L} \bmod \mathfrak{M}_1)$, and so $\mathfrak{Q}_2$ is not a S_q-subgroup of $N(\mathfrak{Q}_1)$. This is not possible, so $D(\bar{\mathfrak{L}}) = \langle 1 \rangle$. We have in fact shown that if $\mathfrak{L}_1 \lhd \mathfrak{M}$, and $\mathfrak{M}_1 \subset \mathfrak{L}_1 \subseteq \mathfrak{L}$, then $C_{\mathfrak{B}}(\mathfrak{L}_1)$ is of order q.

Since $\bar{\mathfrak{L}}$ is abelian, $\mathfrak{L}$ normalizes $[\mathfrak{B}, \mathfrak{B}] = \Omega_1(Z(\mathfrak{Q}_3))$. It follows that $C_{\mathfrak{B}}(\mathfrak{L}) = \Omega_1(Z(\mathfrak{Q}_3))$.

Let $\mathfrak{L}_1 = \langle \mathfrak{B}^{\mathfrak{M}}, \mathfrak{M}_1 \rangle$, and let $\mathfrak{B}_1 = \mathfrak{B}^M$, $M \in \mathfrak{M}$. Since $\mathfrak{B}$ and $\mathfrak{B}_1$ are conjugate in $\mathfrak{M}$, $[\mathfrak{B}, \mathfrak{B}_1]$ is of order q and is centralized by $\mathfrak{L}$. It follows that $[\mathfrak{B}, \mathfrak{B}_1] = \Omega_1(Z(\mathfrak{Q}_3))$. Since $\bar{\mathfrak{L}}$ is abelian, and since $\mathfrak{B}^{\mathfrak{M}}$ covers $\mathfrak{L}_1/\mathfrak{M}_1 = \bar{\mathfrak{L}}_1$, it follows that $[\mathfrak{B}, \mathfrak{L}_1] = \Omega_1(Z(\mathfrak{Q}_3))$. Let $|\mathfrak{Q}_1| = q^n$, and $|\mathfrak{L}_1 : \mathfrak{M}_1| = q^m$. Since each element of $\bar{\mathfrak{L}}_1^\#$ determines a non trivial homomorphism of $\mathfrak{B}/\Omega_1(Z(\mathfrak{Q}_3))$ into $\Omega_1(Z(\mathfrak{Q}_3))$, it follows that $m \leqq n$. Since $C_{\mathfrak{B}}(\mathfrak{L}_1) = \Omega_1(Z(\mathfrak{Q}_3))$, it also follows that $m \geqq n$. Hence, $m = n$. This implies that $\mathfrak{L}_1 = \mathfrak{L}$, since any q-element of $\operatorname{Aut} \mathfrak{B}$ which centralizes $\bar{\mathfrak{L}}_1$ is in $\bar{\mathfrak{L}}_1$, by 3.10. Here we are invoking the well known fact that $\bar{\mathfrak{L}}_1$ is normal in a S_q-subgroup $\mathfrak{Q}$ of $\operatorname{Aut} \mathfrak{B}$ and is in fact in $\mathscr{SCN}(\mathfrak{Q})$. (This appeal to the "enormous" group $\operatorname{Aut} \mathfrak{B}$ is somewhat curious.)

Returning to $\mathfrak{L}$, let $\mathfrak{B}^*$ be a S_q-subgroup of $\mathfrak{L}$, and let $\mathfrak{W} = \Omega_1(\mathfrak{B}^*)$. Since $\mho^1(\mathfrak{B}^*) \subseteq Z(\mathfrak{B}^*)$, and $Z(\mathfrak{B}^*)$ is cyclic, it is easy to see that $\Omega_1(Z(\mathfrak{W})) = \Omega_1(Z(\mathfrak{Q}_3))$, and that $\mathfrak{W}/\Omega_1(Z(\mathfrak{W}))$ is abelian. Hence, $\mathfrak{W}$ is an extra special group of order q^{2n+1} and exponent q.

We next show that $\mathfrak{M}_1$ is a p'-group. Since $\mathfrak{M}_1 \subseteq C(\mathfrak{Q}_1)$, it suffices to show that no non identity p-element of $N(\mathfrak{Q}_1)$ centralizes $\mathfrak{B}$. This is clear by $D_{p,q}$ in $N(\mathfrak{Q}_1)$, together with the fact that no non identity p-element of $\mathfrak{P}\mathfrak{Q}_2$ centralizes $\mathfrak{B}$.

Since $\mathfrak{M}_1$ is a p'-group, so is $\mathfrak{L}$. Since $\mathfrak{L} \lhd \mathfrak{M}$, we assume without loss of generality that $N_{\mathfrak{P}}(\mathfrak{B})$ normalizes $\mathfrak{B}^*$.

Let $\mathfrak{C} \in \mathscr{SCN}_3(\mathfrak{P})$, and set $\mathfrak{C}_1 = \mathfrak{C} \cap N_{\mathfrak{P}}(\mathfrak{B})$. Since $\mathfrak{P} = O_p(\mathfrak{P}\mathfrak{Q}_2) \cdot \mathfrak{N}_{\mathfrak{P}}(\mathfrak{B})$ and $O_p(\mathfrak{P}\mathfrak{Q}_2)$ is cyclic, $\mathfrak{C}_1$ is non cyclic. Since $\mathfrak{C}_1$ is faithfully represented on $\mathfrak{B}$, it is faithfully represented on $\mathfrak{W} = \Omega_1(\mathfrak{B}^*)$. Since $p > q$, $\mathfrak{C}_1$ centralizes $\Omega_1(Z(\mathfrak{W}))$.

We can now choose C in $\mathfrak{C}_1^\#$ so that $\mathfrak{C}_1$ does not centralize $\mathfrak{W}_1 = C_{\mathfrak{W}}(C)$. Let $\mathfrak{W}_2 = [\mathfrak{W}_1, \mathfrak{C}_1]$. We will show that $\mathfrak{W}_2$ is non abelian. To do this, we first show that $\mathfrak{W}_1$ is extra special. Let $W \in \mathfrak{W}_1 - \Omega_1(\boldsymbol{Z}(\mathfrak{W}))$. Since C centralizes W, C normalizes $C_{\mathfrak{W}}(W)$. Since $p > q$, C acts trivially on $\mathfrak{W}/C_{\mathfrak{W}}(W)$, and so C centralizes some element of $\mathfrak{W} - C_{\mathfrak{W}}(W)$. It follows that $\boldsymbol{Z}(\mathfrak{W}_1) = \boldsymbol{Z}(\mathfrak{W})$, so that $\mathfrak{W}_1$ is extra special. We can now find $\mathfrak{W}_3 \subseteq \mathfrak{W}_1$ so that $\mathfrak{W}_1 = \mathfrak{W}_2\mathfrak{W}_3$ and $\mathfrak{W}_2 \cap \mathfrak{W}_3 \subseteq \boldsymbol{Z}(\mathfrak{W})$; in fact, we take $\mathfrak{W}_3 = C_{\mathfrak{W}_1}(\mathfrak{W}_2)$. By the argument just given, $\mathfrak{W}_3$ is extra special. Since $\mathfrak{W}_1$ is, too, it follows that $\mathfrak{W}_2$ is extra special, hence is non abelian.

For such an element C, let $\mathfrak{X} = \boldsymbol{C}(C) \supseteq \langle \mathfrak{C}, \mathfrak{W}_2 \rangle$. By Lemma 17.5, $\mathfrak{C} \subseteq \boldsymbol{O}_{p',p}(\mathfrak{X})$. Since $\mathfrak{W}_2 = [\mathfrak{W}_2, \mathfrak{C}_1]$, by Lemma 8.11, it follows that $\mathfrak{W}_2 \subseteq \boldsymbol{O}_{p'}(\mathfrak{X})$. It follows now that $\mathsf{N}(\mathfrak{C}; q)$ contains a non abelian group. But now Theorem 17.1 implies that the maximal elements of $\mathsf{N}(\mathfrak{C}; q)$ are non abelian. Since $\mathfrak{Q}_1$ is a maximal element of $\mathsf{N}(\mathfrak{C}; q)$ and $\mathfrak{Q}_1$ is elementary, we have a contradiction, completing the proof of this theorem.

THEOREM 22.7. *If p, $q \in \pi_3$, and $p \sim q$, then $\pi(p) = \pi(q)$.*

Proof. Suppose $p \sim r$. By Theorem 22.4, we can suppose that $r \in \pi_4$. Proceeding by way of contradiction, we can assume that a S_q-subgroup $\mathfrak{Q}$ of $\mathfrak{G}$ centralizes every element of $\mathsf{N}(\mathfrak{Q}; r)$, by Theorem 22.1. By Theorem 19.1, a S_p-subgroup $\mathfrak{P}$ of $\mathfrak{G}$ does not centralize every element of $\mathsf{N}(\mathfrak{P}; r)$. Applying Theorem 22.2, we can suppose that $\mathfrak{P}$ centralizes every element of $\mathsf{N}(\mathfrak{P}; q)$.

Let $\mathfrak{Q}_1$ be a maximal element of $\mathsf{N}(\mathfrak{P}; q)$ and let $\mathfrak{R}_1$ be a maximal element of $\mathsf{N}(\mathfrak{P}; r)$. Let $\mathfrak{R}_2$ be a S_r-subgroup of $N(\mathfrak{R}_1)$ permutable with $\mathfrak{P}$ and let $\mathfrak{R}_3$ be a S_r-subgroup of $\mathfrak{G}$ containing $\mathfrak{R}_2$. Let $\mathfrak{A} \in \mathscr{SCN}_3(\mathfrak{P})$. By Theorem 22.6, $\boldsymbol{O}_p(\mathfrak{P}\mathfrak{R}_2) = \langle 1 \rangle$, so $\mathfrak{A}$ does not centralize $\mathfrak{R}_1$. We can then find A in $\mathfrak{A}^\#$ such that $\mathfrak{R}_1^* = [C_{\mathfrak{R}_1}(A), \mathfrak{A}] \neq \langle 1 \rangle$.

Suppose $\mathfrak{Q}_1$ is non cyclic. Then by $C_{p,q}^*$, $\mathfrak{Q}_1$ contains an element of $\mathscr{W}(q)$. Let $\mathfrak{H} = \boldsymbol{C}(A) \supseteq \langle \mathfrak{A}, \mathfrak{R}_1^*, \mathfrak{Q}_1 \rangle = \mathfrak{X}$, and let $\mathfrak{K}$ be an $\mathfrak{A}$-invariant $S_{q,r}$-subgroup of $\boldsymbol{O}_{p'}(\mathfrak{X})$ with Sylow system $\mathfrak{K}_r$, $\mathfrak{Q}_1$. By Theorem 22.3, $\mathfrak{Q}_1 \lhd \mathfrak{K}$. Since $N(\mathfrak{Q}_1) = \boldsymbol{O}^p(N(\mathfrak{Q}_1))$, it follows that $\gamma\mathfrak{A}\mathfrak{K}_r = \langle 1 \rangle$ by Lemma 8.11 and the fact that $N(\mathfrak{Q}_1)$ does not contain an elementary subgroup of order r^3. This violates the fact that $\mathfrak{R}_1^* = \gamma\mathfrak{R}_1^*\mathfrak{A} \neq \langle 1 \rangle$, by $D_{p,r}$ in $\mathfrak{H}$. Hence, $\mathfrak{Q}_1$ is cyclic.

Since $\gamma\mathfrak{Q}_1\mathfrak{P} = \langle 1 \rangle$, $\mathfrak{P}_1 = \boldsymbol{O}_p(\mathfrak{P}\mathfrak{Q}_2) \neq \langle 1 \rangle$, where $\mathfrak{Q}_2$ is a S_q-subgroup of $\mathfrak{G}$ permutable with $\mathfrak{P}$ and containing $\mathfrak{Q}_1$, which exists by $C_{p,q}^*$. Since $N(\mathfrak{P}_1) = \boldsymbol{O}^q(N(\mathfrak{P}_1))$, it follows that $\mathfrak{Q}_1 \subseteq \boldsymbol{Z}(\mathfrak{Q}_2)$, $\mathfrak{Q}_1$ being a S_q-subgroup of $\boldsymbol{O}_{p'}(N(\mathfrak{P}_1))$.

Let $\mathfrak{B} = \boldsymbol{V}(ccl_{\mathfrak{G}}(\mathfrak{A}); \mathfrak{P})$, and $\mathfrak{N}_1 = \boldsymbol{N}(\boldsymbol{Z}(\mathfrak{B}))$. By Lemma 17.3, $\mathfrak{N}_1 =$

$O_{p'}(\mathfrak{N}_1)\cdot\mathfrak{N}_1 \cap N(\mathfrak{N}_1)$. Since $\mathfrak{Q}_1$ is a S_q-subgroup of $O_{p'}(\mathfrak{N}_1)$, it follows readily that $\mathfrak{N}_1 \cap N(\mathfrak{N}_1)$ contains an element of $\mathscr{W}(q)$. In particular, $N(\mathfrak{N}_1)$ contains an element of $\mathscr{W}(q)$. If $\mathfrak{L}$ is a $S_{q,r}$-subgroup of $N(\mathfrak{N}_1)$ with Sylow system $\mathfrak{L}_q$, $\mathfrak{L}_r$, then $\mathfrak{L}_q \lhd \mathfrak{L}$, by Theorem 22.3. By Theorem 22.6, $\mathfrak{N}_1$ contains an element $\mathfrak{E}$ of $\mathscr{SCN}_3(\mathfrak{N}_3)$. By Corollary 17.3, $\mathsf{N}(\mathfrak{E})$ is trivial. Since $\mathfrak{L}_q \in \mathsf{N}(\mathfrak{E})$, we have a contradiction, completing the proof of this theorem.

23. Preliminary Results about the Maximal Subgroups of $\mathfrak{G}$

Hypothesis 23.1.
(i) *ϖ is a non empty subset of π_3.*
(ii) *For at least one p in ϖ, $\varpi = \pi(p)$.*

We remark that by Theorem 22.7, Hypothesis 23.1 (ii) is equivalent to

(ii)′ *$\pi(p) = \varpi$ for all p in ϖ.*

Under Hypothesis 23.1, Theorem 22.5 implies that $\mathfrak{G}$ contains a S_ϖ-subgroup $\mathfrak{H}$. Since $\mathfrak{H}$ also satisfies E_{ϖ_1} for all subsets ϖ_1 of ϖ, $\mathfrak{H}$ is a proper subgroup of $\mathfrak{G}$ by P. Hall's characterization of solvable groups [15]. This section is devoted to a study of $\mathfrak{H}$ and its normalizer $\mathfrak{M} = N(\mathfrak{H})$. All results of this section assume that Hypothesis 23.1 holds. Let $\varpi = \{p_1, \cdots, p_n\}$, $n \geqq 1$, and let $\mathfrak{P}_1, \cdots, \mathfrak{P}_n$ be a Sylow system for $\mathfrak{H}$.

LEMMA 23.1. *$\mathfrak{M}$ is a maximal subgroup of $\mathfrak{G}$ and is the unique maximal subgroup of $\mathfrak{G}$ containing $\mathfrak{H}$.*

Proof. Let $\mathfrak{K}$ be any proper subgroup of $\mathfrak{G}$ containing $\mathfrak{H}$. We must show that $\mathfrak{K} \subseteq \mathfrak{M}$. Since $\mathfrak{K}$ is solvable we assume without loss of generality that $\mathfrak{K}$ is a ϖ, q-group for some $q \notin \varpi$. Let $\mathfrak{P}_1, \cdots, \mathfrak{P}_n$, $\mathfrak{Q}$ be a Sylow system for $\mathfrak{K}$. It suffices to show that $\mathfrak{P}_1 \lhd \mathfrak{P}_1\mathfrak{Q}$.

Since $q \notin \varpi$, $p_1 \not\sim q$. Theorem 22.1 implies that $\mathfrak{P}_1$ centralizes $O_q(\mathfrak{P}_1\mathfrak{Q})$. By Lemma 17.5, $\mathfrak{B} \lhd \mathfrak{P}_1\mathfrak{Q}$, where $\mathfrak{B} = V(ccl_{\mathfrak{G}}(\mathfrak{A}); \mathfrak{P}_1)$ and $\mathfrak{A} \in \mathscr{SCN}_3(\mathfrak{P}_1)$. By Lemma 17.2, $\mathfrak{N}_1 = N(Z(\mathfrak{B})) = O^{p_1}(\mathfrak{N}_1)$. Since $\mathfrak{N}_1$ does not contain an elementary subgroup of order q^3, Lemma 8.13 implies that $\mathfrak{P}_1$ centralizes every q-factor of $\mathfrak{P}_1\mathfrak{Q}$ and so $\mathfrak{P}_1 \lhd \mathfrak{P}_1\mathfrak{Q}$, completing the proof of this lemma.

LEMMA 23.2. *If $p_i \in \pi(F(\mathfrak{H}))$, and $\mathfrak{A}_i \in \mathscr{SCN}_3(\mathfrak{P}_i)$, then $C(\mathfrak{A}_i) \subseteq \mathfrak{M}$.*

Proof. We can assume that $i = 1$. By $C^*_{p_1,p_j}$ $\mathfrak{H}$ contains a S_{p_j}-subgroup of $C(\mathfrak{A}_1)$ for each $j = 2, \cdots, n$. Thus, it suffices to show that if $q \notin \varpi$, and $\mathfrak{Q}$ is a S_q-subgroup of $C(\mathfrak{A}_1)$ permutable with $\mathfrak{P}_1$, then $\mathfrak{Q} \subseteq \mathfrak{M}$.

By the preceding argument, $\mathfrak{P}_1 \lhd \mathfrak{P}_1\mathfrak{Q}$. Since $\mathfrak{P}_1$ normalizes $C(\mathfrak{A}_1) = \mathfrak{A}_1 \times \mathfrak{D}$, $\mathfrak{D}$ being a p_1'-group, it follows that $\mathfrak{P}_1\mathfrak{Q} = \mathfrak{P}_1 \times \mathfrak{Q}$.

Since $F(\mathfrak{H}) \cap \mathfrak{P}_1 \neq \langle 1 \rangle$, it follows that $\mathfrak{M} = N(F(\mathfrak{H}) \cap \mathfrak{P}_1)$, since $F(\mathfrak{H}) \cap \mathfrak{P}_1$ char $\mathfrak{H} \lhd \mathfrak{M}$ and $\mathfrak{M}$ is the unique maximal subgroup of $\mathfrak{G}$ containing $\mathfrak{H}$. The lemma follows since $N(F(\mathfrak{H}) \cap \mathfrak{P}_1) \supseteq C(F(\mathfrak{H}) \cap \mathfrak{P}_1) \supseteq \mathfrak{Q}$.

LEMMA 23.3. *Let* $1 \leq i \leq n$, *and let* $\mathfrak{A}_i \in \mathscr{SCN}_3(\mathfrak{P}_i)$, $\mathfrak{B}_i = V(ccl_{\mathfrak{G}}(\mathfrak{A}_i); \mathfrak{P}_i)$. *If* $C(\mathfrak{A}_i) \subseteq \mathfrak{M}$, *then* $N(\mathfrak{B}_i) \subseteq \mathfrak{M}$.

Proof. We can assume that $i = 1$. If $F(\mathfrak{H})$ is a p_1-group, then Lemma 17.5 implies that $\mathfrak{B}_1 \lhd \mathfrak{H}$ and so $\mathfrak{B}_1 \lhd \mathfrak{M}$, since $\mathfrak{B}_1$ is weakly closed in $F(\mathfrak{H}) \cap \mathfrak{P}_1$. In this case, $N(\mathfrak{B}_1) = \mathfrak{M}$ and we are done.

We can suppose that $F(\mathfrak{H})$ is not a p_1-group, and so $\mathfrak{T} = O_{p'}(\mathfrak{H}) \neq \langle 1 \rangle$. Let $\mathfrak{T}_2, \cdots, \mathfrak{T}_n$ be a $\mathfrak{P}_1$-invariant Sylow system for $\mathfrak{T}$, where $\mathfrak{T}_i$ is a S_{p_i}-subgroup of $\mathfrak{T}$ and we allow $\mathfrak{T}_i = \langle 1 \rangle$. By $C^*_{p_1, p_i}$, $\mathfrak{T}_i$ is a maximal element of $\mathsf{M}(\mathfrak{P}_1; p_i)$.

Let $N \in N(\mathfrak{B}_1)$. Then by Theorem 17.1, $\mathfrak{T}_i^N = \mathfrak{T}_i^{C_i}$ where $C_2, \cdots, C_n$ are in $C(\mathfrak{A}_i) \subseteq \mathfrak{M}$. Since $\mathfrak{T}$ char $\mathfrak{H} \lhd \mathfrak{M}$, each $\mathfrak{T}_i^{C_i}$ is contained in $\mathfrak{T}$ and so $\mathfrak{T}^N = \mathfrak{T}$. Since $\mathfrak{T} \neq \langle 1 \rangle$, $\mathfrak{M} = N(\mathfrak{T}) \supseteq N(\mathfrak{B}_1)$, as required.

LEMMA 23.4. *Let* $1 \leq i \leq n$, $\mathfrak{A}_i \in \mathscr{SCN}_3(\mathfrak{P}_i)$, $\mathfrak{B}_i = V(ccl_{\mathfrak{G}}(\mathfrak{A}_i); \mathfrak{P}_i)$. *If* $\langle C(\mathfrak{A}_i), N(\mathfrak{B}_i) \rangle \subseteq \mathfrak{M}$, *then* $\mathfrak{M}$ *is the unique maximal subgroup of* $\mathfrak{G}$ *containing* $\mathfrak{P}_i$.

Proof. We can assume that $i = 1$. Let $\mathfrak{Q}$ be a q-subgroup of $\mathfrak{G}$ permutable with $\mathfrak{P}_1$. It suffices to show that $\mathfrak{Q} \subseteq \mathfrak{M}$.

Since $\mathfrak{Q} = O_q(\mathfrak{P}_1\mathfrak{Q}) \cdot N_{\mathfrak{Q}}(\mathfrak{B}_1)$, it suffices to show that $\mathfrak{Q}_1 = O_q(\mathfrak{P}_1\mathfrak{Q}) \subseteq \mathfrak{M}$. If $\mathfrak{Q}$ is centralized by $\mathfrak{P}_1$, then by hypothesis $\mathfrak{Q} \subseteq \mathfrak{M}$. Otherwise we apply Theorem 22.1 and conclude that $q \in \varpi$. By Theorem 17.1, $\mathfrak{Q}_1^C \subseteq \mathfrak{H}$ for suitable $C \in C(\mathfrak{A}_1) \subseteq \mathfrak{M}$, and the lemma follows.

LEMMA 23.5. *For each* $i = 1, \cdots, n$, *if* $\mathfrak{A}_i \in \mathscr{SCN}_3(\mathfrak{P}_i)$, *then* $C(\mathfrak{A}_i) \subseteq \mathfrak{M}$, *and* $\mathfrak{M}$ *is the unique maximal subgroup of* $\mathfrak{G}$ *containing* $\mathfrak{P}_i$.

Proof. First, suppose $p_i \in \pi(F(\mathfrak{H}))$. Then $C(\mathfrak{A}_i) \subseteq \mathfrak{M}$, by Lemma 23.2. Then by Lemma 23.3, $N(\mathfrak{B}_i) \subseteq \mathfrak{M}$, $\mathfrak{B}_i = V(ccl_{\mathfrak{G}}(\mathfrak{A}_i); \mathfrak{P}_i)$, and then by Lemma 23.4, this lemma follows. We can suppose that $p_i \notin \pi(F(\mathfrak{H}))$.

We assume that $i = 1$. Let $C(\mathfrak{A}_1) = \mathfrak{A}_1 \times \mathfrak{D}$, where $\mathfrak{D}$ is a p_1'-group. It suffices to show that for each q in $\pi(\mathfrak{D})$, $\mathfrak{M}$ contains a S_q-subgroup $\mathfrak{Q}$ of $\mathfrak{D}$. If $q \in \varpi$, this is the case by $C^*_{p_1, q}$, so we can suppose that $q \notin \varpi$.

Since $p_1 \notin \pi(F(\mathfrak{H}))$, $\mathfrak{A}_1$ does not centralize $F(\mathfrak{H})$. If $F(\mathfrak{H})$ were cyclic, and $p = \max\{p_1, \cdots, p_n\}$, then a S_p-subgroup of $\mathfrak{H}$ would be contained in $F(\mathfrak{H})$ and so be cyclic. Since this is not the case, $F(\mathfrak{H})$ is non cyclic, so we can assume that $F(\mathfrak{H}) \cap \mathfrak{P}_2$ is non cyclic. We can then find A in $\mathfrak{A}_1^{\#}$ so that $C(A) \cap F(\mathfrak{H}) \cap \mathfrak{P}_2$ contains an element of $\mathscr{W}(p_2)$, say $\mathfrak{W}$.

Let $\mathfrak{L}^* = \langle \mathfrak{D}, \mathfrak{W}, \mathfrak{A}_1 \rangle \subseteq C(A)$, and let $\mathfrak{L}$ be a $S_{p_1,p_2,q}$-subgroup of $\mathfrak{L}^*$ with Sylow system $\mathfrak{L}_{p_1}, \mathfrak{L}_{p_2}, \mathfrak{L}_q$, where $\mathfrak{A}_1 \subseteq \mathfrak{L}_{p_1}$ and $\mathfrak{Q} \subseteq \mathfrak{L}_q$. Since $\mathfrak{A}_1 \subseteq O_{p_1',p_1}(\mathfrak{L}^*)$ by Lemma 17.5, it follows that $\mathfrak{A}_1 O_{p_1'}(\mathfrak{L}^*)/O_{p_1'}(\mathfrak{L}^*)$ is a central factor of $\mathfrak{L}^*$. Hence, $\mathfrak{A}_1$ is a S_{p_1}-subgroup of $\mathfrak{L}^*$ and so $\mathfrak{L}^* = \mathfrak{A}_1 \cdot O_{p_1'}(\mathfrak{L}^*)$.

We apply Theorem 22.3 and conclude that $\mathfrak{L}_{p_2} \lhd \mathfrak{L}$. If $\tilde{\mathfrak{L}}$ is a maximal element of $\mathsf{M}(\mathfrak{A}_1; p_2, q)$ containing $\mathfrak{L}_{p_2} \cdot \mathfrak{L}_q$, it follows that $\tilde{\mathfrak{L}}_{p_2} \lhd \tilde{\mathfrak{L}}$, where $\tilde{\mathfrak{L}}_{p_2}$ is a maximal element of $\mathsf{M}(\mathfrak{A}_1; p_2)$. By construction, $\tilde{\mathfrak{L}}$ contains $\mathfrak{Q}$. By Theorem 17.1, there is an element C in $C(\mathfrak{A}_1)$ such that $\tilde{\mathfrak{L}}_{p_2}^{C} = O_{p_2}(\mathfrak{P}_1\mathfrak{P}_2)$. Since $\mathfrak{Q}^C$ normalizes $\tilde{\mathfrak{L}}_{p_2}^{C}$, it follow that $N(O_{p_2}(\mathfrak{P}_1\mathfrak{P}_2))$ contains a S_q-subgroup of $C(\mathfrak{A}_1)$. But $p_2 \in \pi(F(\mathfrak{H}))$, so by what is already proved, we have $N(O_{p_2}(\mathfrak{P}_1\mathfrak{P}_2)) \subseteq \mathfrak{M}$, and so $\mathfrak{M}$ contains $C(\mathfrak{A}_1)$. We apply Lemmas 23.3 and 23.4 and complete the proof of this lemma.

24. Further Linking Theorems

LEMMA 24.1. *If $p \in \pi_4$, $q \in \pi_3 \cup \pi_4$ and $q \sim p$, then $\pi(q) \subseteq \pi(p)$.*

Proof. If $q = p$, there is nothing to prove, so suppose $q \neq p$. Corollary 19.1 implies that $q \in \pi_3$. Let $r \sim q$, $r \neq q$, $r \neq p$. We must show that $r \sim p$.

If $r \in \pi_4$ and $\mathfrak{Q}$ is a S_q-subgroup of $\mathfrak{G}$, then $\mathfrak{Q}$ does not centralize every element of $\mathsf{M}(\mathfrak{Q}; p)$ and $\mathfrak{Q}$ does not centralize every element of $\mathsf{M}(\mathfrak{Q}; r)$. By Theorem 22.2, we have $p \sim r$.

If $r \in \pi_3$, then since also $q \in \pi_3$, we have $r \sim p$, by Theorem 22.7. This completes the proof of this lemma.

If $p \in \pi_4$ and $p_1 \in \pi(p)$, $p_1 \neq p$, let $\pi(p_1) = \{p, p_1, \cdots, p_n\}$. By Theorem 22.7 and Lemma 24.1, $\pi(p_i) = \pi(p_j)$, $1 \leq i, j \leq n$. It follows from $p \in \pi_4$ that $p_i \in \pi_3$, $1 \leq i \leq n$. By Theorem 22.5, $\mathfrak{G}$ satisfies $C_{\pi_3(p_1)}$. Let $\mathfrak{H}$ be a $S_{\pi_3(p_1)}$-subgroup of $\mathfrak{G}$. Clearly, $\mathfrak{H} \subset \mathfrak{G}$ since $p \notin \pi_3(p_1)$.

It is easy to see that $F(\mathfrak{H})$ is non cyclic. Choose i so that the S_{p_i}-subgroup of $F(\mathfrak{H})$ is non cyclic. Let $\mathfrak{P}_1, \cdots, \mathfrak{P}_n$ be a Sylow system for $\mathfrak{H}$, $\mathfrak{P}_j$ being a S_{p_j}-subgroup of $\mathfrak{H}$. Thus, $\mathfrak{P}_i \cap F(\mathfrak{H})$ is non cyclic, so that $\mathfrak{P}_i \cap F(\mathfrak{H})$ contains a subgroup $\mathfrak{B}$ of type (p, p) which is normal in $\mathfrak{P}_i$. Let $\mathfrak{A}$ be an element of $\mathscr{SCN}_3(\mathfrak{P}_i)$ which contains $\mathfrak{B}$. Let $\mathfrak{P}_0$ be a maximal element of $\mathsf{M}(\mathfrak{A}; p)$. By Lemma 24.1 and Theorem 22.6, $\mathfrak{P}_0 \neq \langle 1 \rangle$. Let $C(\mathfrak{A}) = \mathfrak{A} \times \mathfrak{D}$, $\mathfrak{D}$ being a p_i'-group.

THEOREM 24.1. *$\langle \mathfrak{P}_0, O_{p_i'}(\mathfrak{H}), \mathfrak{D} \rangle$ is a p_i'-group.*

Proof. Let $\mathscr{P}$ be the set of $\mathfrak{A}$-invariant subgroups $\tilde{\mathfrak{P}}_0$ of $\mathfrak{P}_0$ such that $\langle\tilde{\mathfrak{P}}_0, O_{p_i'}(\mathfrak{H}), \mathfrak{D}\rangle \subset \mathfrak{G}$. Since $\langle O_{p_i'}(\mathfrak{H}), \mathfrak{D}\rangle \subseteq C(\mathfrak{B})$, it follows that $\langle 1\rangle \in \mathscr{P}$.

Suppose $\tilde{\mathfrak{P}}_0 \in \mathscr{P}$, and $\mathfrak{T} = \langle\tilde{\mathfrak{P}}_0, O_{p_i'}(\mathfrak{H}), \mathfrak{D}\rangle$. Since $\mathfrak{A}$ normalizes $\mathfrak{T}$, $\langle\mathfrak{A}, \mathfrak{T}\rangle = \mathfrak{A}\mathfrak{T} = \mathfrak{L} \subset \mathfrak{G}$. By Lemma 17.6, $\mathfrak{A} \subseteq O_{p_i', p_i}(\mathfrak{L})$.

Let $\bar{\mathfrak{A}}$ be the image of $\mathfrak{A}$ under the projection of $O_{p_i', p_i}(\mathfrak{L})$ onto $O_{p_i', p_i}(\mathfrak{L})/O_{p_i'}(\mathfrak{L})$. Since $\bar{\mathfrak{A}} \cong \mathfrak{A}$, we see that $\bar{\mathfrak{A}}$ is a self centralizing subgroup of $O_{p_i', p_i}(\mathfrak{L})/O_{p_i'}(\mathfrak{L})$, and it follows readily that $O_{p_i', p_i}(\mathfrak{L})/O_{p_i'}(\mathfrak{L})$ is centralized by $\tilde{\mathfrak{P}}_0, O_{p_i'}(\mathfrak{H})$ and $\mathfrak{D}$. By Lemma 1.2.3 of [21], we have $\langle\tilde{\mathfrak{P}}_0, O_{p_i'}(\mathfrak{H}), \mathfrak{D}\rangle \subseteq O_{p_i'}(\mathfrak{L})$ and hence $\mathfrak{T} = O_{p_i'}(\mathfrak{L})$ is a p_i'-group.

Let $\mathfrak{T}_1, \cdots, \mathfrak{T}_m$ be an $\mathfrak{A}$-invariant Sylow system of $\mathfrak{T}$, $\mathfrak{T}_j$ being a S_{q_j}-subgroup of $\mathfrak{T}$. If $q_j \in \{p_1, \cdots, p_n\}$, it follows from $C^*_{p_i, q_j}$ that $\mathfrak{T}_j$ is a maximal element of $\mathsf{M}(\mathfrak{A}; q_j)$. Since $\mathfrak{D} \subseteq \mathfrak{T}$, this implies that $O_{p_i'}(\mathfrak{H})$ is a S-subgroup of $\mathfrak{T}$. If $q_j \neq p$, $q_j \notin \{p_1, \cdots, p_n\}$, then Theorem 22.1 implies that $\mathfrak{A}$ centralizes $\mathfrak{T}_j$, so that $\mathfrak{T}_j \subseteq \mathfrak{D}$. Finally, if $q_j = p$, then there is an element D of $\mathfrak{D}$ such that $\mathfrak{T}_j^D \subseteq \mathfrak{P}_0$, by Theorem 17.1.

Let $\mathfrak{K}$ be a fixed $S_{p'}$-subgroup of $\langle\mathfrak{D}, O_{p_i'}(\mathfrak{H})\rangle$. By the preceding paragraph, $\mathfrak{K}$ is a $S_{p'}$-subgroup of $\mathfrak{T}$, and $\mathfrak{P}_0 \cap \mathfrak{T}$ is a S_p-subgroup of $\mathfrak{T}$. Since $\tilde{\mathfrak{P}}_0 \subseteq \mathfrak{P}_0 \cap \mathfrak{T}$, it follows that $\langle\tilde{\mathfrak{P}}_0 \mid \tilde{\mathfrak{P}}_0 \in \mathscr{P}\rangle = \mathfrak{P}^*$ is permutable with $\mathfrak{K}$ so that $\mathfrak{P}^*\mathfrak{K}$ is a proper p_i'-subgroup of $\mathfrak{G}$. This means that $\mathscr{P}$ contains a unique maximal element. Since $C_{\mathfrak{P}_0}(B)$ is $\mathfrak{A}$-invariant for each $B \in \mathfrak{B}^\sharp$, since $\mathfrak{P}_0 = \langle C_{\mathfrak{P}_0}(B) \mid B \in \mathfrak{B}^\sharp\rangle$, and since $\langle C_{\mathfrak{P}_0}(B), O_{p_i'}(\mathfrak{H}), \mathfrak{D}\rangle \subseteq C(B) \subset \mathfrak{G}$, the theorem follows.

THEOREM 24.2. *Let* $\mathfrak{R} = \langle\mathfrak{P}_0, O_{p_i'}(\mathfrak{H}), \mathfrak{D}\rangle$, *and* $\mathfrak{M} = N(\mathfrak{R})$. *Then* $\mathfrak{M}$ *contains* $\mathfrak{H}$, $\mathfrak{M}$ *is a maximal subgroup of* $\mathfrak{G}$ *and* $\mathfrak{M}$ *is the only maximal subgroup of* $\mathfrak{G}$ *containing* $\mathfrak{P}_i$.

Proof. Since $\mathfrak{P}_0 \neq \langle 1\rangle$, $\mathfrak{M}$ is a proper subgroup of $\mathfrak{G}$. We first show that $\mathfrak{M}$ contains $\mathfrak{P}_i$. Let $\mathfrak{Q}$ be an $\mathfrak{A}$-invariant S_q-subgroup of $\mathfrak{R}$, so that $\mathfrak{Q}$ is a maximal element of $\mathsf{M}(\mathfrak{A}; q)$, either by virtue of $q \in \pi(p_i)$, or by virtue of $q \notin \pi(p_i)$ so that $\mathfrak{A}$ centralizes $\mathfrak{Q}$. For P in $\mathfrak{P}_i$, $\mathfrak{Q}^P = \mathfrak{Q}^D$ for some D in $\mathfrak{D}$ by Theorem 17.1 together with $C(\mathfrak{A}) = \mathfrak{A} \times \mathfrak{D}$. Since $\mathfrak{D} \subseteq \mathfrak{R}$, $\mathfrak{Q}^P$ is a S_q-subgroup of $\mathfrak{R}$. Hence, $\mathfrak{R}^P \subseteq \mathfrak{R}$, and so $\mathfrak{R}^P = \mathfrak{R}$. Thus, $\mathfrak{P}_i \subseteq \mathfrak{M}$.

To show that $\mathfrak{H} \subseteq \mathfrak{M}$, we use the fact that $\mathfrak{H} = O_{p_i'}(\mathfrak{H}) \cdot N_{\mathfrak{H}}(\mathfrak{B})$, where $\mathfrak{B} = V(ccl_{\mathfrak{G}}(\mathfrak{A}); \mathfrak{P}_i)$. Since $O_{p_i'}(\mathfrak{H}) \subseteq \mathfrak{R}$, it suffices to show that $N_{\mathfrak{H}}(\mathfrak{B}) \subseteq \mathfrak{M}$. We will in fact show that $N(\mathfrak{B}) \subseteq \mathfrak{M}$. Let $\mathfrak{Q}$ be a $\mathfrak{B}$-invariant S_q-subgroup of $\mathfrak{R}$. If $N \in N(\mathfrak{B})$, then $\mathfrak{A}^{N^{-1}} \subseteq \mathfrak{B}$, so that $\mathfrak{A}^{N^{-1}}$ normalizes $\mathfrak{Q}$. Hence, $\mathfrak{A}$ normalizes $\mathfrak{Q}^N = \mathfrak{Q}^D$, $D \in \mathfrak{D}$, and we see that $\mathfrak{R}^N = \mathfrak{R}$. Thus, $\mathfrak{M}$ contains $\mathfrak{H}$ and $N(\mathfrak{B})$.

Let $\mathfrak{M}_1$ be a maximal subgroup of $\mathfrak{G}$ containing $\mathfrak{M}$. It is easy to see that $\mathfrak{R} = O_{p_i'}(\mathfrak{M}_1)$ by Lemma 7.3, so that $\mathfrak{M}_1 \subseteq \mathfrak{M}$, and $\mathfrak{M}$ is a maximal subgroup of $\mathfrak{G}$.

Let $\mathfrak{K}$ be any proper subgroup of $\mathfrak{G}$ containing $\mathfrak{P}_i$. To show that $\mathfrak{K} \subseteq \mathfrak{M}$, it suffices to treat the case that $\mathfrak{K}$ is a q, p_i-group. Let $\mathfrak{K}_q$ be a S_q-subgroup of $\mathfrak{K}$ permutable with $\mathfrak{P}_i$. Since $N(\mathfrak{B}) \subseteq \mathfrak{M}$, it suffices to show that $O_{p_i'}(\mathfrak{K}) \subseteq \mathfrak{M}$. This is clear by $C^*_{p_i,p}$ if $q \in \{p_1, \cdots, p_n\}$. If $q = p$, this is also clear, by Theorem 17.1, since $C(\mathfrak{A}) \subseteq \mathfrak{M}$ and $\mathfrak{P}_0 \subseteq \mathfrak{M}$. If $q \notin \{p, p_1, \cdots, p_n\}$, then $\mathfrak{P}_i$ centralizes $O_q(\mathfrak{P}_i\mathfrak{K}_q)$ by Theorem 22.1, and we are done, since $C(\mathfrak{A}) \subseteq \mathfrak{M}$.

If $q \in \pi_3 \cup \pi_4$, and $\mathfrak{Q}$ is a S_q-subgroup of $\mathfrak{G}$, we define

$$\mathscr{A}_1(\mathfrak{Q}) = \{\mathfrak{Q}_0 \mid \mathfrak{Q}_0 \subseteq \mathfrak{Q}, \mathfrak{Q}_0 \text{ contains some element of } \mathscr{SCN}_3(\mathfrak{Q})\} ,$$

$$\mathscr{A}_i(\mathfrak{Q}) = \{\mathfrak{Q}_0 \mid \mathfrak{Q}_0 \subseteq \mathfrak{Q}, \mathfrak{Q}_0 \text{ contains a subgroup } \mathfrak{Q}_1 \text{ of type } (q, q)$$
$$\text{such that } C_{\mathfrak{Q}}(Q) \in \mathscr{A}_{i-1}(\mathfrak{Q}) \text{ for each } Q \text{ in } \mathfrak{Q}_1\},\ i = 2, 3, 4 .$$

LEMMA 24.2. *If $q \in \pi_3 \cup \pi_4$ and $\mathfrak{Q}$ is a S_q-subgroup of $\mathfrak{G}$, then every subgroup $\mathfrak{Q}_0$ of $\mathfrak{Q}$ which contains a subgroup of type (q, q, q) is in $\mathscr{A}_3(\mathfrak{Q})$.*

Proof. Let $\mathfrak{B} \in \mathscr{U}(\mathfrak{Q})$, $\mathfrak{Q}_0^* = C_{\mathfrak{Q}_0}(\mathfrak{B})$, so that $\mathfrak{Q}_0^*$ is non cyclic. Let $\mathfrak{Q}_1$ be a subgroup of $\mathfrak{Q}_0^*$ of type (q, q). If $Q \in \mathfrak{Q}_1$, then $C_{\mathfrak{Q}}(Q) \supseteq \mathfrak{B}$. Since $\mathfrak{B}$ is contained in an element of $\mathscr{SCN}_3(\mathfrak{Q})$, it follows that $C_{\mathfrak{Q}}(Q)$ is in $\mathscr{A}_2(\mathfrak{Q})$.

THEOREM 24.3. *If $q \in \pi_3$, $\mathfrak{Q}$ is a S_q-subgroup of $\mathfrak{G}$, and $\mathfrak{Q}$ is contained in a unique maximal subgroup of $\mathfrak{G}$, then each element of $\mathscr{A}_1(\mathfrak{Q})$ is contained in a unique maximal subgroup of $\mathfrak{G}$.*

Proof. Let $\mathfrak{M}$ be the unique maximal subgroup of $\mathfrak{G}$ containing $\mathfrak{Q}$. We remark that if this theorem is proved for the pair $(\mathfrak{Q}, \mathfrak{M})$, then it will also be proved for all pairs $(\mathfrak{Q}^M, \mathfrak{M})$ where $M \in \mathfrak{M}$. This prompts the following definition: $\mathscr{A}_1{}^*(\mathfrak{Q})$ is the set of all subgroups $\mathfrak{Q}_0$ of $\mathfrak{Q}$ such that $\mathfrak{Q}_0$ contains $\mathfrak{C}^M$ for some $\mathfrak{C}$ in $\mathscr{SCN}_3(\mathfrak{Q})$ and some $M \in \mathfrak{M}$. Clearly $\mathscr{A}_1(\mathfrak{Q}) \subseteq \mathscr{A}_1{}^*(\mathfrak{Q})$.

Suppose some element of $\mathscr{A}_1{}^*(\mathfrak{Q})$ is contained in a maximal subgroup of $\mathfrak{G}$ different from $\mathfrak{M}$. Among all such elements $\mathfrak{Q}_0$ of $\mathscr{A}_1{}^*(\mathfrak{Q})$, let $|\mathfrak{Q}_0|$ be maximal. By hypothesis, $\mathfrak{Q}_0 \subset \mathfrak{Q}$. Let $\mathfrak{M}_1$ be a maximal subgroup of $\mathfrak{G}$ different from $\mathfrak{M}$ which contains $\mathfrak{Q}_0$ and let $\mathfrak{Q}_0^*$ be a S_q-subgroup of $\mathfrak{M}_1$ which contains $\mathfrak{Q}_0$. If $\mathfrak{Q}_0 \subset \mathfrak{Q}_0^*$, then $\mathfrak{Q}_0 \subset N_{\mathfrak{Q}_0^*}(\mathfrak{Q}_0)$. Since $\mathfrak{Q}_0 \subset N_{\mathfrak{Q}}(\mathfrak{Q}_0)$, maximality of $|\mathfrak{Q}_0|$ implies that $N_{\mathfrak{G}}(\mathfrak{Q}_0) \subseteq \mathfrak{M}$, so that $N_{\mathfrak{Q}_0^*}(\mathfrak{Q}_0) \subseteq \mathfrak{Q}^M$ for some M in $\mathfrak{M}$. Since $\mathfrak{M}_1 \neq \mathfrak{M}$, so also $\mathfrak{M}_1^{M^{-1}} \neq \mathfrak{M}$. But $N_{\mathfrak{Q}_0^*}(\mathfrak{Q}_0)^{M^{-1}} \in \mathscr{A}_1{}^*(\mathfrak{Q})$, and maximality of $|\mathfrak{Q}_0|$ is violated. Hence, $\mathfrak{Q}_0$ is a S_q-subgroup of $\mathfrak{M}_1$.

Let $\mathfrak{C} \in \mathscr{SCN}_3(\mathfrak{Q})$ be chosen so that $\mathfrak{C}^M \subseteq \mathfrak{Q}_0$ for some $M \in \mathfrak{M}$. Since every element of $\mathsf{M}(\mathfrak{C})$ is contained in $\mathfrak{M}$, every element of

$\mathsf{N}(\mathfrak{C}^M)$ is contained in $\mathfrak{M}^M = \mathfrak{M}$. Hence $O_{q'}(\mathfrak{M}_1) \subseteq \mathfrak{M}$. If $\mathfrak{B} = V(ccl_{\mathfrak{G}}(\mathfrak{C}); \mathfrak{Q}_0)$, then $\mathfrak{Q}_0 \subset N_{\mathfrak{Q}}(\mathfrak{B})$, so $N_{\mathfrak{M}_1}(\mathfrak{B}) \subseteq \mathfrak{M}$, by maximality of $|\mathfrak{Q}_0|$. Since $\mathfrak{M}_1 = O_{q'}(\mathfrak{M}_1) \cdot N_{\mathfrak{M}_1}(\mathfrak{B})$ by Lemma 17.6, we find that $\mathfrak{M}_1 \subseteq \mathfrak{M}$, contrary to assumption. The theorem is proved.

THEOREM 24.4. *Let $q \in \pi_3 \cup \pi_4$, and let $\mathfrak{Q}$ be a S_q-subgroup of $\mathfrak{G}$. If each element of $\mathscr{A}_1(\mathfrak{Q})$ is contained in a unique maximal subgroup $\mathfrak{M}$ of $\mathfrak{G}$, then for each $i = 2, 3, 4$, and each element $\mathfrak{Q}_0$ of $\mathscr{A}_i(\mathfrak{Q})$, $\mathfrak{M}$ is the unique maximal subgroup of $\mathfrak{G}$ containing $\mathfrak{Q}_0$.*

Proof. For $i = 2, 3, 4$, let $\mathscr{A}_i{}^*(\mathfrak{Q})$ be the set of subgroups $\mathfrak{Q}_0$ of $\mathfrak{Q}$ such that $\mathfrak{Q}_0$ contains a subgroup $\mathfrak{Q}_1$ of type (q, q) such that $C_{\mathfrak{M}}(Q)$ contains an element of $\mathscr{A}_{i-1}^*(\mathfrak{Q}^M)$ for some $M \in \mathfrak{M}$ and all $Q \in \mathfrak{Q}_1$. Here $\mathscr{A}_i{}^*(\mathfrak{Q}^M)$ denotes the set of $\mathfrak{Q}_0^M$, $\mathfrak{Q}_0 \in \mathscr{A}_i{}^*(\mathfrak{Q})$. Suppose $i = 2, 3$, or 4 is minimal with the property that some element of $\mathscr{A}_i{}^*(\mathfrak{Q})$ is contained in at least two maximal subgroups of $\mathfrak{G}$. This implies that $\mathscr{A}_{i-1}^*(\mathfrak{Q}^M)$ does not contain any elements which are contained in two maximal subgroups of $\mathfrak{G}$, M being an arbitrary element of $\mathfrak{M}$. Choose $\mathfrak{Q}_0$ in $\mathscr{A}_i{}^*(\mathfrak{Q})$ with $|\mathfrak{Q}_0|$ maximal subject to the condition that $\mathfrak{Q}_0$ is contained in a maximal subgroup $\mathfrak{M}_1$ of $\mathfrak{G}$ with $\mathfrak{M}_1 \neq \mathfrak{M}$. We see that $\mathfrak{Q}_0$ is a S_q-subgroup of $\mathfrak{M}_1$. Let $\mathfrak{Q}_1$ be a subgroup of $\mathfrak{Q}_0$ of type (q, q) with the property that $C_{\mathfrak{M}}(Q)$ contains an element of $\mathscr{A}_{i-1}^*(\mathfrak{Q}^M)$ for suitable M in $\mathfrak{M}$, and each Q in $\mathfrak{Q}_1$. (We allow M to depend on Q.) Since $O_{q'}(\mathfrak{M}_1)$ is generated by its subgroups $O_{q'}(\mathfrak{M}_1) \cap C(Q)$, $Q \in \mathfrak{Q}_1^\sharp$, it follows that $O_{q'}(\mathfrak{M}_1) \subseteq \mathfrak{M}$.

Let $\mathfrak{C}$ be an element of $\mathscr{SCN}_3(\mathfrak{Q})$. Then $\mathfrak{C} \nsubseteq \mathfrak{Q}_0$, or we are done. Let $\tilde{\mathfrak{Q}}_0 = \mathfrak{Q}_0 \cap O_{q',q}(\mathfrak{M}_1)$. Since $\tilde{\mathfrak{Q}}_0 \cap \mathfrak{C} = \mathfrak{Q}_0 \cap \mathfrak{C}$ by (B), it follows that $N_{\mathfrak{Q}}(\tilde{\mathfrak{Q}}_0) \supset \mathfrak{Q}_0$. Hence, $N(\tilde{\mathfrak{Q}}_0) \subseteq \mathfrak{M}$, by maximality of $|\mathfrak{Q}_0|$. Since $\mathfrak{M}_1 = O_{q'}(\mathfrak{M}_1) \cdot N_{\mathfrak{M}_1}(\tilde{\mathfrak{Q}}_0)$, we have $\mathfrak{M}_1 \subseteq \mathfrak{M}$, completing the proof of this theorem, since $\mathscr{A}_i(\mathfrak{Q}) \subseteq \mathscr{A}_i{}^*(\mathfrak{Q})$, $i = 2, 3, 4$.

THEOREM 24.5. *If $q \in \pi_3$ and $\mathfrak{Q}$ is a S_q-subgroup of $\mathfrak{G}$, then $\mathfrak{Q}$ is contained in a unique maximal subgroup of $\mathfrak{G}$.*

Proof. If $\pi(q) \subseteq \pi_3$, this theorem follows from Lemma 23.5. Suppose $p \in \pi(q) \cap \pi_4$. Let $\pi(q) = \{p, p_1, \cdots, p_n\}$, where $q = p_1$, and let $\mathfrak{H}$ be a $S_{\pi_3(p_1)}$-subgroup of $\mathfrak{G}$ containing $\mathfrak{Q}$. If $\mathfrak{Q} \cap F(\mathfrak{H})$ is non cyclic, we are done by Theorem 24.2, so we suppose that $\mathfrak{Q} \cap F(\mathfrak{H})$ is cyclic.

Let $\mathfrak{M}$ be the unique maximal subgroup of $\mathfrak{G}$ containing $\mathfrak{H}$. Suppose we are able to show that $C(\mathfrak{C}) \subseteq \mathfrak{M}$ for some $\mathfrak{C}$ in $\mathscr{SCN}_3(\mathfrak{Q})$. Since $F(\mathfrak{H}) \cap \mathfrak{Q}$ is cyclic, $F(\mathfrak{M}) \cap \mathfrak{Q}$ is also cyclic. Hence, $O_{q'}(\mathfrak{M}) \neq 1$. If $\mathfrak{B} = V(ccl_{\mathfrak{G}}(\mathfrak{C}); \mathfrak{Q})$, then $N(\mathfrak{B})$ normalizes $O_{q'}(\mathfrak{M})$, by Theorem 17.1, together with $C(\mathfrak{C}) \subseteq \mathfrak{M}$. Since $\mathfrak{K} = O_{q'}(\mathfrak{K}) \cdot N_{\mathfrak{K}}(\mathfrak{B})$ for every proper subgroup $\mathfrak{K}$ of $\mathfrak{G}$ which contains $\mathfrak{Q}$, it suffices to show that every element of $\mathscr{M}(\mathfrak{Q})$ is contained in $\mathfrak{M}$. This follows readily by C^*_{q,p_j}, Theorem 22.1 and $C(\mathfrak{C}) \subseteq \mathfrak{M}$.

Thus, it suffices to show that $C(\mathfrak{C}) \subseteq \mathfrak{M}$. Choose i such that $\mathfrak{F}_i$, a S_{p_i}-subgroup of $F(\mathfrak{H})$, is non cyclic, and let $\mathfrak{P}_i$ be a S_{p_i}-subgroup of $\mathfrak{H}$ permutable with $\mathfrak{Q}$. It suffices to show that $C_{\mathfrak{F}_i}(C) \in \mathscr{A}_4(\mathfrak{P}_i)$ for some $C \in \mathfrak{C}^*$, by Theorems 24.3 and 24.4 together with the fact that $\mathfrak{M}$ is the unique maximal subgroup of $\mathfrak{G}$ containing $\mathfrak{P}_i$.

Let $\mathfrak{F}_i^* = O_{p_i}(\mathfrak{Q}\mathfrak{P}_i)$, so that $\mathfrak{F}_i^*$ is a maximal element of $\mathscr{M}(\mathfrak{Q}; p_i)$. By Lemma 17.3, $\mathfrak{Q} \subseteq N(\mathfrak{F}_i^*)'$. Since $\mathfrak{P}_i$ is contained in $\mathfrak{M}$ and no other maximal subgroup, $\mathfrak{Q} \subseteq \mathfrak{M}'$. Thus, if $\Omega_1(Z_2(\mathfrak{F}_i))$ is generated by two elements, then $\mathfrak{Q}$ centralizes $Z_2(\mathfrak{F}_i)$ and we are done. If $Z(\mathfrak{F}_i)$ is non cyclic, then every subgroup of $Z(\mathfrak{F}_i)$ of type (p_i, p_i) is contained in $\mathscr{A}_4(\mathfrak{P}_i)$. Since $\mathfrak{C}$ contains a subgroup of type (q, q, q), $C(C) \cap Z(\mathfrak{F}_i)$ is non cyclic for some C in $\mathfrak{C}^*$, and we are done in this case. There remains the possibility that $Z(\mathfrak{F}_i)$ is cyclic, while $\Omega_1(Z_2(\mathfrak{F}_i))$ is not generated by two elements. Since every subgroup of $\Omega_1(Z_2(\mathfrak{F}_i))$ of type (p_i, p_i) which contains $\Omega_1(Z(\mathfrak{F}_i))$ is contained in $\mathscr{A}_4(\mathfrak{P}_i)$, by Lemma 24.2, and since $C(C)$ contains such a subgroup for some C in $\mathfrak{C}^*$, we are done.

The preceding theorems give precise information regarding the S_q-subgroups of the maximal subgroups of $\mathfrak{G}$ for q in π_3.

THEOREM 24.6. *Let $q \in \pi_3$ and let $\mathfrak{M}$ be a maximal subgroup of $\mathfrak{G}$. If $\mathfrak{Q}$ is a S_q-subgroup of $\mathfrak{M}$ and $\mathfrak{Q}$ is not a S_q-subgroup of $\mathfrak{G}$, then $\mathfrak{Q}$ contains a cyclic subgroup of index at most q.*

Proof. Let $\mathfrak{Q}^*$ be a S_q-subgroup of $\mathfrak{G}$ containing $\mathfrak{Q}$, let $\mathfrak{B} \in \mathscr{U}(\mathfrak{Q}^*)$ and let $\mathfrak{Q}_0 = C_{\mathfrak{Q}}(\mathfrak{B})$ so that $|\mathfrak{Q} : \mathfrak{Q}_0| = 1$ or q. If $\mathfrak{Q}_0$ is non cyclic, then $\mathfrak{Q}_0 \in \mathscr{A}_4(\mathfrak{Q}^*)$, and so $\mathfrak{Q}_0$ is contained in a unique maximal subgroup of $\mathfrak{G}$, which must be $\mathfrak{M}$, since $\mathfrak{Q}_0 \subseteq \mathfrak{M}$. But $\mathfrak{Q}^* \not\subseteq \mathfrak{M}$, a contradiction, so $\mathfrak{Q}_0$ is cyclic, as required.

Theorem 24.6 is of interest in its own right, and plays an important role in the study of π_4, to which all the preceding results are now turned.

Hypothesis 24.1.

1. $3 \in \pi_4$.
2. *$\mathfrak{P}$ is a S_3-subgroup of $\mathfrak{G}$.*
3. *$\mathfrak{K}$ is a proper subgroup of $\mathfrak{G}$ such that*
   (i) $\mathfrak{P} \subseteq \mathfrak{K}$.
   (ii) *If $\mathfrak{H} = O_3(\mathfrak{K})$, there is a subgroup $\mathfrak{C}$ of $\mathfrak{H}$ chosen in*

accordance with Lemma 8.2 *such that* $\boldsymbol{Z}(\mathfrak{C})$ *is generated by two elements.*

THEOREM 24.7. *Under Hypothesis* 24.1, $\mathfrak{P}$ *is contained in a unique maximal subgroup* $\mathfrak{M}$ *of* $\mathfrak{G}$, *and* $\mathfrak{M}$ *centralizes* $\boldsymbol{Z}(\mathfrak{P})$.

Proof. Let $\mathfrak{L}$ be any proper subgroup of $\mathfrak{G}$ containing $\mathfrak{P}$. We must show that $\mathfrak{L}$ centralizes $\boldsymbol{Z}(\mathfrak{P})$.

By Lemma 8.2, $\ker(\mathfrak{K} \rightarrow \operatorname{Aut}\mathfrak{C})$ is a 3-group, so is contained in $\mathfrak{H}$. It follows that $C_{\mathfrak{K}}(\mathfrak{C}) = \boldsymbol{Z}(\mathfrak{C})$ and in particular $C_{\mathfrak{P}}(\mathfrak{C}) = \boldsymbol{Z}(\mathfrak{C})$.

Suppose $\mathfrak{C} \subseteq \boldsymbol{O}_3(\mathfrak{L})$. Then $\boldsymbol{Z}(\boldsymbol{O}_3(\mathfrak{L})) \subseteq C_{\mathfrak{P}}(\mathfrak{C}) \subseteq \boldsymbol{Z}(\mathfrak{C})$, so $\boldsymbol{Z}(\boldsymbol{O}_3(\mathfrak{L}))$ is generated by two elements. Since $|\mathfrak{L}|$ is odd, a $S_{3'}$-subgroup of $\mathfrak{L}$ centralizes $\boldsymbol{Z}(\boldsymbol{O}_3(\mathfrak{L}))$, so centralizes $\boldsymbol{Z}(\mathfrak{P})$. Since $\mathfrak{P}$ also centralizes $\boldsymbol{Z}(\mathfrak{P})$, we have $\mathfrak{L} \subseteq \boldsymbol{C}(\boldsymbol{Z}(\mathfrak{P}))$.

Suppose $\mathfrak{C} \not\subseteq \boldsymbol{O}_3(\mathfrak{L})$. Since $\boldsymbol{Z}(\mathfrak{C})$ is a normal abelian subgroup of $\mathfrak{P}$ we have $\boldsymbol{Z}(\mathfrak{C}) \subseteq \boldsymbol{O}_3(\mathfrak{L})$. Since $\gamma^2\mathfrak{P}\mathfrak{C}^2 \subseteq \boldsymbol{Z}(\mathfrak{C})$, we conclude that $\mathfrak{C} \subseteq \boldsymbol{O}_{3,3',3}(\mathfrak{L})$. By the preceding paragraph, $\boldsymbol{N}(\mathfrak{P} \cap \boldsymbol{O}_{3,3',3}(\mathfrak{L}))$ centralizes $\boldsymbol{Z}(\mathfrak{P})$. Thus, it suffices to show that $\mathfrak{P}\boldsymbol{O}_{3,3'}(\mathfrak{L}) = \mathfrak{L}_1$ centralizes $\boldsymbol{Z}(\mathfrak{P})$. Since $\mathfrak{L}_1 = \boldsymbol{N}_{\mathfrak{L}_1}(\boldsymbol{O}_3(\mathfrak{L})\mathfrak{C}) \cdot [\boldsymbol{O}_{3,3'}(\mathfrak{L}), \mathfrak{C}]$, and since $\mathfrak{P}$ normalizes $\boldsymbol{O}_3(\mathfrak{L})\mathfrak{C}$, it suffices to show that $[\boldsymbol{O}_{3,3'}(\mathfrak{L}), \mathfrak{C}]$ centralizes $\boldsymbol{Z}(\mathfrak{P})$. Let $\mathfrak{Z} = \boldsymbol{Z}(\boldsymbol{O}_3(\mathfrak{L}))$, so that $\mathfrak{Z}$ contains $\boldsymbol{Z}(\mathfrak{P})$. Since $\mathfrak{Z}$ is a normal abelian subgroup of $\mathfrak{P}$, (B) implies that $\mathfrak{Z} \subseteq \boldsymbol{O}_3(\mathfrak{K})$. Hence, $\gamma^2\mathfrak{Z}\mathfrak{C}^2 = 1$, which implies that $[\boldsymbol{O}_{3,3'}(\mathfrak{L}), \mathfrak{C}]$ induces only 3-automorphisms on $\mathfrak{Z}$, and suffices to complete the proof.

Hypothesis 24.2.

1. $3 \in \pi_4$.
2. $\mathfrak{P}$ *is a* S_3*-subgroup of* $\mathfrak{G}$.
3. *If* $\mathfrak{K}$ *is any proper subgroup of* $\mathfrak{G}$ *containing* $\mathfrak{P}$, *and if* $\mathfrak{H} = \boldsymbol{O}_3(\mathfrak{K})$, *then every subgroup* $\mathfrak{C}$ *of* $\mathfrak{H}$ *chosen in accordance with Lemma* 8.2 *satisfies* $m(\boldsymbol{Z}(\mathfrak{C})) \geq 3$.

REMARK. If $3 \in \pi_4$, then Hypothesis 24.1 and Hypothesis 24.2 exhaust all possibilities.

LEMMA 24.3. *Under Hypothesis* 24.2, $\mathfrak{P}$ *contains an element* $\mathfrak{B}$ *of* $\mathscr{U}(\mathfrak{P})$ *such that the normal closure of* $\mathfrak{B}$ *in* $\boldsymbol{C}(\Omega_1(\boldsymbol{Z}(\mathfrak{P})))$ *is abelian.*

Proof. If $\boldsymbol{Z}(\mathfrak{P})$ is non cyclic, every element of $\mathscr{U}(\mathfrak{P})$ satisfies this lemma. Otherwise, set $\mathfrak{K} = \boldsymbol{C}(\Omega_1(\boldsymbol{Z}(\mathfrak{P})))$, and let $\mathfrak{A}$ be a non cyclic normal abelian subgroup of $\mathfrak{K}$. Since $\mathfrak{A} \lhd \mathfrak{P}$, $\mathfrak{A}$ contains an element $\mathfrak{B}$ of $\mathscr{U}(\mathfrak{P})$ which meets the demands of this lemma.

THEOREM 24.8. *Let* $p \in \pi_4$ *and let* $\mathfrak{P}$ *be a* S_p*-subgroup of* $\mathfrak{G}$. *If* $p = 3$, *assume that* $\mathscr{U}(\mathfrak{P})$ *contains an element* $\mathfrak{B}$ *such that the normal* *closure of $\mathfrak{B}$ in $C(\Omega_1(Z(\mathfrak{P})))$ is abelian. If $p \geqq 5$, let $\mathfrak{B}$ be any element of $\mathscr{U}(\mathfrak{P})$. If $\mathfrak{K}$ is any proper subgroup of $\mathfrak{G}$ such that $O_{p'}(\mathfrak{K}) = 1$ and if $\mathfrak{K}_p$ is a S_p-subgroup of $\mathfrak{K}$, then $\mathfrak{K} = \mathfrak{L} \cdot N_{\mathfrak{K}}(\mathfrak{V})$, where $\mathfrak{V} = V(ccl_{\mathfrak{G}}(\mathfrak{B}); \mathfrak{K}_p)$, and $\mathfrak{L}$ is the largest normal subgroup of $\mathfrak{K}$ which centralizes $Z(\mathfrak{K}_p)$.*

Proof. Observe that $\mathfrak{L}$ contains $O_p(\mathfrak{K})$.

Since $O_p(\mathfrak{K} \bmod \mathfrak{L}) = \mathfrak{L} \cdot (\mathfrak{K}_p \cap O_p(\mathfrak{K} \bmod \mathfrak{L}))$, maximality of $\mathfrak{L}$ guarantees that $\mathfrak{L} = O_p(\mathfrak{K} \bmod \mathfrak{L})$. If $\mathfrak{V} \subseteq \mathfrak{L}$, then Sylow's theorem yields this theorem since $\mathfrak{V}$ is weakly closed in $\mathfrak{L} \cap \mathfrak{K}_p$.

Suppose by way of contradiction that $\mathfrak{V} \not\subseteq \mathfrak{L}$. Let $\mathfrak{L}_1 = O_{p'}(\mathfrak{K} \bmod \mathfrak{L})$. By Lemma 1.2.3 of [21], $\gamma\mathfrak{V}\mathfrak{L}_1 \not\subseteq \mathfrak{L}$.

Let $\mathfrak{P}_1 = \mathfrak{K}_p \cap \mathfrak{L}$, and let $\tilde{\mathfrak{L}}_1 = \mathfrak{L}_1 \cap N_{\mathfrak{K}}(\mathfrak{P}_1)$. Let $\mathfrak{V}_1$ be the normal closure of $\mathfrak{V}$ in $N_{\mathfrak{K}}(\mathfrak{P}_1)$. Suppose $\gamma\tilde{\mathfrak{L}}_1\mathfrak{V}_1 \subseteq C(Z(\mathfrak{K}_p))$. Since $\gamma\tilde{\mathfrak{L}}_1\mathfrak{V}_1 \lhd N_{\mathfrak{K}}(\mathfrak{P}_1)$, and since $\mathfrak{K} = \mathfrak{L} \cdot N_{\mathfrak{K}}(\mathfrak{P}_1)$ by Sylow's theorem, we see that $\mathfrak{L} \cdot \gamma\tilde{\mathfrak{L}}_1\mathfrak{V}_1 \lhd \mathfrak{K}$. Maximality of $\mathfrak{L}$ implies that $\gamma\tilde{\mathfrak{L}}_1\mathfrak{V}_1 \subseteq \mathfrak{L}$. In particular, $\gamma\tilde{\mathfrak{L}}_1\mathfrak{V} \subseteq \mathfrak{L}$. Since $\mathfrak{L}_1 = \mathfrak{L} \cdot \tilde{\mathfrak{L}}_1$, by Sylow's theorem we have $\mathfrak{V} \subseteq O_p(\mathfrak{K} \bmod \mathfrak{L})$, which is not the case. Hence, $\gamma\tilde{\mathfrak{L}}_1\mathfrak{V}_1 \not\subseteq C(Z(\mathfrak{K}_p))$. Since $Z(\mathfrak{P}_1) \supseteq Z(\mathfrak{K}_p)$, we also have $\gamma\tilde{\mathfrak{L}}_1\mathfrak{V}_1 \not\subseteq C(Z(\mathfrak{P}_1))$. Since $\langle \mathfrak{V}_1, \tilde{\mathfrak{L}}_1 \rangle \subseteq N(Z(\mathfrak{P}_1))$, the identity $[X, YZ] = [X, Z][X, Y]^Z$ implies that $\mathfrak{V}_1$ contains a conjugate $\mathfrak{B}_1 = \mathfrak{B}^G$ of $\mathfrak{B}$ such that $\gamma\mathfrak{L}_1\mathfrak{B}_1 \not\subseteq C(Z(\mathfrak{P}_1))$. Since $\tilde{\mathfrak{L}}_1 \subseteq N_{\mathfrak{K}}(\mathfrak{P}_1)$, application of Theorem C of [21] to $\mathfrak{B}_1\tilde{\mathfrak{L}}_1/\tilde{\mathfrak{L}}_1 \cap C(Z(\mathfrak{P}_1))$ yields a special q-group $\bar{\mathfrak{Q}} = \mathfrak{Q}/\mathfrak{L}_1 \cap C(Z(\mathfrak{P}_1))$ such that $\mathfrak{B}_1$ acts irreducibly and non trivially on $\bar{\mathfrak{Q}}/D(\bar{\mathfrak{Q}})$. Since $\bar{\mathfrak{Q}}$ is a p'-group, and $\bar{\mathfrak{Q}}$ does not centralize $Z(\mathfrak{P}_1)$, $\bar{\mathfrak{Q}}$ does not centralize $\mathfrak{W} = \Omega_1(Z(\mathfrak{P}_1))$. Furthermore, $\mathfrak{W} = \mathfrak{W}_1 \times \mathfrak{W}_2$, where $\mathfrak{W}_1 = C_{\mathfrak{W}}(\mathfrak{Q})$ and $\mathfrak{W}_2 = \gamma\mathfrak{W}\mathfrak{Q}$, and $\mathfrak{W}_i$ is invariant under $\mathfrak{B}_1\mathfrak{Q}$, $i = 1, 2$.

Since $\mathfrak{W}_2$ is a p-group and $\mathfrak{W}_2 \neq 1$, we have $\mathfrak{W}_3 \neq 1$, where $\mathfrak{W}_3 = C_{\mathfrak{W}_2}(\mathfrak{B}_0)$ and $\mathfrak{B}_0 = \ker(\mathfrak{B}_1 \rightarrow \operatorname{Aut} \tilde{\mathfrak{Q}}) \neq 1$. If $p \geqq 5$, Lemma 18.1 gives an immediate contradiction. If $p = 3$, and $\gamma^2\mathfrak{W}_3\mathfrak{B}_1^2 = 1$, we also have a contradiction with (B), since $\gamma\mathfrak{W}_3\mathfrak{Q} \neq 1$. If $\gamma^2\mathfrak{W}_3\mathfrak{B}_1^2 \neq 1$, Lemma 16.3 implies that $Z(\mathfrak{P})$ is cyclic, and that $\mathfrak{B}_0 = \Omega_1(Z(\mathfrak{P}^G))$. However, the normal closure of $\mathfrak{B}_1$ in $C(\Omega_1(Z(\mathfrak{P}^G)))$ is abelian, and so $\gamma^2\mathfrak{W}_3\mathfrak{B}_1^2 = 1$, the desired contradiction, completing the proof of this theorem.

REMARK. Except for the case $p = 3$, and the side conditions $O_{p'}(\mathfrak{K}) = 1$ and $\mathfrak{L} \lhd \mathfrak{K}$, Theorem 24.8 is a repetition of Lemma 18.1.

Hypothesis 24.3.

1. $p \in \pi_4$, $q \in \pi(p)$, $q \neq p$.
2. *$\mathfrak{Q}$ is a S_q-subgroup of $\mathfrak{G}$, $\mathfrak{P}_0$ is a maximal element of $\mathsf{M}(\mathfrak{Q}; p)$, and $\mathfrak{P}_1$ is a S_p-subgroup of $N(\mathfrak{P}_0)$ permutable with $\mathfrak{Q}$.*
3. *$\mathfrak{P}$ is a S_p-subgroup of $\mathfrak{G}$ containing $\mathfrak{P}_1$, and $\mathfrak{B} \in \mathscr{U}(\mathfrak{P})$, where*

for $p = 3$, *the normal closure of* $\mathfrak{B}$ *in* $C(\Omega_1(Z(\mathfrak{P})))$ *is abelian.*

4. $\mathfrak{B} = V(ccl_{\mathfrak{G}}(\mathfrak{B}); \mathfrak{P}_1)$.

THEOREM 24.9. *Under Hypothesis* 24.3, *either* $N_{\mathfrak{Q}}(\mathfrak{B})$ *contains an element of* $\mathscr{A}_4(\mathfrak{Q})$ *or* $C_{\mathfrak{Q}}(Z(\mathfrak{P}_0))$ *contains an element of* $\mathscr{A}_4(\mathfrak{Q})$. *Furthermore,* $\mathfrak{P}_1 = \mathfrak{P}$ *and* $\mathfrak{G}$ *satisfies* $C_{\pi(q)}$.

Proof. Let $\mathfrak{L}$ be the largest normal subgroup of $\mathfrak{K} = N(\mathfrak{P}_0)$ which centralizes $Z(\mathfrak{P}_0)$. Then $\mathfrak{K} = \mathfrak{L} \cdot N_{\mathfrak{K}}(\mathfrak{B})$, by Theorem 24.8. Since $\mathfrak{L} \lhd \mathfrak{K}$, $\mathfrak{L} \cap \mathfrak{Q} \lhd \mathfrak{Q}$. If $\mathfrak{L} \cap \mathfrak{Q}$ is non cyclic, then $\mathfrak{L} \cap \mathfrak{Q} \in \mathscr{A}_4(\mathfrak{Q})$.

Suppose $\mathfrak{L} \cap \mathfrak{Q}$ is a non identity cyclic group. By Lemma 17.6, $\mathfrak{Q} \subseteq \mathfrak{K}'$. Since a Sylow q-subgroup of $\mathfrak{L}$ is cyclic, it follows that $\mathfrak{K}'$ centralizes $\mathfrak{Q} \cap \mathfrak{L} \cdot \mathfrak{L}_0/\mathfrak{L}_0$, where $\mathfrak{L}_0 = O_{q'}(\mathfrak{L})$, and so $\mathfrak{Q} \cap \mathfrak{L} \subseteq Z(\mathfrak{Q})$. If $\mathfrak{B}$ centralizes $\mathfrak{Q} \cap \mathfrak{L} \cdot \mathfrak{L}_0/\mathfrak{L}_0$, then $N_{\mathfrak{K}}(\mathfrak{B})$ contains a S_q-subgroup of $\mathfrak{K}$. In this case, $\mathfrak{Q}$ normalizes $\mathfrak{B}^K$ for some K in $\mathfrak{K}$. Let $\langle \mathfrak{Q}, \mathfrak{P}_1^* \rangle$ be a $S_{q,p}$-subgroup of $\mathfrak{K}$ containing $\mathfrak{Q}\mathfrak{B}^K$, with $\mathfrak{B}^K \subseteq \mathfrak{P}_1^*$. By the conjugacy of Sylow systems in $\mathfrak{K}$, we have $\mathfrak{P}_1^{*K_1} = \mathfrak{P}_1$, $\mathfrak{Q}^{K_1} = \mathfrak{Q}$ for suitable K_1 in $\mathfrak{K}$. Hence, $\mathfrak{Q}$ normalizes $\mathfrak{B}^{KK_1}$ and $\mathfrak{B}^{KK_1} \subseteq \mathfrak{P}_1$. Since $\mathfrak{B}$ is weakly closed in $\mathfrak{P}_1$, $\mathfrak{B} = \mathfrak{B}^{KK_1}$ and we are done. If $\mathfrak{B}$ does not centralize $\mathfrak{Q} \cap \mathfrak{L} \cdot \mathfrak{L}_0/\mathfrak{L}_0$, then $N(\mathfrak{B}) \cap \mathfrak{L}$ is a q'-group, since $\mathfrak{Q} \cap \mathfrak{L}$ is cyclic. In this case the factorization, $\mathfrak{K} = \mathfrak{L} \cdot N_{\mathfrak{K}}(\mathfrak{B})$, together with $\mathfrak{Q} \cap \mathfrak{L} \subseteq Z(\mathfrak{Q})$, yields that $\mathfrak{Q} = \mathfrak{Q} \cap \mathfrak{L} \times \mathfrak{Q}_1$, for some subgroup $\mathfrak{Q}_1$ of $\mathfrak{Q}$. This in turn implies that every non cyclic subgroup of $\mathfrak{Q}$ is in $\mathscr{A}_4(\mathfrak{Q})$.

Since $\mathfrak{K} = \mathfrak{L} \cdot N_{\mathfrak{K}}(\mathfrak{B})$ and $\mathfrak{L} \cap \mathfrak{Q}$ is cyclic, the S_q-subgroups of $N_{\mathfrak{K}}(\mathfrak{B})$ are non cyclic. Hence, $\mathfrak{Q}$ contains a non cyclic subgroup $\mathfrak{Q}_0$ such that $\mathfrak{Q}_0$ normalizes $\mathfrak{B}^K$ for some K in $\mathfrak{K}$. By the conjugacy of Sylow systems, we can find K_1 in $\mathfrak{K}$ such that $\mathfrak{B}^{KK_1} \subseteq \mathfrak{P}_1$ and $\mathfrak{Q}_0^{K_1} \subseteq \mathfrak{Q}$. Since $\mathfrak{B}$ is weakly closed in $\mathfrak{P}_1$, $\mathfrak{B} = \mathfrak{B}^{KK_1}$, and we are done, since every non cyclic subgroup of $\mathfrak{Q}$ is contained in $\mathscr{A}_4(\mathfrak{Q})$.

Suppose $\mathfrak{L} \cap \mathfrak{Q} = \langle 1 \rangle$. Then $\mathfrak{L}$ is a q'-group. From $\mathfrak{K} = \mathfrak{L} \cdot N_{\mathfrak{K}}(\mathfrak{B})$, we conclude that $\mathfrak{Q}$ normalizes $\mathfrak{B}^K$ for some K in $\mathfrak{K}$ and the conjugacy of Sylow systems, together with the fact that $\mathfrak{B}$ is weakly closed in $\mathfrak{P}_1$, imply that $\mathfrak{Q}$ normalizes $\mathfrak{B}$. This completes the proof of the first assertion of the theorem.

If $\mathfrak{P}_1 \subset \mathfrak{P}$, then $\mathfrak{P}_1 \subset N_{\mathfrak{P}}(\mathfrak{B})$. Since every element of $\mathscr{A}_4(\mathfrak{Q})$ is contained in a unique maximal subgroup $\mathfrak{M}$ of $\mathfrak{G}$, by Theorem 24.3, if $N(\mathfrak{B})$ contains an element of $\mathscr{A}_4(\mathfrak{Q})$, then $\mathfrak{P}_1$ is not a S_p-subgroup of $\mathfrak{M}$. But $\mathfrak{P}_1\mathfrak{Q}$ is a maximal p, q-subgroup of $\mathfrak{G}$, by Lemma 7.3. If $C(Z(\mathfrak{P}_0))$ contains an element of $\mathscr{A}_4(\mathfrak{Q})$, then since $Z(\mathfrak{P}_0) \supseteq Z(\mathfrak{P})$ by (B) and Theorem 22.7, we see that $C(Z(\mathfrak{P}))$ contains an element of $\mathscr{A}_4(\mathfrak{Q})$. Hence, $\mathfrak{P} \subseteq \mathfrak{M}$. Thus, in all cases, $\mathfrak{P} \subseteq \mathfrak{M}$. Since $\mathfrak{M}$ also contains a $S_{\pi_3(q)}$-subgroup of $\mathfrak{G}$, $\mathfrak{G}$ satisfies $E_{\pi(q)}$. Since $\mathfrak{Q}$ is contained in $\mathfrak{M}$ and no other maximal subgroup of $\mathfrak{G}$, $\mathfrak{G}$ satisfies $C_{\pi(q)}$ as required.

Hypothesis 24.4.

1. $3 \in \pi_4$.
2. $\mathfrak{P}$ *is a* S_3*-subgroup of* $\mathfrak{G}$.
3. $\mathfrak{P}$ *contains a subgroup* $\mathfrak{A}$ *which is elementary of order* 27 *with the property that* $\gamma^2 C(A)\mathfrak{A}^2 = 1$ *for all* $A \in \mathfrak{A}^{\#}$.

Hypothesis 24.5.

1. $p \in \pi_4$.
2. *A* S_p*-subgroup* $\mathfrak{P}$ *of* $\mathfrak{G}$ *is contained in at least two maximal subgroups of* $\mathfrak{G}$.

LEMMA 24.4. *Assume that Hypothesis* 24.5 *is satisfied and that if* $p = 3$, *Hypothesis* 24.4 *is also satisfied. If* $p \geqq 5$, *let* $\mathfrak{A}$ *be an arbitrary element of* $\mathscr{S}\mathscr{C}\mathscr{N}_3(\mathfrak{P})$. *If* $p = 3$, *let* $\mathfrak{A}$ *be the subgroup given in Hypothesis* 24.4. *Let* $\mathfrak{W}$ *be the weak closure of* $\mathfrak{A}$ *in* $\mathfrak{P}$, *and let* $\mathfrak{W}^*$ *be the subgroup of* $\mathfrak{P}$ *generated by its subgroups* $\mathfrak{B}$ *such that* $\mathfrak{B} \subseteq \mathfrak{A}^G$ *and* $\mathfrak{A}^G/\mathfrak{B}$ *is cyclic for suitable* G *in* $\mathfrak{G}$. *Let* $\mathfrak{M}$ *be a proper subgroup of* $\mathfrak{G}$ *containing* $\mathfrak{P}$, *with the properties that* $\mathfrak{M}$ *is a* p, q*-group for some prime* q *and* $\mathfrak{M}$ *has* p*-length at most two. Let* $(\mathfrak{X}, \mathfrak{Y})$ *be any one of the pairs* $(\boldsymbol{Z}(\mathfrak{P}), \mathfrak{W})$, $(\boldsymbol{Z}(\mathfrak{W}^*), \mathfrak{W})$, $(\boldsymbol{Z}(\mathfrak{P}), \mathfrak{W}^*)$. *Then* $\mathfrak{M} = \mathfrak{M}_1\mathfrak{M}_2$, *where* $\mathfrak{M}_1$ *normalizes* $\mathfrak{X}$ *and* $\mathfrak{M}_1/C_{\mathfrak{M}_1}(\mathfrak{X})$ *is a* p*-group, and* $\mathfrak{M}_2$ *normalizes* $\mathfrak{Y}$.

Proof. Let $\mathfrak{Q}$ be a S_q-subgroup of $\mathfrak{M}$, and let $\mathfrak{H} = \boldsymbol{O}_p(\mathfrak{M})$. Then $\mathfrak{H}\mathfrak{Q} \lhd \mathfrak{M}$. The lemma will follow immediately if we can show that $\gamma\mathfrak{Q}\mathfrak{Y}$ normalizes $\mathfrak{X}$ and induces only p-automorphisms on $\mathfrak{X}$.

Suppose by way of contradiction that either some element of $\gamma\mathfrak{Q}\mathfrak{Y}$ induces a non trivial q-automorphism on $\mathfrak{X}$, or $\gamma\mathfrak{Q}\mathfrak{Y}$ does not normalize $\mathfrak{X}$. If $\mathfrak{Y} = \mathfrak{W}$, we can find $\mathfrak{B} = \mathfrak{A}^G \subseteq \mathfrak{Y}$ such that either some element of $\gamma\mathfrak{Q}\mathfrak{B}$ induces a non trivial q-automorphism of $\mathfrak{X}$ or else $\gamma\mathfrak{Q}\mathfrak{B}$ does not normalize $\mathfrak{X}$. Similarly, if $\mathfrak{Y} = \mathfrak{W}^*$, we can find $\mathfrak{B} \subseteq \mathfrak{Y}$ and G in $\mathfrak{G}$ such that $\mathfrak{B} \subseteq \mathfrak{A}^G$, $\mathfrak{A}^G/\mathfrak{B}$ is cyclic and such that either some element of $\gamma\mathfrak{Q}\mathfrak{B}$ induces a non trivial q-automorphism of $\boldsymbol{Z}(\mathfrak{P})$ or else $\gamma\mathfrak{Q}\mathfrak{B}$ does not normalize $\boldsymbol{Z}(\mathfrak{P})$.

Let $\bar{\mathfrak{Q}} = \mathfrak{Q}\mathfrak{H}/\mathfrak{H}$, so that $\gamma\bar{\mathfrak{Q}}\mathfrak{B} = (\gamma\mathfrak{Q}\mathfrak{B})\mathfrak{H}/\mathfrak{H}$. Since $\gamma\bar{\mathfrak{Q}}\mathfrak{B}$ is generated by the subgroups $\gamma\bar{\mathfrak{Q}}_1\mathfrak{B}$ which have the property that $\mathfrak{B}$ acts irreducibly and non trivially on $\bar{\mathfrak{Q}}_1/\boldsymbol{D}(\bar{\mathfrak{Q}}_1)$, we can find $\bar{\mathfrak{Q}}_1 = \mathfrak{Q}_1\mathfrak{H}/\mathfrak{H}$ such that $\gamma\mathfrak{Q}_1\mathfrak{B}$ either does not normalize $\mathfrak{X}$ or some element of $\gamma\mathfrak{Q}_1\mathfrak{B}$ induces a non trivial q-automorphism on $\mathfrak{X}$, and with the additional property that $\mathfrak{B}$ acts irreducibly on $\bar{\mathfrak{Q}}_1/\boldsymbol{D}(\bar{\mathfrak{Q}}_1)$.

Let $\mathfrak{B}_0 = \ker(\mathfrak{B} \rightarrow \operatorname{Aut}\bar{\mathfrak{Q}}_1) = \ker(\mathfrak{B} \rightarrow \operatorname{Aut}\bar{\mathfrak{Q}}_1/\boldsymbol{D}(\bar{\mathfrak{Q}}_1))$, so that $\mathfrak{B}/\mathfrak{B}_0$ is cyclic. Let $\mathfrak{M}_1 = \mathfrak{H}\mathfrak{B}\mathfrak{Q}_1$, and $\mathfrak{H}_1 = \boldsymbol{O}_p(\mathfrak{M}_1)$. Since $\mathfrak{H}\mathfrak{B} \subseteq \mathfrak{P}$, and since

$\mathfrak{Z}(\mathfrak{P}) \subseteq \mathfrak{H}$, it follows that $Z(\mathfrak{P}) \subseteq Z(\mathfrak{H}_1)$. Also, since $Z(\mathfrak{W}^*)$ is a normal abelian subgroup of $\mathfrak{P}$, we have $Z(\mathfrak{W}^*) \subseteq \mathfrak{H}$.

Suppose that $\mathfrak{X} = Z(\mathfrak{P})$. If $p = 3$, then since $\mathfrak{A}^G/\mathfrak{B}_0$ is generated by two elements, it follows that $\mathfrak{B}_0 \neq \langle 1 \rangle$. Hence, $Z(\mathfrak{H}_1) \subseteq C(\mathfrak{B}_0)$. Since the normal closure of $\mathfrak{A}^G$ in $C(\mathfrak{B}_0)$ is abelian, we have $\gamma^2 Z(\mathfrak{H}_1)\mathfrak{B}^2 = \langle 1 \rangle$, and (B) implies that a S_q-subgroup of $\mathfrak{H}\mathfrak{Q}_1$ centralizes $Z(\mathfrak{H}_1)$, so centralizes $Z(\mathfrak{P})$.

Suppose $p \geqq 5$. We first treat the case that $\mathfrak{H}_1 \cap \mathfrak{U} \neq \langle 1 \rangle$ for some $\mathfrak{U} \in \mathscr{U}(\mathfrak{P}^G)$, $\mathfrak{U} \subseteq \mathfrak{A}^G$. Then $\langle Z(\mathfrak{H}_1), \mathfrak{A}^G \rangle \subseteq C(\mathfrak{H}_1 \cap \mathfrak{U}) = \mathfrak{C}$ and $\mathfrak{P}^G \cap \mathfrak{C}$ is of index at most p in $\mathfrak{P}^G$. If $\mathfrak{P}^*$ is a S_p-subgroup of $\mathfrak{C}$ containing $\mathfrak{P}^G \cap \mathfrak{C}$, then $\mathfrak{P}^G \cap \mathfrak{C} \lhd \mathfrak{P}^*$. Hence, $\gamma\mathfrak{P}^*\mathfrak{B} \subseteq \gamma\mathfrak{P}^*\mathfrak{A}^G \subseteq \mathfrak{P}^G \cap \mathfrak{C}$, and so $\gamma\mathfrak{P}^*\mathfrak{B}^3 = \langle 1 \rangle$. It follows that $\mathfrak{B} \subseteq O_p(\mathfrak{C})$. (Note that $O_{p'}(\mathfrak{C}) = \langle 1 \rangle$ since $\mathfrak{A}^G \subseteq \mathfrak{C}$.) Hence, $\gamma Z(\mathfrak{H}_1)\mathfrak{B} \subseteq O_p(\mathfrak{C})$, and so $\gamma^4 Z(\mathfrak{H}_1)\mathfrak{B}^4 = \langle 1 \rangle$, so that a S_q-subgroup of $\mathfrak{H}\mathfrak{Q}_1$ centralizes $Z(\mathfrak{H}_1)$ and so centralizes $Z(\mathfrak{P})$.

We can now suppose that $\mathfrak{H}_1 \cap \mathfrak{U} = \langle 1 \rangle$ for all $\mathfrak{U}$ such that $\mathfrak{U} \in \mathscr{U}(\mathfrak{P}^G)$, $\mathfrak{U} \subseteq \mathfrak{A}^G$. In this case, since $\mathfrak{A}^G/\mathfrak{B}_0$ is generated by two elements, there is a normal elementary subgroup $\mathfrak{E}$ of $\mathfrak{P}^G$ of order p^3 such that $\mathfrak{E} \subseteq \mathfrak{A}^G$. Hence, $\mathfrak{E} \cap \mathfrak{B}_0 \neq \langle 1 \rangle$. Since $\mathfrak{E} \cap \mathfrak{B}_0 \subseteq \mathfrak{E} \cap \mathfrak{H}_1$, we can find E in $\mathfrak{E} \cap \mathfrak{H}_1^\sharp$. Consider $C(E) \supseteq \langle Z(\mathfrak{H}_1), C_{\mathfrak{P}^G}(E) \rangle$. Since $\mathfrak{A}^G/\mathfrak{B}$ is cyclic, if $\mathfrak{U} \in \mathscr{U}(\mathfrak{P}^G)$ and $\mathfrak{U} \subseteq \mathfrak{A}^G$, then $\mathfrak{B} \cap \mathfrak{U} = \mathfrak{U}_0 \neq \langle 1 \rangle$. Let $U \in \mathfrak{U}_0^\sharp$. Let $\mathfrak{P}^*$ be a S_p-subgroup of $C(E)$ containing $C_{\mathfrak{P}^G}(E)$, so that $|\mathfrak{P}^* : C_{\mathfrak{P}^G}(E)| = 1$, p or p^2. We have $\gamma^2\mathfrak{P}^*\mathfrak{B}^2 \subseteq C_{\mathfrak{P}^G}(E)$, and so $\gamma^4\mathfrak{P}^*\mathfrak{B}^4 = \langle 1 \rangle$. This implies that $\mathfrak{B} \subseteq O_p(C(E))$. Let $Z \in Z(\mathfrak{H}_1)$; then $[Z, U] \in O_p(C(E))$, so that $[Z, U, U, U] \in C_{\mathfrak{P}^G}(E)$. Since $U \in \mathfrak{U}_0 \subseteq \mathfrak{U} \in \mathscr{U}(\mathfrak{P}^G)$, it follows that $[Z, U, U, U, U] \in Z(\mathfrak{P}^G)$. Since $\mathfrak{H}_1 \cap \mathfrak{U} = \langle 1 \rangle$, and since $[Z, U, U, U, U] \in Z(\mathfrak{P}^G) \cap \mathfrak{H}_1$, we have $[Z, U, U, U, U] = \langle 1 \rangle$. This shows that a S_q-subgroup of $\mathfrak{H}\mathfrak{Q}_1$ centralizes $Z(\mathfrak{H}_1)$ and so centralizes $Z(\mathfrak{P})$.

Suppose now that $\mathfrak{X} = Z(\mathfrak{W}^*)$, so that $\mathfrak{Y} = \mathfrak{W}$. In this case, $\mathfrak{B} = \mathfrak{A}^G$. Hence, $\mathfrak{B}_0 \subseteq \mathfrak{W}^*$, since $\mathfrak{B}/\mathfrak{B}_0$ is cyclic. Since $Z(\mathfrak{W}^*)$ is contained in $\mathfrak{H}_1$, if $\mathfrak{B}^*$ denotes the normal closure of $\mathfrak{B}_0$ in $\mathfrak{H}\mathfrak{B}\mathfrak{Q}_1$, then $Z(\mathfrak{W}^*)$ centralizes $\mathfrak{B}^*$, $\mathfrak{B}^*$ being a subgroup of $\mathfrak{W}^*$.

Let $\mathfrak{C}^* = C(\mathfrak{B}^*) \cap \mathfrak{H}_1$ so that $\mathfrak{C}^*$ is normal in $\mathfrak{H}\mathfrak{B}\mathfrak{Q}_1$. If $p = 3$, we have $\gamma^2\mathfrak{C}^*\mathfrak{B}^2 = \langle 1 \rangle$, since $\mathfrak{B}_0 \neq \langle 1 \rangle$, and it follows that a S_q-subgroup of $\mathfrak{M}_1$ centralizes $\mathfrak{C}^*$. Namely, if $\mathfrak{C}^* = \mathfrak{C}_1^* \supset \mathfrak{C}_2^* \supset \cdots$ is part of a chief series for $\mathfrak{M}_1$, then $\mathfrak{H}_1$ centralizes each $\mathfrak{C}_i^*/\mathfrak{C}_{i+1}^*$, so that a S_q-subgroup of $\mathfrak{M}_1$ centralizes each $\mathfrak{C}_i^*/\mathfrak{C}_{i+1}^*$, so centralizes $\mathfrak{C}^*$. If $p \geqq 5$, then $\mathfrak{B}_0 \cap \mathfrak{U} \neq \langle 1 \rangle$ for some $\mathfrak{U} \in \mathscr{U}(\mathfrak{P}^G)$, $\mathfrak{U} \subseteq \mathfrak{B}$, and we have $\gamma^4\mathfrak{C}^*\mathfrak{B}^4 = \langle 1 \rangle$, and we are done.

THEOREM 24.10. *Under Hypothesis* 24.5, $p = 3$ *and* $\pi(3) = \{3\}$. *Furthermore, Hypothesis* 24.4 *is not satisfied.*

Proof. Suppose that either $p \geqq 5$ or Hypothesis 24.4 is satisfied. Let $\mathfrak{A}$ be any element of $\mathscr{SCN}_3(\mathfrak{P})$ in case $p \geqq 5$ and let $\mathfrak{A}$ be the subgroup given by Hypothesis 24.4 in case $p = 3$. Let $\mathfrak{W}, \mathfrak{W}^*$ be as in Lemma 24.4. Let $\mathfrak{N}_1 = N(Z(\mathfrak{P})), \mathfrak{N}_2 = N(\mathfrak{W}), \mathfrak{N}_3 = N(Z(\mathfrak{W}^*))$, and let $\mathfrak{H}$ be any proper subgroup of $\mathfrak{G}$ containing $\mathfrak{P}$. Then by Lemma 24.4 and Lemma 7.7, we have $\mathfrak{H} = (\mathfrak{H} \cap \mathfrak{N}_1)(\mathfrak{H} \cap \mathfrak{N}_2) = (\mathfrak{H} \cap \mathfrak{N}_1)(\mathfrak{H} \cap \mathfrak{N}_3) = (\mathfrak{H} \cap \mathfrak{N}_2)(\mathfrak{H} \cap \mathfrak{N}_3)$. Taking $\mathfrak{H} = \mathfrak{N}_1$, we get $\mathfrak{N}_1 \subseteq \mathfrak{N}_2\mathfrak{N}_3$, $\mathfrak{N}_1 \subseteq \mathfrak{N}_3\mathfrak{N}_2$. Taking $\mathfrak{H} = \mathfrak{N}_2$, we get $\mathfrak{N}_2 \subseteq \mathfrak{N}_1\mathfrak{N}_3$, $\mathfrak{N}_2 \subseteq \mathfrak{N}_3\mathfrak{N}_1$. Taking $\mathfrak{H} = \mathfrak{N}_3$, we get $\mathfrak{N}_3 \subseteq \mathfrak{N}_1\mathfrak{N}_2$, $\mathfrak{N}_3 \subseteq \mathfrak{N}_2\mathfrak{N}_1$. By Lemma 8.6, we conclude that $\mathfrak{N}_1\mathfrak{N}_2$ is a group and so $\mathfrak{H} \subseteq \mathfrak{N}_1\mathfrak{N}_2$ for every proper subgroup $\mathfrak{H}$ of $\mathfrak{G}$ containing $\mathfrak{P}$. If $\mathfrak{N}_1\mathfrak{N}_2 = \mathfrak{G}$, then $O_p(\mathfrak{N}_1)$ is contained in every conjugate of $\mathfrak{N}_2$, against the simplicity of $\mathfrak{G}$. Hence, $\mathfrak{N}_1\mathfrak{N}_2$ is the unique maximal subgroup of $\mathfrak{G}$ containing $\mathfrak{P}$.

We can now suppose that $p = 3$ and that Hypothesis 24.4 is not satisfied. Suppose $q \in \pi(3), q \neq 3$. Let $\mathfrak{Q}$ be a S_q-subgroup of $\mathfrak{G}$ permutable with $\mathfrak{P}$ and let $\mathfrak{M}$ be the unique maximal subgroup of $\mathfrak{G}$ containing $\mathfrak{Q}$. If $\mathfrak{H} = O_3(\mathfrak{M})$ and $\mathfrak{C}$ is a subgroup of $\mathfrak{H}$ chosen in accordance with Lemma 8.2, then Theorem 24.7 yields that $m(Z(\mathfrak{C})) \geqq 3$. Let $\mathfrak{F}$ be a subgroup of $\mathfrak{Q}$ of type (q, q, q) and let $\Omega_1(Z(\mathfrak{C})) = \mathfrak{C}_1 \times \cdots \times \mathfrak{C}_r$, each $\mathfrak{C}_i$ being a minimal $\mathfrak{F}$-invariant subgroup. If $\mathfrak{F}$ centralizes $Z(\mathfrak{C})$, then any subgroup of $Z(\mathfrak{C})$ of type $(3, 3, 3)$ will serve as $\mathfrak{A}$. This is so, since in this case, $C(A) \subseteq \mathfrak{M}$ for all A in $\mathfrak{A}^{\#}$. Otherwise, $|\mathfrak{C}_i| \geqq 27$ for some i, and since $\mathfrak{F}/C_{\mathfrak{F}}(\mathfrak{C}_i)$ is cyclic, $C_{\mathfrak{F}}(\mathfrak{C}_i) \in \mathscr{A}_4(\mathfrak{Q})$, so we let $\mathfrak{A}$ be any subgroup of $\mathfrak{C}_i$ of type $(3, 3, 3)$. The proof is complete.

25. The Isolated Prime

Hypothesis 25.1.

1. $3 \in \pi_4$.

2. *A S_3-subgroup $\mathfrak{P}$ of $\mathfrak{G}$ is contained in at least two maximal subgroups of $\mathfrak{G}$.*

THEOREM 25.1. *Under Hypothesis 25.1, there is a q-subgroup $\mathfrak{Q}$ of $\mathfrak{G}$ permutable with $\mathfrak{P}$ such that if $\mathfrak{H} = \mathfrak{P}\mathfrak{Q}$ and if $\bar{\mathfrak{P}}, \bar{\mathfrak{Q}}$ are the images of $\mathfrak{P}, \mathfrak{Q}$ respectively in $\mathfrak{H}/O_3(\mathfrak{H})$, then $\bar{\mathfrak{P}} \neq 1$ is cyclic, $\bar{\mathfrak{P}}$ is faithfully and irreducibly represented on $\bar{\mathfrak{Q}}/D(\bar{\mathfrak{Q}})$, and $\mathfrak{Q}$ does not centralize $\mathfrak{B} = \Omega_1(Z(O_3(\mathfrak{H})))$.*

Proof. There is at least one proper subgroup of $\mathfrak{G}$ containing $\mathfrak{P}$ and not normalizing $Z(\mathfrak{P})$, since otherwise $N(Z(\mathfrak{P}))$ is the unique maximal subgroup of $\mathfrak{G}$ containing $\mathfrak{P}$. Let $\mathfrak{H}$ be minimal with these two properties. Then $\mathfrak{H} = \mathfrak{P}\mathfrak{Q}$ for some q-group $\mathfrak{Q}$. Since $3 \in \pi_4$, $O_q(\mathfrak{H}) = 1$. Since $\mathscr{SCN}_3(\mathfrak{Q})$ is empty, $\mathfrak{H}$ has q-length 1. Hence,

$O_3(\mathfrak{H})\mathfrak{Q} \lhd \mathfrak{H}$. By Lemma 8.13, $\bar{\mathfrak{P}}$ is abelian. By minimality of $\mathfrak{H}$, $\bar{\mathfrak{P}}$ acts faithfully and irreducibly on $\bar{\mathfrak{Q}}/D(\bar{\mathfrak{Q}})$. If $\bar{\mathfrak{P}} = 1$, then $\mathfrak{P} \lhd \mathfrak{H}$, and $\mathfrak{Q}$ normalizes $Z(\mathfrak{P})$, which is not the case.

Since $\mathfrak{Q}$ does not normalize $Z(\mathfrak{P})$, $\mathfrak{Q}$ does not centralize $Z(O_3(\mathfrak{H}))$ so does not centralize $\Omega_1(Z(O_3(\mathfrak{H})))$. The proof is complete.

We will now show that Hypothesis 24.4 is satisfied. $\bar{\mathfrak{P}}\bar{\mathfrak{Q}}$ is represented on $\mathfrak{V} = \Omega_1(Z(O_3(\mathfrak{H})))$, and it follows from (B) that the minimal polynomial of a generator of $\bar{\mathfrak{P}}$ is $(x-1)^{|\bar{\mathfrak{P}}|}$. Hence, there is an elementary subgroup $\mathfrak{A}$ of $\mathfrak{V}$ of order 27 on which $\mathfrak{P}$ acts indecomposably. Let $\mathfrak{P}_0 = C_{\mathfrak{P}}(\mathfrak{A})$ and let $\mathfrak{E} = \Omega_1(Z(\mathfrak{P}_0))$ so that $\mathfrak{A} \subseteq \mathfrak{E}$. Choose $A \in \mathfrak{A}^{\#}$. and set $\mathfrak{C} = C(A)$. Let $\mathfrak{P}^*$ be a S_3-subgroup of $\mathfrak{C}$ containing $\mathfrak{P}_0$. (It may occur that $\mathfrak{P} = \mathfrak{P}^*$ but this makes no difference in the following argument.) If $\mathfrak{P}_0 = \mathfrak{P}^*$, then $\gamma^2\mathfrak{C}\mathfrak{A}^2 = 1$. Suppose $|\mathfrak{P}^* : \mathfrak{P}_0| = 3$. Then $\langle\mathfrak{P}, \mathfrak{P}^*\rangle \subseteq N(\mathfrak{P}_0)$, so that $\langle\mathfrak{P}, \mathfrak{P}^*\rangle$ normalizes $\mathfrak{E}$. Since $\mathfrak{P}$ and $\mathfrak{P}^*$ are conjugate in $N(\mathfrak{P}_0)$, any element of $\mathfrak{P}^* - \mathfrak{P}_0$ has minimal polynomial $(x-1)^3$ on $\mathfrak{E}$.

Let $\mathfrak{R} = O_3(\mathfrak{C})$. Then $|\mathfrak{R} : \mathfrak{R} \cap \mathfrak{P}_0| = 1$ or 3, so that $\gamma\mathfrak{R}\mathfrak{E} \subseteq \mathfrak{P}_0$, and $\gamma^2\mathfrak{R}\mathfrak{E}^2 = 1$. By (B), $\mathfrak{E} \subseteq \mathfrak{R}$. If $\mathfrak{R} \subseteq \mathfrak{P}_0$, then $\mathfrak{E} \subseteq Z(\mathfrak{R})$, and $\gamma^2\mathfrak{C}\mathfrak{A}^2 = 1$. Suppose $|\mathfrak{R} : \mathfrak{R} \cap \mathfrak{P}_0| = 3$. Then $D(\mathfrak{R}) \subseteq \mathfrak{P}_0$, so that $\mathfrak{E} \subseteq C_{\mathfrak{R}}(D(\mathfrak{R}))$. If $C_{\mathfrak{R}}(D(\mathfrak{R})) \subseteq \mathfrak{P}_0$, then $\mathfrak{E} \subseteq Z(C_{\mathfrak{R}}(D(\mathfrak{R})))$, and once again $\gamma^2\mathfrak{C}\mathfrak{A}^2 = 1$. Hence we can suppose that $C_{\mathfrak{R}}(D(\mathfrak{R}))$ contains an element K of $\mathfrak{R} - \mathfrak{R} \cap \mathfrak{P}_0$. Since $\mathfrak{R} \subseteq \mathfrak{P}^*$, it follows from the preceding paragraph that the class of $C_{\mathfrak{R}}(D(\mathfrak{R}))$ is at least three. On the other hand, if X and Y are in $C_{\mathfrak{R}}(D(\mathfrak{R}))$, then $[X, Y] \in C_{\mathfrak{R}}(D(\mathfrak{R})) \cap \mathfrak{R}'$. Since $\mathfrak{R}' \subseteq D(\mathfrak{R})$, we have $[X, Y, Z] = 1$ for all X, Y, Z in $C_{\mathfrak{R}}(D(\mathfrak{R}))$. This contradiction shows that $\gamma^2\mathfrak{C}\mathfrak{A}^2 = 1$ for all A in $\mathfrak{A}^{\#}$. Combining this result with the results of Section 24 yields the following theorem.

THEOREM 25.2. *If $p \in \pi_4$, and $\mathfrak{P}$ is a S_p-subgroup of $\mathfrak{G}$, then $\mathfrak{P}$ is contained in a unique maximal subgroup of $\mathfrak{G}$.*

THEOREM 25.3. *Let $p \in \pi_4$ and let $\mathfrak{P}$ be a S_p-subgroup of $\mathfrak{G}$. Then each element of $\mathscr{M}_1(\mathfrak{P})$ is contained in a unique maximal subgroup of $\mathfrak{G}$.*

Proof. First, assume that if $p = 3$, then $\mathscr{U}(\mathfrak{P})$ contains an element $\mathfrak{B}$ whose normal closure in $C(Z(\mathfrak{P}))$ is abelian, while if $p \geqq 5$, $\mathfrak{B}$ is an arbitrary element of $\mathscr{U}(\mathfrak{P})$.

Let $\mathfrak{M}$ be the unique maximal subgroup of $\mathfrak{G}$ containing $\mathfrak{P}$. Let $\mathscr{M}_1^*(\mathfrak{P})$ be the set of subgroups $\mathfrak{P}_0$ of $\mathfrak{P}$ such that $\mathfrak{P}_0$ contains $\mathfrak{C}^M$ for suitable $\mathfrak{C}$ in $\mathscr{SCN}_3(\mathfrak{P})$, M in $\mathfrak{M}$. Suppose by way of contradiction that some element $\mathfrak{P}_0$ of $\mathscr{M}_1^*(\mathfrak{P})$ is contained in a maximal subgroup $\mathfrak{M}_1$ of $\mathfrak{G}$ different from $\mathfrak{M}$, and that $|\mathfrak{P}_0|$ is maximal. It follows readily that $\mathfrak{P}_0$ is a S_p-subgroup of $\mathfrak{M}_1$. Since $\mathfrak{P}_0$ contains $\mathfrak{C}^M$ for suitable $\mathfrak{C}$ in $\mathscr{SCN}_3(\mathfrak{P})$, M in $\mathfrak{M}$, $\boldsymbol{O}_{p'}(\mathfrak{M}_1) = 1$. Thus the hypotheses of Theorem 24.8 are satisfied, $\mathfrak{M}_1$ playing the role of $\mathfrak{K}$ and $\mathfrak{P}_0$ the role of $\mathfrak{K}_p$, $\mathfrak{B} = \boldsymbol{V}(\mathrm{ccl}_{\mathfrak{G}}(\mathfrak{B}); \mathfrak{P}_0)$. Since $\boldsymbol{N}_{\mathfrak{P}}(\mathfrak{B}) \supset \mathfrak{P}_0$, and since $\mathfrak{P}_0 \supseteq \mathfrak{C}^M \supseteq \boldsymbol{Z}(\mathfrak{P})$ ($\mathfrak{C}^M$ being self centralizing), we conclude from the factorization given in Theorem 24.8 and from the maximality of $\mathfrak{P}_0$ that $\mathfrak{M}_1 \subseteq \mathfrak{M}$.

There remains the possibility that for every $\mathfrak{B}$ in $\mathscr{U}(\mathfrak{P})$, the normal closure of $\mathfrak{B}$ in $\mathfrak{M} = \boldsymbol{C}(\boldsymbol{Z}(\mathfrak{P}))$ is non abelian, and $p = 3$.

Let $\mathfrak{H} = \boldsymbol{O}_3(\mathfrak{M})$. If $\mathfrak{H}$ contains a non cyclic characteristic abelian subgroup $\mathfrak{A}$, then $\mathfrak{A}$ contains an element $\mathfrak{B}$ of $\mathscr{U}(\mathfrak{P})$, and $\mathfrak{B}^{\mathfrak{M}}$ is abelian. Since we are assuming there are no such elements, every characteristic abelian subgroup of $\mathfrak{H}$ is cyclic. The structure of $\mathfrak{H}$ is given by 3.5. If $\mathfrak{C}$ is any element of $\mathscr{SCN}_3(\mathfrak{P})$, then $\mathfrak{C} \subseteq \mathfrak{H}$, by (B), so $\mathfrak{C} \in \mathscr{SCN}_3(\mathfrak{H})$.

As before, let $\mathfrak{P}_0 \in \mathscr{A}_1^*(\mathfrak{P})$ be chosen so that $\mathfrak{P}_0$ is contained in a maximal subgroup $\mathfrak{M}_1$ of $\mathfrak{G}$ different from $\mathfrak{M}$, with $|\mathfrak{P}_0|$ maximal. Then $\mathfrak{P}_0$ is a S_3-subgroup of $\mathfrak{M}_1$ and $\boldsymbol{O}_{3'}(\mathfrak{M}_1) = 1$.

Let $\mathfrak{T} = \boldsymbol{O}_3(\mathfrak{M}_1)$. Since $\gamma^2\mathfrak{T}\mathfrak{C}^2 = 1$, (B) implies that $\mathfrak{T} \cap \mathfrak{C} = \mathfrak{P}_0 \cap \mathfrak{C}$. Since $\mathfrak{P}_0 = \boldsymbol{N}_{\mathfrak{P}}(\mathfrak{T})$ by maximality of $\mathfrak{P}_0$, we conclude that $\mathfrak{C} \subseteq \mathfrak{T}$. We need to show that $\mathfrak{H} \subseteq \mathfrak{P}_0$. Consider $\mathfrak{H} \cap \mathfrak{P}_0 = \mathfrak{H}_0$. Since $\gamma^2\mathfrak{P}_0\mathfrak{H}_0^2 \subseteq \Omega_1(\boldsymbol{Z}(\mathfrak{P}))$, we conclude that $\mathfrak{H}_0 \subseteq \boldsymbol{O}_{3\,3',3}(\mathfrak{M}_1)$, and maximality of $|\mathfrak{P}_0|$ implies that $\boldsymbol{N}(\mathfrak{P}_0 \cap \boldsymbol{O}_{3,3'\,3}(\mathfrak{M}_1)) \subseteq \mathfrak{M}$ so it suffices to show that $\mathfrak{T}_0 = \mathfrak{P}_0\boldsymbol{O}_{3,3'}(\mathfrak{M}_1) \subseteq \mathfrak{M}$, and it follows readily from $\mathfrak{T}_0 = \boldsymbol{N}_{\mathfrak{T}_0}(\mathfrak{T}\mathfrak{H}_0)\cdot\gamma\mathfrak{H}_0\boldsymbol{O}_{3,3'}(\mathfrak{M}_1)$ that it suffices to show that $\gamma\mathfrak{H}_0\boldsymbol{O}_{3,3'}(\mathfrak{M}_1) \subseteq \mathfrak{M}$. Since $\mathfrak{C} \subseteq \mathfrak{T}$, we have $\boldsymbol{Z}(\mathfrak{T}) \subseteq \mathfrak{C}$, so that $\gamma^2\boldsymbol{Z}(\mathfrak{T})\mathfrak{H}_0^2 = 1$, and $\gamma\mathfrak{H}_0\boldsymbol{O}_{3,3'}(\mathfrak{M}_1)$ induces only 3-automorphisms on $\boldsymbol{Z}(\mathfrak{T})$, so centralizes $\boldsymbol{Z}(\mathfrak{P})$, and $\mathfrak{M}_1 \subseteq \mathfrak{M}$ follows in case $\mathfrak{H}_0 \subset \mathfrak{H}$.

Suppose $\mathfrak{H} \subseteq \mathfrak{P}_0$. If $\mathfrak{H} \cap \mathfrak{T} \supset \mathfrak{C}$, then $\Omega_1(\boldsymbol{Z}(\mathfrak{P})) \subseteq \mathfrak{T}'$, and since $\gamma^2\mathfrak{T}\mathfrak{H}^2 \subseteq \Omega_1(\boldsymbol{Z}(\mathfrak{P})) \subseteq \mathfrak{T}' \subseteq \boldsymbol{D}(\mathfrak{T})$, (B) implies that $\mathfrak{H} \subseteq \mathfrak{T}$. In this case, $\Omega_1(\boldsymbol{Z}(\mathfrak{T})) = \Omega_1(\boldsymbol{Z}(\mathfrak{P})) \lhd \mathfrak{M}_1$, so $\mathfrak{M}_1 \subseteq \mathfrak{M}$. There remains the possibility that $\mathfrak{H} \cap \mathfrak{T} = \mathfrak{C}$.

If $\mathfrak{T} = \mathfrak{C}$, then $\gamma^2\mathfrak{T}\mathfrak{H}^2 = 1$ and (B) is violated. Hence, $\mathfrak{T} \supset \mathfrak{C}$, so that $\mathfrak{T}' \neq 1$. Hence, $\mathfrak{T}' \cap \boldsymbol{Z}(\mathfrak{T}) \neq 1$. If $\Omega_1(\boldsymbol{Z}(\mathfrak{P})) \subseteq \mathfrak{T}'$, then $\gamma^2\mathfrak{T}\mathfrak{H}^2 \subseteq \mathfrak{T}'$ and we are done. If $\Omega_1(\boldsymbol{Z}(\mathfrak{P})) \not\subseteq \mathfrak{T}'$, we conclude that $\mathfrak{H}$ centralizes $\mathfrak{T}' \cap \boldsymbol{Z}(\mathfrak{T})$, since $\mathfrak{T}' \cap \boldsymbol{Z}(\mathfrak{H}) \subseteq \mathfrak{C}$. This is absurd, since $\Omega_1(\boldsymbol{C}_{\mathfrak{P}}(\mathfrak{H})) = \Omega_1(\boldsymbol{Z}(\mathfrak{P}))$ by (B) applied to $\mathfrak{M}$, completing the proof of this theorem.

Before combining all these results, we require an additional result about π_4.

THEOREM 25.4. *Let $p \in \pi_4$, let $\mathfrak{P}$ be a S_p-subgroup of $\mathfrak{G}$ and let $\mathfrak{M}$ be the unique maximal subgroup of $\mathfrak{G}$ containing $\mathfrak{P}$. Then $\mathfrak{P} \subseteq \mathfrak{M}'$.*

Proof. Let $\mathfrak{C} \in \mathscr{SCN}_3(\mathfrak{P})$, and suppose G in $\mathfrak{G}$ has the property that $\mathfrak{C}^G \subseteq \mathfrak{P}$. Then $\mathfrak{C} \subseteq \mathfrak{M}^{G^{-1}}$. By Theorem 25.3, we have $\mathfrak{M}^{G^{-1}} = \mathfrak{M}$, so that $G \in \mathfrak{M}$. Hence $\mathfrak{B} = V(ccl_{\mathfrak{G}}(\mathfrak{C}); \mathfrak{P}) = V(ccl_{\mathfrak{M}}(\mathfrak{C}); \mathfrak{P})$. By (B) and $p \in \pi_4$, $\mathfrak{C}^M \subseteq O_p(\mathfrak{M})$ for each M in $\mathfrak{M}$. Hence, $\mathfrak{B} \lhd \mathfrak{M}$, so maximality of $\mathfrak{M}$ implies $\mathfrak{M} = N(\mathfrak{B})$. By uniqueness of $\mathfrak{M}$ (or because $\mathfrak{B}$ is weakly closed in $\mathfrak{P}$), we have $\mathfrak{M} \supseteq N(\mathfrak{P})$. Furthermore, by Theorem 25.3, if $\mathfrak{M}^G \neq \mathfrak{M}$, then $\mathfrak{C} \not\subseteq \mathfrak{M}^G$. Thus, $\mathfrak{C}$ is not in the kernel $\mathfrak{K}(G)$ of the permutation representation of $\mathfrak{P}$ on the cosets of $\mathfrak{P}$ in $\mathfrak{M}G\mathfrak{P}$. We can then find C in $\mathfrak{C}$ such that $\mathfrak{K}(G)C$ has order p in $Z(\mathfrak{P}/\mathfrak{K}(G))$, so Theorem 14.4.1 in [12] yields this theorem.

We are now in a position to let π_3 and π_4 coalesce, that is, we set $\pi_0 = \pi_3 \cup \pi_4$.

THEOREM 25.5. *Let $\mathfrak{M}$ be a maximal subgroup of $\mathfrak{G}$. If $p \in \pi_0$ and $\mathfrak{M}_p$ is a S_p-subgroup of $\mathfrak{M}$, then either $\mathfrak{M}_p$ is a S_p-subgroup of $\mathfrak{G}$ or $\mathfrak{M}_p$ has a cyclic subgroup of index at most p, and $\mathfrak{M}_p \notin \mathscr{A}_4(\mathfrak{P})$ for every S_p-subgroup $\mathfrak{P}$ of $\mathfrak{G}$. If ϖ is the largest subset of π_0 with the property that $\mathfrak{M}$ contains a S_ϖ-subgroup $\mathfrak{S}$ of $\mathfrak{G}$, then $\mathfrak{S} \lhd \mathfrak{M}$, and $\mathfrak{S} \subseteq \mathfrak{M}'$.*

Proof. Let $\mathfrak{P}$ be a S_p-subgroup of $\mathfrak{G}$ containing $\mathfrak{M}_p$. Suppose $\mathfrak{M}_p \subset \mathfrak{P}$. Then $\mathfrak{M}_p \notin \mathscr{A}_4(\mathfrak{P})$, by Theorems 24.3, 24.5, and 25.3. Thus, if $\mathfrak{B} \in \mathscr{U}(\mathfrak{P})$, then $C(\mathfrak{B}) \cap \mathfrak{M}_p$ is cyclic. Since $|\mathfrak{M}_p : C(\mathfrak{B}) \cap \mathfrak{M}_p| = 1$ or p the first assertion follows.

Let $\mathfrak{S}_q$ be a S_q-subgroup of $\mathfrak{S}$ for q in ϖ. (If ϖ is empty there is no more to prove.) If $q \in \pi_3$, then $\mathfrak{S}_q \subseteq \mathfrak{M}'$ by uniqueness of $\mathfrak{M}$ and Lemma 17.2. If $q \in \pi_4$, then $\mathfrak{S}_q \subseteq \mathfrak{M}'$ by uniqueness of $\mathfrak{M}$ and Theorem 25.5. Hence, $\mathfrak{S} \subseteq \mathfrak{M}'$. If $r \in \pi(\mathfrak{M})$, $r \notin \varpi$, then $\mathfrak{M}'$ centralizes every chief r-factor of $\mathfrak{M}$, by Lemma 8.13. Since $\mathfrak{S} \subseteq \mathfrak{M}'$, we conclude that $\mathfrak{S} \lhd \mathfrak{M}$.

THEOREM 25.6. *π_0 is partitioned into non empty subsets $\sigma_1, \cdots, \sigma_n$, $n \geq 1$, with the following properties:*

(i) *If $\tau \subseteq \pi_0$, then $\mathfrak{G}$ satisfies E_τ if and only if $\tau \subseteq \sigma_i$ for some $i = 1, \cdots, n$.*

(ii) *If $\mathfrak{H}_i$ is a S_{σ_i}-subgroup of $\mathfrak{G}$, then $\mathfrak{N}_i = N(\mathfrak{H}_i)$ is a maximal subgroup of $\mathfrak{G}$, $\mathfrak{H}_i \subseteq \mathfrak{N}_i'$, and $\mathfrak{H}_i \cap \mathfrak{H}_i^G$ is of square free order for each $G \in \mathfrak{G} - \mathfrak{N}_i$, $i = 1, \cdots, n$.*

(iii) *If $p_i \in \sigma_i$ and $\mathfrak{P}_i$ is a S_{p_i}-subgroup of $\mathfrak{H}_i$, and if $\mathfrak{P}_i \cap \mathfrak{P}_i^G = \mathfrak{D}_i \neq 1$ for some $G \in \mathfrak{G} - \mathfrak{N}_i$, then $\mathfrak{D}_i$ is of order p_i and $C_{\mathfrak{P}_i}(\mathfrak{D}_i) = \mathfrak{D}_i \times \mathfrak{E}_i$, where $\mathfrak{E}_i$ is cyclic, $i = 1, 2, \cdots, n$.*

Proof. By Lemma 8.5, π_0 is non empty. By Corollary 19.1, Theorems 24.3, 24.4, 24.5, 25.2 and 25.3 $\sim$ is an equivalence relation on π_0 and if $\sigma_1, \cdots, \sigma_n$ are the equivalence classes of π_0 under $\sim$, then (i) holds.

Let $\mathfrak{H} = \mathfrak{H}_i$ be a S_{σ_i}-subgroup of $\mathfrak{G}$ and let $\mathfrak{P} = \mathfrak{P}_i$ be a S_{p_i}-subgroup of $\mathfrak{H}$ for $p = p_i \in \sigma_i$. By Theorem 25.5, $N(\mathfrak{H}) = \mathfrak{N}$ is a maximal subgroup of $\mathfrak{G}$, and $\mathfrak{H} \subseteq \mathfrak{N}'$.

Suppose $G \in \mathfrak{G} - \mathfrak{N}$ and $\mathfrak{P} \cap \mathfrak{P}^G = \mathfrak{D} \neq 1$. If $\mathfrak{D}_1$ is any non identity characteristic subgroup of $\mathfrak{D}$, then either $N(\mathfrak{D}_1) \cap \mathfrak{P} \notin \mathscr{A}_4(\mathfrak{P})$ or $N(\mathfrak{D}_1) \cap \mathfrak{P}^G \notin \mathscr{A}_4(\mathfrak{P}^G)$, by Theorems 24.3, 24.4, 24.5, 25.5 and 25.3. Since $N(D(\mathfrak{D}))$ contains every element of both $\mathscr{U}(\mathfrak{P})$ and $\mathscr{U}(\mathfrak{P}^G)$, we conclude that $\mathfrak{D}$ is elementary of order p or p^2. Suppose $|\mathfrak{D}| = p^2$. If $\mathfrak{D}$ contains $\Omega_1(Z(\mathfrak{P}))$ then $N(\mathfrak{D})$ contains an element of $\mathscr{U}(\mathfrak{P})$, so that $N(\mathfrak{D}) \cap \mathfrak{P} \in \mathscr{A}_4(\mathfrak{P})$. If $\mathfrak{D}$ does not contain $\Omega_1(Z(\mathfrak{P}))$ then $N(\mathfrak{D}) \cap \mathfrak{P}$ contains an elementary subgroup of order p^3, so once again $N(\mathfrak{D}) \cap \mathfrak{P} \in \mathscr{A}_4(\mathfrak{P})$. The same argument applies to $\mathfrak{P}^G$, so that $\mathfrak{P}^G \subseteq \mathfrak{N}$. Hence $\mathfrak{P}^G = \mathfrak{P}^N$ for some N in $\mathfrak{N}$. Hence $GN^{-1} \in N(\mathfrak{P}) \subseteq \mathfrak{N}$, so $G \in \mathfrak{N}$, contrary to hypothesis. Hence, $\mathfrak{D}$ is of order p.

If $C_{\mathfrak{P}}(\mathfrak{D}) \in \mathscr{A}_4(\mathfrak{P})$, then $N(\mathfrak{D}) \subseteq \mathfrak{N}$, so that $\mathfrak{P}^G \cap \mathfrak{N} \supset \mathfrak{D}$, contrary to the fact that $\mathfrak{P}^G \cap \mathfrak{P}^N$ has order 1 or p for all N in $\mathfrak{N}$, by the preceding paragraph. Hence, $C_{\mathfrak{P}}(\mathfrak{D}) \notin \mathscr{A}_4(\mathfrak{P})$. If $\mathfrak{B} \in \mathscr{U}(\mathfrak{P})$, and $C_{\mathfrak{P}}(\mathfrak{D}) \cap C_{\mathfrak{P}}(\mathfrak{B}) = \mathfrak{E}$, then $\mathfrak{E}$ is of index at most p in $C_{\mathfrak{P}}(\mathfrak{D})$ and $\mathfrak{E}$ is disjoint from $\mathfrak{D}$, since $C_{\mathfrak{P}}(\mathfrak{D}) \notin \mathscr{A}_4(\mathfrak{P})$. Hence, $C_{\mathfrak{P}}(\mathfrak{D}) = \mathfrak{D} \times \mathfrak{E}$. This proves (iii), the cyclicity of $\mathfrak{E}$ following from $C_{\mathfrak{P}}(\mathfrak{D}) \notin \mathscr{A}_4(\mathfrak{P})$. The proof is complete.

26. The Maximal Subgroups of $\mathfrak{G}$

The purpose of this section is to use the preceding results, notably Theorems 25.5 and 25.6, to complete the proofs of the results stated in Section 14.

LEMMA 26.1. *If $p \in \pi_1 \cup \pi_2$ and $\mathfrak{P}$ is a S_p-subgroup of $\mathfrak{G}$, then $\mathfrak{P} \subseteq N(\mathfrak{P})'$.*

Proof. If $\mathfrak{P}$ is abelian, the lemma follows from Grün's theorem and the simplicity of $\mathfrak{G}$. If $\mathfrak{P}$ is non abelian, $\mathfrak{P}$ is not metacyclic, by 3.8. Also, $p \geqq 5$, as already observed several times. Thus, from 3.4 we see that $\Omega_1(\mathfrak{P})$ is a non abelian group of order p^3. The hypotheses of Lemma 8.10 are satisfied, so $\mathfrak{P} \subseteq N(\Omega_1(Z(\mathfrak{P})))'$ by Theorem 14.4.2 in [12] and the simplicity of $\mathfrak{G}$. Since $N(\mathfrak{P}) \subseteq N(\Omega_1(Z(\mathfrak{P})))$, and since $N(\Omega_1(Z(\mathfrak{P})))$ has p-length one, the lemma follows.

LEMMA 26.2. *If $p \in \pi_2$ and $\mathfrak{P}$ is a S_p-subgroup of $\mathfrak{G}$, then $\mathfrak{P}$ is abelian or is a central product of a cyclic group and a non abelian group of order p^3 and exponent p.*

Proof. We only need to show that $\mathfrak{P}$ is not isomorphic to (iii) in 3.4. Suppose false. Let $\mathfrak{P}_1 = \Omega_1(\mathfrak{P})$, and let $\mathfrak{R}$ be a fixed $S_{p'}$-subgroup of $N(\mathfrak{P})$. Set $\mathfrak{R}_1 = \mathfrak{R}/C_{\mathfrak{R}}(\mathfrak{P})$. The oddness of $|N(\mathfrak{P})|$ guarantees that $\mathfrak{R}_1$ is abelian.

Let $\mathscr{C}$ be a chief series for $\mathfrak{P}$, one of whose terms is $\mathfrak{P}_1$ and which is $\mathfrak{R}$-admissible. Let α_i be the character of $\mathfrak{R}_1$ on the ith term of $\mathscr{C}$ modulo the $(i+1)$st, where $i = 1, \cdots, \ell + 3$, and $|\mathfrak{P}:\mathfrak{P}_1| = p^{\ell}$. Since $\mathfrak{P}/\mathfrak{P}_1$ is cyclic, $\alpha_1 = \cdots = \alpha_{\ell}$. From 3.4, we see that $\alpha_1 = \alpha_{\ell+3}$. Furthermore, $\alpha_{\ell+2} = \alpha_1\alpha_{\ell+1}$, and $\alpha_{\ell+3} = \alpha_{\ell+1}\alpha_{\ell+2}$. Combining these equalities yields $\alpha_{\ell+1}^2 = 1$, so $\alpha_{\ell+1} = 1$, and Lemma 26.1 is violated.

If $\mathfrak{B}$ normalizes $\mathfrak{A}$ we say that $\mathfrak{B}$ *is prime on* $\mathfrak{A}$ provided any two elements of $\mathfrak{B}^\#$ have the same fixed points on $\mathfrak{A}$. If $|\mathfrak{B}|$ is a prime, $\mathfrak{B}$ is necessarily prime on $\mathfrak{A}$. If $\mathfrak{A}$ is solvable, then $\mathfrak{B}$ is prime on $\mathfrak{A}$ if and only if for each prime p, there is a S_p-subgroup $\mathfrak{A}_p$ of $\mathfrak{A}$ which is normalized by $\mathfrak{B}$ and such that $\mathfrak{B}$ is prime on $\mathfrak{A}_p$.

The next two lemmas are restatements of Lemma 13.12.

LEMMA 26.3. *Suppose $\mathfrak{A}$ is a solvable π-group, and $\mathfrak{B}$ is a cyclic π'-subgroup of* Aut$(\mathfrak{A})$ *which is prime on $\mathfrak{A}$. Assume also that $|\mathfrak{A}|\cdot|\mathfrak{B}|$ is odd. If $|\mathfrak{B}|$ is not a prime, if the centralizer of $\mathfrak{B}$ in $\mathfrak{A}$ is a Z-group, and if $\mathfrak{B}$ has no fixed points on $\mathfrak{A}/\mathfrak{A}'$, then $\mathfrak{A}$ is nilpotent.*

LEMMA 26.4. *Suppose $\mathfrak{A}$ is a solvable π-group and $\mathfrak{B}$ is a π'-subgroup of* Aut$(\mathfrak{A})$ *of prime order. Assume also that $|\mathfrak{A}|\cdot|\mathfrak{B}|$ is odd. If the centralizer of $\mathfrak{B}$ in $\mathfrak{A}$ is a Z-group, and if $\mathfrak{B}$ has no fixed points on $\mathfrak{A}/\mathfrak{A}'$, then $\mathfrak{A}/F(\mathfrak{A})$ is nilpotent.*

$\mathscr{X}$ denotes the set of all proper subgroups of $\mathfrak{G}$, $\mathscr{X}_0$ denotes those subgroups $\mathfrak{A}$ of $\mathfrak{G}$ such that, for all $p \in \pi_0$, $\mathfrak{A}$ does not contain an element of $\mathscr{S}\!\mathscr{C}_4(\mathfrak{P})$ for any S_p-subgroup $\mathfrak{P}$ of $\mathfrak{G}$; $\mathscr{X}_1 = \mathscr{X} - \mathscr{X}_0$. $\mathscr{M}$ denotes the set of maximal subgroups of $\mathfrak{G}$, $\mathscr{M}_i = \mathscr{M} \cap \mathscr{X}_i$, $i = 0, 1$.

If $\mathfrak{R} \in \mathscr{X}_0$, then $\mathfrak{R}$ does not contain an elementary subgroup of order p^3 for any prime p, so $\mathfrak{R}'$ is nilpotent. Furthermore, if $\pi(\mathfrak{R}) = \{p_1, \cdots, p_n\}$, $p_1 > p_2 > \cdots > p_n$, then $\mathfrak{R}$ has a Sylow series of complexion $(p_1, \cdots, p_n)$.

Suppose $p \in \pi_0$ and $\mathfrak{P}_0$ is a subgroup of type (p, p) with $\mathfrak{P}_0 \in \mathscr{X}_0$. Let $\mathfrak{P}_1, \cdots, \mathfrak{P}_n$ be the distinct S_p-subgroups of $\mathfrak{G}$ which contain $\mathfrak{P}_0$. Since $\mathfrak{P}_0 \notin \mathscr{S}\!\mathscr{C}_4(\mathfrak{P}_i)$, $1 \leq i \leq n$, it follows that $\mathfrak{P}_0 \supseteq \Omega_1(Z(\mathfrak{P}_i))$, and that $N(\mathfrak{P}_0) - C(\mathfrak{P}_0)$ contains an element of order p centralizing $\Omega_1(Z(\mathfrak{P}_i))$. Since $N(\mathfrak{P}_0)/C(\mathfrak{P}_0)$ is p-closed, this implies that $\Omega_1(Z(\mathfrak{P}_i)) = \Omega_1(Z(\mathfrak{P}_j))$, $1 \leq i, j \leq n$. This fact is very important, since it shows that the $p + 1$ subgroups of $\mathfrak{P}_0$ of order p are contained in two conjugate classes in $\mathfrak{G}$, one class containing $\Omega_1(Z(\mathfrak{P}_1))$, the remaining p subgroups lying in a single conjugate class.

If $\mathfrak{M} \in \mathscr{M}_0$, $\boldsymbol{H}(\mathfrak{M})$ denotes the largest normal nilpotent S-subgroup of $\mathfrak{M}$. Note that by Lemma 8.5, $\boldsymbol{H}(\mathfrak{M}) \neq 1$. More explicitly, $\pi(\boldsymbol{H}(\mathfrak{M}))$ contains the largest prime in $\pi(\mathfrak{M})$. Note also that $\boldsymbol{H}(\mathfrak{M})$ is a S-subgroup of $\mathfrak{G}$.

If $\mathfrak{M} \in \mathscr{M}_1$, $\boldsymbol{H}_1(\mathfrak{M})$ denotes the unique S_σ-subgroup of $\mathfrak{M}$, where $\sigma = \sigma(\mathfrak{M})$ is the equivalence class of π_0 under $\sim$ associated with $\mathfrak{M}$. That is, $p \in \sigma$ if and only if $p \in \pi_0$ and $\mathfrak{M}$ contains a S_p-subgroup of $\mathfrak{G}$. Or again, $p \in \sigma$ if and only if $\mathfrak{M}$ contains an elementary subgroup of order p^3. Or again, $p \in \sigma$ if and only if $p \in \pi_0$ and $\mathfrak{M}$ contains an element of $\mathscr{S}_4(\mathfrak{P})$ for some S_p-subgroup $\mathfrak{P}$ of $\mathfrak{G}$.

Suppose $\mathfrak{M} \in \mathscr{M}_1$, $q \in \pi(\mathfrak{M}) - \sigma(\mathfrak{M})$ and a S_q-subgroup $\mathfrak{Q}$ of $\mathfrak{M}$ centralizes $\boldsymbol{H}_1(\mathfrak{M})$. Since $\mathfrak{M}$ is the unique maximal subgroup of $\mathfrak{G}$ containing $\boldsymbol{H}_1(\mathfrak{M})$, it follows that $N(\mathfrak{Q}) \subseteq \mathfrak{M}$, so that $\mathfrak{Q}$ is a S_q-subgroup of $\mathfrak{G}$. Then by Lemma 26.1, $\mathfrak{Q} \subseteq \mathfrak{M}'$. Since the derived group of $\mathfrak{M}/\boldsymbol{H}_1(\mathfrak{M})$ is nilpotent, we have $\mathfrak{Q} \lhd \mathfrak{M}$. Thus, if τ is the largest subset of $\pi(\mathfrak{M}) - \sigma(\mathfrak{M})$ such that some S_τ-subgroup of $\mathfrak{M}$ centralizes $\boldsymbol{H}_1(\mathfrak{M})$, then $\mathfrak{M}$ contains a unique S_τ-subgroup $\boldsymbol{E}_1(\mathfrak{M})$, $\boldsymbol{E}_1(\mathfrak{M})$ is a normal nilpotent S-subgroup of $\mathfrak{M}$, $\boldsymbol{E}_1(\mathfrak{M})$ is a S-subgroup of $\mathfrak{G}$, and the structure of the S_q-subgroups of $\boldsymbol{E}_1(\mathfrak{M})$ is given by Lemma 26.2. We set $\boldsymbol{H}(\mathfrak{M}) = \langle \boldsymbol{E}_1(\mathfrak{M}), \boldsymbol{H}_1(\mathfrak{M}) \rangle = \boldsymbol{E}_1(\mathfrak{M}) \times \boldsymbol{H}_1(\mathfrak{M})$. Since $\boldsymbol{E}_1(\mathfrak{M}) \lhd \mathfrak{M}$ and $\boldsymbol{E}_1(\mathfrak{M})$ centralizes $\boldsymbol{H}_1(\mathfrak{M})$, and since $\mathfrak{M}$ is the unique maximal subgroup of $\mathfrak{G}$ containing $\boldsymbol{H}_1(\mathfrak{M})$, it follows that $\boldsymbol{E}_1(\mathfrak{M})$ is a T.I. set in $\mathfrak{G}$.

If $p \in \pi_0 \cap \pi^*$ and $\mathfrak{P}$ is a S_p-subgroup of $\mathfrak{G}$, then the definitions of π_0 and π^* imply that $\Omega_1(\boldsymbol{Z}_2(\mathfrak{P}))$ is of type (p, p). In this case, we set $\boldsymbol{T}(\mathfrak{P}) = \boldsymbol{C}_{\mathfrak{P}}(\Omega_1(\boldsymbol{Z}_2(\mathfrak{P})))$, and remark that $\boldsymbol{T}(\mathfrak{P})$ char $\mathfrak{P}$, $|\mathfrak{P} : \boldsymbol{T}(\mathfrak{P})| = p$. Furthermore, if P is an element of order p in $\boldsymbol{T}(\mathfrak{P})$, then $\boldsymbol{C}_{\mathfrak{P}}(P)$ contains an elementary subgroup of order p^3. If $q \in \pi_0 - \pi^*$, set $\boldsymbol{T}(\mathfrak{Q}) = \mathfrak{Q}$, $\mathfrak{Q}$ being any S_q-subgroup of $\mathfrak{G}$. The relevance of $\boldsymbol{T}(\mathfrak{Q})$ lies in the fact that if Q is any element of $\boldsymbol{T}(\mathfrak{Q})$ of order q, then $\boldsymbol{C}(Q)$ is contained in only one maximal subgroup of $\mathfrak{G}$, namely, the one that contains $\mathfrak{Q}$. This statement is an immediate consequence of the theorems proved about $\mathscr{S}_4(\mathfrak{Q})$, explicitly stated in Theorem 25.5.

If $\mathfrak{A} \in \mathscr{X}_1$, then $\mathfrak{A}$ is contained in a unique maximal subgroup $\mathfrak{M}$ of $\mathfrak{G}$, so we set $\boldsymbol{M}(\mathfrak{A}) = \mathfrak{M}$. The existence of the mapping $\boldsymbol{M}$ from $\mathscr{X}_1$ to $\mathscr{M}_1$ is naturally crucial.

If $\mathfrak{M} \in \mathscr{M}_0$, set $\hat{\boldsymbol{H}}(\mathfrak{M}) = \boldsymbol{H}(\mathfrak{M})^\#$. If $\mathfrak{M} \in \mathscr{M}_1$, let $\hat{\boldsymbol{H}}(\mathfrak{M})$ consist of all elements H in $\boldsymbol{H}(\mathfrak{M})^\#$ with the property that some power of H, say $H_1 = H^n$ is either in $\boldsymbol{E}_1(\mathfrak{M})^\#$ or is in $\boldsymbol{T}(\mathfrak{Q})^\#$ for some S_q-subgroup $\mathfrak{Q}$ of $\mathfrak{M}$ with $q \in \pi(\boldsymbol{H}_1(\mathfrak{M}))$.

Let $q \in \pi_0$ and let $\mathfrak{Q}$ be a S_q-subgroup of $\mathfrak{G}$ with $\boldsymbol{T}(\mathfrak{Q}) \subset \mathfrak{Q}$; let $\mathscr{T}(\mathfrak{Q})$ denote the set of subgroups $\mathfrak{Q}_0$ of $\mathfrak{Q}$ of type (q, q) such that

$\mathfrak{Q}_0 = \Omega_1(C_{\mathfrak{Q}}(Q))$ for some element Q in $\mathfrak{Q}_0$. If $\mathfrak{Q}_0 \in \mathscr{T}(\mathfrak{Q})$, then $\mathfrak{Q}_0 \supseteq \Omega_1(\boldsymbol{Z}(\mathfrak{Q}))$. Furthermore, if $q \in \pi_0$ and $\mathfrak{Q}_1$ is a subgroup of $\mathfrak{G}$ of type (q, q), and if $\mathfrak{Q}_1$ is contained in at least two maximal subgroups of $\mathfrak{G}$, then $\mathfrak{Q}_1 \in \mathscr{T}(\mathfrak{Q})$ for every S_q-subgroup $\mathfrak{Q}$ of $\mathfrak{G}$ which contains $\mathfrak{Q}_1$.

LEMMA 26.5.
(i) *If* $\mathfrak{M} \in \mathscr{M}_0$, *then* $\boldsymbol{H}(\mathfrak{M})'$ *is a T.I. set in* $\mathfrak{G}$.
(ii) *If* $\mathfrak{M} \in \mathscr{M}_1$, *then* $\hat{\boldsymbol{H}}(\mathfrak{M})$ *is a T.I. set in* $\mathfrak{G}$.

Proof.
(i) $\boldsymbol{H}(\mathfrak{M})'$ is cyclic and normal in $\mathfrak{M}$, by Lemma 26.2. Hence, if $H \in \boldsymbol{H}(\mathfrak{M})'^{\#} \cap \boldsymbol{H}(\mathfrak{M}^G)'^{\#}$ for some G in $\mathfrak{G}$, then $N(\langle H\rangle) \supseteq \langle \mathfrak{M}, \mathfrak{M}^G\rangle$, so $G \in \mathfrak{M}$, as required.

(ii) It is immediate from the definition that $\hat{\boldsymbol{H}}(\mathfrak{M})$ is a normal subset of $\mathfrak{M}$, so $\hat{\boldsymbol{H}}(\mathfrak{M})$ is a T.I. set in $\mathfrak{M}$. Suppose $G \in \mathfrak{G}$ and $H \in \hat{\boldsymbol{H}}(\mathfrak{M}) \cap \hat{\boldsymbol{H}}(\mathfrak{M})^G$. Choose n so that $K = H^n$ is in either $\boldsymbol{E}_1(\mathfrak{M})^{\#}$ or $\boldsymbol{T}(\mathfrak{Q})^{\#}$ for some S_q-subgroup $\mathfrak{Q}$ of $\boldsymbol{H}(\mathfrak{M})$, and such that K is of prime order. If $K \in \boldsymbol{E}_1(\mathfrak{M})^{\#}$, then since $(|\boldsymbol{E}_1(\mathfrak{M})|, |\boldsymbol{H}_1(\mathfrak{M})|) = 1$, it follows that $K \in \boldsymbol{E}_1(\mathfrak{M})^{G\#}$. Hence $C(K) \supseteq \langle \boldsymbol{H}_1(\mathfrak{M}), \boldsymbol{H}_1(\mathfrak{M})^G\rangle$, and so $G \in \mathfrak{M}$. Suppose $K \in \boldsymbol{H}_1(\mathfrak{M})^{\#}$. Then $C_{\mathfrak{Q}}(K) \in \mathscr{S}\!\mathscr{C}_4(\mathfrak{Q})$ and so $C(K) \subseteq \mathfrak{M}$. This implies that $\boldsymbol{H}_1(\mathfrak{M}) \cap \boldsymbol{H}_1(\mathfrak{M})^G$ contains non cyclic S_q-subgroups. By Theorem 25.6 (ii), we again have $G \in \mathfrak{M}$. The lemma is proved.

With Lemma 26.5 at hand, it is fairly clear that the one remaining obstacle in this chapter is π^*. In dealing with π^*, we will repeatedly use the assumption that $|\mathfrak{G}|$ is odd.

LEMMA 26.6. *Let* $p \in \pi_0$, *let* $\mathfrak{P}$ *be a* S_p*-subgroup of* $\mathfrak{G}$, *and let* $\mathfrak{M} = \boldsymbol{M}(\mathfrak{P})$. *If* $\mathfrak{P}_1$ *is any non identity subgroup of* $\boldsymbol{T}(\mathfrak{P})$ *and* $\mathfrak{P}_1$ *is contained in the p-subgroup* $\mathfrak{P}^*$ *of* $\mathfrak{G}$, *then* $N(\mathfrak{P}^*) \subseteq \mathfrak{M}$.

Proof. In any case, $\mathfrak{P}^* \subseteq \mathfrak{M}$, by Theorem 25.6 (iii). If $\mathfrak{P}^*$ is non cyclic, then $N(\Omega_1(\mathfrak{P}^*))$ contains an element of $\mathscr{S}\!\mathscr{C}_4(\mathfrak{P}_0)$ for some S_p-subgroup $\mathfrak{P}_0$ of $\mathfrak{M}$ and we are done. Otherwise, $\Omega_1(\mathfrak{P}^*) = \Omega_1(\mathfrak{P}_1)$, so $N(\Omega_1(\mathfrak{P}^*))$ contains an element of $\mathscr{S}\!\mathscr{C}_4(\mathfrak{P})$, and we are done.

LEMMA 26.7. *Suppose* $p, q \in \pi_1 \cup \pi_2$, $p \neq q$, $\mathfrak{Q}$ *is a* S_q*-subgroup of* $\mathfrak{G}$ *and* $\mathfrak{P}$ *is a* S_p*-subgroup of* $N(\mathfrak{Q})$. *If* $\mathfrak{P}$ *is cyclic, then* $\mathfrak{P}$ *is prime on* $\mathfrak{Q}$.

Proof. Suppose false. Then $q \equiv \pm 1 \pmod p$, and every p, q-subgroup $\mathfrak{K}$ of $\mathfrak{G}$ is q-closed. Also $\Omega_1(\mathfrak{P}) \subseteq \boldsymbol{Z}(\mathfrak{P}^*)$ for some S_p-subgroup $\mathfrak{P}^*$ of $\mathfrak{G}$, by Lemma 26.2 and $|\mathfrak{P}| > p$. If $\mathfrak{P}^*$ is cyclic, or if $\mathfrak{P}^*$ is non abelian, then $\mathfrak{P} \subseteq N(\Omega_1(\mathfrak{P}))'$, by Lemma 26.1. Since every chief

q-factor of $N(\Omega_1(\mathfrak{P}))$ is centralized by $N(\Omega_1(\mathfrak{P}))'$, it follows that $\mathfrak{P}$ centralizes $C_{\mathfrak{Q}}(\Omega_1(\mathfrak{P}))$ and we are done.

If $\mathfrak{P}^*$ is abelian and non cyclic, then $\mathfrak{P}^*$ normalizes some S_q-subgroup $\mathfrak{Q}^*$ of $N(\Omega_1(\mathfrak{P}))$. Since the lemma is assumed false, $C_{\mathfrak{Q}}(\Omega_1(\mathfrak{P})) \neq 1$, so $\mathfrak{Q}^* \neq 1$. If $\mathfrak{K}$ is a maximal p, q-subgroup of $\mathfrak{G}$ containing $\mathfrak{P}^*\mathfrak{Q}^*$, then $\mathfrak{K}$ is q-closed, so contains a S_q-subgroup of $\mathfrak{G}$. This violates the hypothesis of this lemma.

LEMMA 26.8. *Let $p \in \pi_0$, $q \in \pi(\mathfrak{G})$ and suppose that $q \in \pi_1 \cup \pi_2$ or $p \not\sim q$. If $\mathfrak{K}$ is any p, q-subgroup of $\mathfrak{G}$ and $\mathfrak{K}$ contains an element of $\mathscr{SCN}_3(\mathfrak{P})$ for some S_p-subgroup $\mathfrak{P}$ of $\mathfrak{G}$, then $\mathfrak{K}$ is p-closed.*

Proof. Let $\mathfrak{M} = M(\mathfrak{K})$. The hypotheses imply that $p \,||\, H_1(\mathfrak{M})|$ and $q \nmid |H_1(\mathfrak{M})|$. The lemma follows.

LEMMA 26.9. *Let $p \in \pi_0$, $q \in \pi(\mathfrak{G})$ and suppose that $q \in \pi_1 \cup \pi_2$ or $p \not\sim q$. If $\mathfrak{Q}$ is a q-subgroup of $\mathfrak{G}$ which is normalized by the cyclic p-subgroup $\mathfrak{P}$ of $\mathfrak{G}$, then $\mathfrak{P}$ is prime on $\mathfrak{Q}$.*

Proof. If $|\mathfrak{P}| = p$, the lemma is trivial. Otherwise, the lemma follows from Lemma 26.8, since $N(\Omega_1(\mathfrak{P}))$ contains an element of $\mathscr{SCN}_3(\mathfrak{P}_0)$ for some S_p-subgroup $\mathfrak{P}_0$ of $\mathfrak{G}$.

LEMMA 26.10. *Let $\mathfrak{M} \in \mathscr{M}$, and let $\mathfrak{P}$ be a S_p-subgroup of $\mathfrak{M}$ for some prime p. If $\mathfrak{P}$ is non abelian and $\mathfrak{P} \nsubseteq \mathfrak{M}'$, then $\mathfrak{P}$ does not contain a cyclic subgroup of index p.*

Proof. We can suppose that $\mathfrak{P} \in \mathscr{X}_0$, for if $\mathfrak{P} \in \mathscr{X}_1$, then $\mathfrak{M} = M(\mathfrak{P})$ and $\mathfrak{P} \subseteq \mathfrak{M}'$ by Theorem 25.6 (ii). Hence, proceeding by way of contradiction we can suppose that $\mathfrak{P} = gp\langle P_0, P_1 \mid P_0^{p^n} = P_1^p = 1, P_1^{-1}P_0P_1 = P_0^{1+p^{n-1}}\rangle$, where $n \geqq 2$. Note that $\mathfrak{P}' = \langle P_0^{p^{n-1}}\rangle$.

If $\mathfrak{M}'$ is nilpotent, then $\mathfrak{P}' \lhd \mathfrak{M}$, so $\mathfrak{M} = N(\mathfrak{P}')$ by maximality of $\mathfrak{M}$. This implies that $\mathfrak{P}$ is a S_p-subgroup of $\mathfrak{G}$ which is not the case. Hence, $\mathfrak{M}'$ is not nilpotent. In particular, $\mathfrak{M} \in \mathscr{M}_1$. It follows that $p \not\sim q$ for all q in $\pi(H_1(\mathfrak{M}))$.

We first show that $E_1(\mathfrak{M}) = 1$. For $\mathfrak{P}'$ centralizes $E_1(\mathfrak{M})$, so if $\mathfrak{M}_1$ is an element of $\mathscr{M}$ containing $N(\mathfrak{P}')$, then $E_1(\mathfrak{M})$ normalizes some S_p-subgroup $\mathfrak{P}_0$ of $\mathfrak{M}_1$ with $\mathfrak{P} \subseteq \mathfrak{P}_0$. It follows from Lemma 8.16 that $E_1(\mathfrak{M})$ centralizes $\mathfrak{P}_0$. If $E_1(\mathfrak{M}) \neq 1$, then $\mathfrak{P}_0 \subseteq \mathfrak{M}$, which is not the case, so $E_1(\mathfrak{M}) = 1$.

Choose q in $\pi(H(\mathfrak{M}))$ and let $\mathfrak{Q}$ be a S_q-subgroup of $\mathfrak{M}$ normalized by $\mathfrak{P}$. We can now choose $\mathfrak{A} \subseteq T(\mathfrak{Q})$ such that $\mathfrak{A}$ is normalized by $\Omega_1(\mathfrak{P})$, is centralized by some non identity element P of $\Omega_1(\mathfrak{P})$, but is not centralized by $\Omega_1(\mathfrak{P})$. For otherwise, $\Omega_1(\mathfrak{P})$ centralizes $T(\mathfrak{Q})$, and

$N(\Omega_1(\mathfrak{P})) \subseteq \mathfrak{M}$, which is not the case. For such a choice of $\mathfrak{A}$ and P, let $\mathfrak{K}$ be a $S_{p,q}$-subgroup of $C(P)$ which contains $\mathfrak{A}\Omega_1(\mathfrak{P})$. By Lemma 26.7, there is a S_q-subgroup $\mathfrak{K}_q$ of $\mathfrak{K}$ which contains $\mathfrak{A}$ and is contained in $\mathfrak{M}$. Since $\Omega_1(\mathfrak{P})$ does not centralize $\mathfrak{A}$, and since $p \not\sim q$, a S_p-subgroup $\mathfrak{K}_p$ of $\mathfrak{K}$ is contained in $\mathscr{X}_0$, by Lemma 26.8.

We wish to show that $\mathfrak{K}_q \lhd \mathfrak{K}$. This is clear if $\mathfrak{K}_q$ contains an element of $\mathscr{A}_4(\mathfrak{Q}^*)$ for some S_q-subgroup $\mathfrak{Q}^*$ of $\mathfrak{G}$, by Lemma 26.6. Otherwise, Lemma 8.5 implies that $\mathfrak{K}_q \lhd \mathfrak{K}$, since $q > p$. By Lemma 26.6, $\mathfrak{K} \subseteq \mathfrak{M}$, so $\mathfrak{M}$ contains a S_p-subgroup of $C(P)$. This implies that $\langle P \rangle \neq \langle P_0^{p^{n-1}} \rangle$. Since the p subgroups of $\mathfrak{P}$ of order p different from $\langle P_0^{p^{n-1}} \rangle$ are conjugate in $\mathfrak{P}$, and since $\hat{H}(\mathfrak{M})$ is a normal subset of $\mathfrak{M}$, we can suppose that $P = P_1$.

Let $\mathfrak{P}^*$ be a S_p-subgroup of $\mathfrak{G}$ containing $\mathfrak{P}$ and let $\mathfrak{W} = \Omega_1(Z_2(\mathfrak{P}^*))$, so that $\mathfrak{W} \cap \mathfrak{P} = \langle P_0^{p^{n-1}} \rangle$, or else $p \in \pi_2$. It follows that $P_0 W$ centralizes P_1 for some W in $\mathfrak{W}$. But $\mathfrak{M}$ contains a S_p-subgroup of $C(P_1)$, so $C(P_1) \cap \mathfrak{M}$ contains an element of order equal to that of $P_0 W$. Since $P_0 W$ and P_0 have the same order, a S_p-subgroup of $C(P_1) \cap \mathfrak{M}$ has exponent p^n, which is not the case. The proof is complete.

LEMMA 26.11. *Let $\mathfrak{M} \in \mathscr{M}$ and let $\mathfrak{P}$ be a S_p-subgroup of $\mathfrak{M}$ for some prime p. If $\mathfrak{P}$ is non abelian, then $\mathfrak{P} \subseteq \mathfrak{M}'$.*

Proof. First, suppose $p \in \pi_0$. If $\mathfrak{P} \in \mathscr{X}_1$, we are done. Otherwise, $\mathfrak{P}$ contains a cyclic subgroup of index p and we are done by the preceding lemma.

We can now suppose that $p \in \pi_2$. If $\mathfrak{M}'$ is nilpotent, the lemma follows readily from Lemmas 26.1 and 26.2. We can suppose that $\mathfrak{M}'$ is not nilpotent and that $\mathfrak{P} \not\subseteq \mathfrak{M}'$. Since $\mathfrak{P}$ is non abelian, Lemma 26.2 implies that $\Omega_1(\mathfrak{P})$ is of order p^3, or else $\mathfrak{P}$ is metacyclic. In the second case, we are done by the preceding lemma.

We first show that $E_1(\mathfrak{M}) = 1$. Since $\Omega_1(Z(\mathfrak{P}))$ centralizes $E_1(\mathfrak{M})$, it follows readily that $N(E_1(\mathfrak{M}))$ dominates $\mathfrak{P}$, by Sylow's theorem. If $E_1(\mathfrak{M}) \neq 1$, then $\mathfrak{M} = N(E_1(\mathfrak{M}))$, and so $\mathfrak{P} \subseteq \mathfrak{M}'$, by Lemma 26.1, and we are done.

Let $\mathfrak{Q}$ be a S_q-subgroup of $\mathfrak{M}$ which is normalized by $\mathfrak{P}$, with $q \in \pi(H(\mathfrak{M}))$.

We show that $\mathfrak{Q} = T(\mathfrak{Q})$. For otherwise, $\mathfrak{P}'$ centralizes $\mathfrak{Q}$, by Lemma 8.16, so that $N(\mathfrak{P}') \subseteq \mathfrak{M}$. By Lemmas 26.1 and 26.2, $\mathfrak{P} \subseteq N(\mathfrak{P}')'$, contrary to $\mathfrak{P} \not\subseteq \mathfrak{M}'$. Hence, $\mathfrak{Q} = T(\mathfrak{Q})$.

Let $\mathfrak{Z} = Z(\Omega_1(\mathfrak{P}))$. We next show that $\mathfrak{Z}$ has no fixed points on $\mathfrak{Q}^\#$. Let $\mathfrak{Q}_1 = \mathfrak{Q} \cap C(\mathfrak{Z})$, and suppose by way of contradiction that $\mathfrak{Q}_1 \neq 1$. Let $\mathfrak{L} = N(\mathfrak{Z})$, and let $\mathfrak{L}_0$ be the maximal normal subgroup of $\mathfrak{L}$ of order prime to pq. Let $\mathfrak{L}_p$, $\mathfrak{L}_q$ be permutable Sylow subgroups of $\mathfrak{L}$, $\mathfrak{P} \subseteq \mathfrak{L}_p$, $\mathfrak{Q}_1 \subseteq \mathfrak{L}_q$. Since $\mathfrak{L}_p \subseteq \mathfrak{L}'$, it follows that $\mathfrak{L}$ is not contained in any conjugate of $\mathfrak{M}$. This implies that $\mathfrak{L}_q \in \mathscr{H}_0$. This in turn implies that $\mathfrak{L}_p$ centralizes every chief q-factor of $\mathfrak{L}$, by Lemma 8.13. Hence, $\mathfrak{L}_p \lhd \mathfrak{L}_p\mathfrak{L}_q$, and it follows that $N(\mathfrak{L}_q)$ covers $\mathfrak{L}/\mathfrak{L}_0\mathfrak{L}_p$. Since $N(\mathfrak{L}_q) \subseteq \mathfrak{M}$, by Lemma 26.6, we have a contradiction. Hence, $\mathfrak{Q}_1 = 1$.

We next show that if $P \in \Omega_1(\mathfrak{P}) - \mathfrak{Z}$, then $C(P) \subseteq \mathfrak{M}$. This is clear if $C(P) \cap \mathfrak{Q}$ is non cyclic, since $\mathfrak{Q} = T(\mathfrak{Q})$, so suppose $C(P) \cap \mathfrak{Q} = \mathfrak{Q}_1$ is cyclic. We remark that $\mathfrak{Q}_1 \neq 1$, an easy consequence of the preceding paragraph.

Let $\mathfrak{M}_1$ be a maximal subgroup of $\mathfrak{G}$ containing $C(P)$, and let $\mathfrak{Q}^*$ be a S_q-subgroup of $\mathfrak{M}_1$ containing $\mathfrak{Q}_1$. If $\mathfrak{Q}^*$ is non cyclic, then $\mathfrak{Q}^*$ is contained in a unique maximal subgroup $\mathfrak{M}^G$ of $\mathfrak{G}$, $G \in \mathfrak{G}$, and since $\mathfrak{Q}^* \subseteq \mathfrak{M}_1$, we have $\mathfrak{M}_1 = \mathfrak{M}^G$. Since $\mathfrak{M} \cap \mathfrak{M}^G \supseteq \mathfrak{Q}_1$, and since $\mathfrak{Q}_1 \subseteq T(\mathfrak{Q})$, we have $\mathfrak{M} = \mathfrak{M}^G$. Thus, we can suppose that $\mathfrak{Q}^*$ is cyclic.

Since $\mathfrak{Z}$ acts regularly on $\mathfrak{Q}_1$, we can suppose that a S_p-subgroup $\mathfrak{P}^*$ of $\mathfrak{M}_1$ normalizes $\mathfrak{Q}^*$ and that $\langle P, \mathfrak{Z}\rangle \subseteq \mathfrak{P}^*$.

If $\mathfrak{M}_1'$ is nilpotent, then $\Omega_1(\mathfrak{Q}^*) \lhd \mathfrak{M}_1$. Since $\Omega_1(\mathfrak{Q}^*) = \Omega_1(\mathfrak{Q}_1)$, we have $\mathfrak{M} = \mathfrak{M}_1$. Hence, we can suppose that $\mathfrak{M}_1'$ is not nilpotent.

Choose r in $\pi(\boldsymbol{H}_1(\mathfrak{M}_1))$, and let $\mathfrak{R}$ be a S_r-subgroup of $\mathfrak{M}_1$ normalized by $\mathfrak{P}^*\mathfrak{Q}^*$. Since $\mathfrak{Q}^*$ is cyclic, $q \not\sim r$. Since $q \not\sim r$, $\mathfrak{Q}^*$ does not centralize $\mathfrak{R}$. It follows from $\mathfrak{Q}^* \subseteq (\mathfrak{P}^*\mathfrak{Q}^*)'$ that $\mathfrak{R} = T(\mathfrak{R})$, by Lemma 8.16. Since $\mathfrak{Q}^*\mathfrak{Z}$ is a Frobenius group, it follows that $\mathfrak{R}_1 = \mathfrak{R} \cap C(\mathfrak{Z}) \neq 1$. Let $\mathfrak{C} = N(\mathfrak{Z})$.

Let $\mathfrak{K}$ be a $S_{p,r}$-subgroup of $\mathfrak{C}$ which contains $\mathfrak{R}_1$ and $\mathfrak{P}^*$, and let $\mathfrak{K}_r$ be a S_r-subgroup of $\mathfrak{K}$ containing $\mathfrak{R}_1$. If $\mathfrak{K}_r$ is non cyclic, then $\mathfrak{K}_r \in \mathscr{H}_1$, so $\mathfrak{K} \subseteq \mathfrak{M}_1$. If $\mathfrak{K}_r$ is cyclic, then in any case $\mathfrak{K}_r \subseteq \mathfrak{M}_1$, since $\mathfrak{R} = T(\mathfrak{R})$. Let $\mathfrak{K}_p$ be a S_p-subgroup of $\mathfrak{K}$. If $\mathfrak{P}^*$ does not centralize $\mathfrak{R}_1$, then $r > p$, and so $\mathfrak{K}_r \lhd \mathfrak{K}$, and once again $\mathfrak{K} \subseteq \mathfrak{M}_1$. If $\mathfrak{P}^*$ centralizes $\mathfrak{R}_1$ and $\mathfrak{K}_r \ntriangleleft \mathfrak{K}$, then $\mathfrak{K}_p \lhd \mathfrak{K}$. Since the structure of $\mathfrak{K}_p$ is determined by Lemma 26.2, and since $\mathfrak{R}_1$ centralizes $\mathfrak{P}^*$, it follows that $\mathfrak{R}_1$ centralizes $\mathfrak{K}_p$, so that $\Omega_1(\mathfrak{R}_1) \lhd \mathfrak{K}$, and once again $\mathfrak{K} \subseteq \mathfrak{M}_1$. Thus, in any case, we see that $\mathfrak{K} \subseteq \mathfrak{M}_1$. This implies that $\mathfrak{Z} \subseteq \mathfrak{M}_1'$, so $\mathfrak{Z}$ centralizes every chief q-factor of $\mathfrak{M}_1$. This is absurd, since $\mathfrak{Z}\mathfrak{Q}^*$ is a Frobenius group. We conclude that $C(P) \subseteq \mathfrak{M}$ for every P in $\Omega_1(\mathfrak{P}) - \mathfrak{Z}$.

We will now show directly that $N(\Omega_1(\mathfrak{P})) \subseteq \mathfrak{M}$. Choose $N \in N(\Omega_1(\mathfrak{P}))$. Then $\Omega_1(\mathfrak{P})$ normalizes $\mathfrak{Q}$ and $\mathfrak{Q}^N$. Since $\mathfrak{Z}$ has no fixed points on $\mathfrak{Q}^N$, $\mathfrak{Q}^N$ is generated by its subgroups $\mathfrak{Q}^N \cap C(P)$, $P \in \Omega_1(\mathfrak{P}) - \mathfrak{Z}$. By the preceding paragraph, we conclude that $\mathfrak{Q}^N \subseteq \mathfrak{M}$. Since $\mathfrak{M}^N$ is the unique maximal subgroup of $\mathfrak{G}$ containing $\mathfrak{Q}^N$, we have $\mathfrak{M} = \mathfrak{M}^N$, so $N \in \mathfrak{M}$. By Lemma 26.1, $\mathfrak{P} \subseteq N(\Omega_1(\mathfrak{P}))'$, so $\mathfrak{P} \subseteq \mathfrak{M}'$. The proof is complete.

LEMMA 26.12. *Suppose $\mathfrak{M} \in \mathscr{M}$ and $\mathfrak{P}$ is an abelian, non cyclic*

S_p-subgroup of $\mathfrak{M}$ for some prime p. Suppose further that a S_p-subgroup of $\mathfrak{G}$ is non abelian. Then $\mathfrak{P} = \mathfrak{P}_1 \times \mathfrak{P}_2$, where $|\mathfrak{P}_1| = p$, $\mathfrak{P}_1$ centralizes $\boldsymbol{H}(\mathfrak{M})$, $\mathfrak{P}_2\boldsymbol{H}(\mathfrak{M})$ is a Frobenius group with Frobenius kernel $\boldsymbol{H}(\mathfrak{M})$ and $\mathfrak{P}_2$ contains $\Omega_1(\boldsymbol{Z}(\mathfrak{P}^*))$ for every S_p-subgroup $\mathfrak{P}^*$ of $\mathfrak{G}$ which contains $\mathfrak{P}$.

Proof. Let $\mathfrak{P}_0$ be a S_p-subgroup of $\mathfrak{G}$ containing $\mathfrak{P}$. If $p \in \pi_0$, then $\Omega_1(\mathfrak{P}) \in \mathscr{T}(\mathfrak{P}_0)$, and if σ is any automorphism of $\mathfrak{P}_0$ of prime order s, then $s < p$, by Lemma 8.16. The same inequality clearly holds if $p \in \pi_2$.

Choose q in $\pi(\boldsymbol{H}(\mathfrak{M}))$ and let $\mathfrak{Q}$ be a S_q-subgroup of $\mathfrak{M}$ normalized by $\mathfrak{P}$.

Let $\mathfrak{Z} = \Omega_1(\boldsymbol{Z}(\mathfrak{P}_0))$. We will show that $\mathfrak{Q}\mathfrak{Z}$ is a Frobenius group. Let $\mathfrak{C} = \boldsymbol{N}(\mathfrak{Z})$ and suppose by way of contradiction that $\mathfrak{Q}_1 = \mathfrak{Q} \cap \mathfrak{C} \neq 1$. First consider the case that $p \in \pi_0$. Let $\mathfrak{M}_1 = \boldsymbol{M}(\mathfrak{C})$, and let $\mathfrak{P}_{00}$ be a S_p-subgroup of $\mathfrak{M}_1$ normalized by $\mathfrak{Q}_1$ with $\mathfrak{P} \subseteq \mathfrak{P}_{00}$. Then $[\mathfrak{Q}_1, \mathfrak{P}] \subseteq \mathfrak{Q} \cap \mathfrak{P}_{00} = 1$, so $\mathfrak{Q}_1$ centralizes $\mathfrak{P}$. Since $\Omega_1(\mathfrak{P}) \subseteq \mathscr{T}(\mathfrak{P}_{00})$, it follows that $\mathfrak{Q}_1$ centralizes $\mathfrak{P}_{00}$. Thus, if $q \in \pi(\boldsymbol{E}_1(\mathfrak{M}))$ or $\boldsymbol{T}(\mathfrak{Q}) = \mathfrak{Q}$, we conclude that $\mathfrak{P}_{00} \subseteq \mathfrak{M}$, which is contrary to hypothesis. Otherwise, $\boldsymbol{T}(\mathfrak{Q}) \subset \mathfrak{Q}$, or $q \in \pi_1 \cup \pi_2$, so that $q > p$, or $\mathfrak{P}$ centralizes $\mathfrak{Q}$. But in these cases, we at least have $\boldsymbol{N}(\mathfrak{Q}_1) \subseteq \mathfrak{M}_1$, so $\mathfrak{Q}_1 \neq \mathfrak{Q}$, which yields $q > p$, and so a S_q-subgroup of $\mathfrak{M} \cap \mathfrak{M}_1$ is non cyclic, and centralizes $\mathfrak{P}_{00}$. Again we conclude that $\mathfrak{P}_{00} \subseteq \mathfrak{M}$, which is not the case. Hence, we can suppose that $p \in \pi_2$.

Let $\mathfrak{R}$ be a $S_{p,q}$-subgroup of $\mathfrak{C}$ containing $\mathfrak{P}\mathfrak{Q}_1$, $\mathfrak{P} \subseteq \mathfrak{R}_p$, $\mathfrak{Q}_1 \subseteq \mathfrak{R}_q$, and let $\mathfrak{R}^*$ be a maximal p, q-subgroup of $\mathfrak{G}$ containing $\mathfrak{R}$, $\mathfrak{R}_p \subseteq \mathfrak{R}_p^*$, $\mathfrak{R}_q \subseteq \mathfrak{R}_q^*$, where $\mathfrak{R}_p^*$ is a S_p-subgroup of $\mathfrak{R}^*$ and $\mathfrak{R}_q^*$ is a S_q-subgroup of $\mathfrak{R}^*$. Since $\mathfrak{P}_0$ is a S_p-subgroup of $\mathfrak{G}$, $\mathfrak{R}_p = \mathfrak{R}_p^*$ is a S_p-subgroup of $\mathfrak{G}$. If $\mathfrak{R}_q^*$ contains an elementary subgroup of order q^3, then $\mathfrak{R}_q^* \lhd \mathfrak{R}^*$, and maximality of $\mathfrak{R}^*$ implies that $\mathfrak{R}^*$ is contained in a conjugate of $\mathfrak{M}$, contrary to hypothesis. If $\mathfrak{R}_q^*$ does not contain an elementary subgroup of order q^3, then either $q > p$ or $\mathfrak{P}$ centralizes $\mathfrak{Q}_1$. If $q > p$, then $\mathfrak{R}_q^* \lhd \mathfrak{R}^*$, so once again $\mathfrak{R}^* \subseteq \mathfrak{M}^G$ for some $G \in \mathfrak{G}$. If $q < p$, then $\mathfrak{R}_p^* \lhd \mathfrak{R}^*$, and since $\mathfrak{Q}_1$ centralizes $\mathfrak{P}$, $\mathfrak{Q}_1$ centralizes $\mathfrak{R}_p^*$, by Lemma 26.2. In this case, $\boldsymbol{O}_q(\mathfrak{R}^*) \neq 1$. If $\boldsymbol{O}_q(\mathfrak{R}^*)$ is non cyclic, then $\mathfrak{R}^* \subseteq \mathfrak{M}^G$, either by Lemma 26.6, in case $q \in \pi_0$, or because $\mathfrak{Q} \lhd \mathfrak{M}$ in case $q \in \pi_2$. If $\boldsymbol{O}_q(\mathfrak{R}^*)$ is cyclic, then $\mathfrak{Q}_1 \lhd \mathfrak{R}^*$. In this case $\boldsymbol{N}_{\mathfrak{Q}}(\mathfrak{Q}_1)\mathfrak{P}$ is conjugate to a subgroup of $\mathfrak{R}^*$, since $\mathfrak{R}^*$ is a S-subgroup of $\boldsymbol{N}(\mathfrak{Q}_1)$. Since $\mathfrak{R}_p^* \lhd \mathfrak{R}^*$, it follows that $\mathfrak{P}$ centralizes $\boldsymbol{N}_{\mathfrak{Q}}(\mathfrak{Q}_1)$ so that $\boldsymbol{N}_{\mathfrak{Q}}(\mathfrak{Q}_1)$ centralizes some S_p-subgroup of $\boldsymbol{N}(\mathfrak{Q}_1)$. If $q \in \pi(\boldsymbol{E}_1(\mathfrak{M}))$, this is not possible. But if $q \in \pi(\boldsymbol{H}_1(\mathfrak{M}))$, then $\boldsymbol{N}_{\mathfrak{Q}}(\mathfrak{Q}_1)$ is non cyclic, so $\boldsymbol{N}(\boldsymbol{N}_{\mathfrak{Q}}(\mathfrak{Q}_1)) \subseteq \mathfrak{M}$. Thus, in all these cases, $\mathfrak{M}$ contains a S_p-subgroup of $\mathfrak{G}$. Since this is not possible, $\mathfrak{Z}\mathfrak{Q}$ is a Frobenius group, and so $\mathfrak{Z}\boldsymbol{H}(\mathfrak{M})$ is a Frobenius group.

Suppose $\mathfrak{M} \in \mathscr{M}_0$. We will show that if $\mathfrak{Z}_0$ is any subgroup of $\mathfrak{P}$ of order p with $C(\mathfrak{Z}_0) \cap \boldsymbol{H}(\mathfrak{M}) \neq 1$, then $C(\mathfrak{Z}_0) \subseteq \mathfrak{M}$. Let $\mathfrak{M}_1 \in \mathscr{M}$ with $C(\mathfrak{Z}_0) \subseteq \mathfrak{M}_1$. First consider the case $\mathfrak{M}_1 = \mathfrak{M}^G$, for some G in $\mathfrak{G}$. Let $\mathfrak{Q}_1$ be a non identity S_q-subgroup of $C(\mathfrak{Z}_0) \cap \boldsymbol{H}(\mathfrak{M})$ and let $\mathfrak{Q}_2$ be a S_q-subgroup of $C(\mathfrak{Z}_0) \cap \boldsymbol{H}(\mathfrak{M}_1)$ containing $\mathfrak{Q}_1$. If $\mathfrak{Q}_1 \subset \mathfrak{Q}_2$, then Lemma 26.2 implies that $\mathfrak{Q}_2$ is a S_q-subgroup of $\mathfrak{G}$. In this case, since $\mathfrak{M}_1$ and $\mathfrak{M}$ are conjugate and since $\mathfrak{P}$ is a S_p-subgroup of $\mathfrak{M}$, $\mathfrak{P}$ contains a subgroup of order p which centralizes the S_q-subgroup of $\mathfrak{M}$. Since $\mathfrak{Z}\boldsymbol{H}(\mathfrak{M})$ is a Frobenius group, this implies that if $\mathfrak{Z}_1$ is any subgroup of $\mathfrak{P}$ of order p, then either $\mathfrak{Z}_1\boldsymbol{H}_q(\mathfrak{M})$ is a Frobenius group, or $\mathfrak{Z}_1$ centralizes $\boldsymbol{H}_q(\mathfrak{M})$, the S_q-subgroup of $\mathfrak{M}$. This violates the choice of $\mathfrak{Q}_1$. Hence, $\mathfrak{Q}_1 = \mathfrak{Q}_2$. If a S_q-subgroup of $\mathfrak{G}$ is abelian, then $\mathfrak{Q}_1 \lhd \langle \mathfrak{M}, \mathfrak{M}_1 \rangle$, so $\mathfrak{M} = \mathfrak{M}_1$. If some S_q-subgroup of $\mathfrak{G}$ contains $\Omega_1(\mathfrak{Q}_1)$ in its center, then by Lemma 8.10, $\mathfrak{M} = \mathfrak{M}_1$. Hence, we can suppose that $\mathfrak{Q}_1$ is of order q and $\mathfrak{Q}_1 \not\subseteq \boldsymbol{Z}(\boldsymbol{H}(\mathfrak{M}))$. In this case, $N(\mathfrak{Q}_1) \cap \mathfrak{M}_1$ is of index q in $\mathfrak{M}_1$ and $N(\mathfrak{Q}_1) \cap \mathfrak{M}$ is of index q in $\mathfrak{M}$, and $N(\mathfrak{Q}_1) \cap \mathfrak{M}_1$ contains $C(\mathfrak{Z}_0)$.

Let $\mathfrak{L} = N(\mathfrak{Q}_1)$. If $\mathfrak{L}$ is contained in a conjugate of $\mathfrak{M}$, then $N(\mathfrak{Q}_1) \cap \boldsymbol{H}(\mathfrak{M}_1) \lhd \mathfrak{L}$ so $\mathfrak{L} \subseteq \mathfrak{M}_1$, since $N(\mathfrak{Q}_1) \cap \boldsymbol{H}(\mathfrak{M}_1) \lhd \mathfrak{M}_1$. Similarly, $\mathfrak{L} \subseteq \mathfrak{M}$, and we are done. If $\mathfrak{L}$ is contained in an element of $\mathscr{M}_0$, then since $\mathfrak{Z}\boldsymbol{H}(\mathfrak{M})$ is a Frobenius group, we see that $N(\mathfrak{Q}_1) \cap \boldsymbol{H}(\mathfrak{M}) \lhd \langle \mathfrak{L}, \mathfrak{M} \rangle$, and $\mathfrak{L} \subseteq \mathfrak{M}$.

Hence, in showing that $C(\mathfrak{Z}_0) \subseteq \mathfrak{M}$, we can suppose that $C(\mathfrak{Z}_0)$ is contained in an element $\mathfrak{M}_1$ of $\mathscr{M}_1$. Since $\mathfrak{Z} \cdot (C(\mathfrak{Z}_0) \cap \boldsymbol{H}(\mathfrak{M}))$ is a Frobenius group, this implies that $\mathfrak{Z} \not\subseteq \mathfrak{M}_1'$. Since $\mathfrak{P}$ is a S_p-subgroup of $\mathfrak{M}$, we conclude that $\mathfrak{P}$ is a S_p-subgroup of $\mathfrak{M}_1$. By what we have already proved, $\mathfrak{Z}\boldsymbol{H}(\mathfrak{M}_1)$ is a Frobenius group. This implies that $(C(\mathfrak{Z}_0) \cap \boldsymbol{H}(\mathfrak{M}))\boldsymbol{H}_1(\mathfrak{M}_1)$ is nilpotent, so $C(\mathfrak{Z}_0) \cap \boldsymbol{H}(\mathfrak{M})$ centralizes $\boldsymbol{H}_1(\mathfrak{M}_1)$. Since $\mathfrak{M}_1$ is the unique maximal subgroup of $\mathfrak{G}$ containing $\boldsymbol{H}_1(\mathfrak{M}_1)$, it follows that $\boldsymbol{H}(\mathfrak{M})$ centralizes $\boldsymbol{H}_1(\mathfrak{M}_1)$, so that $\mathfrak{M} \subseteq \mathfrak{M}_1$, which is absurd since $\mathfrak{M} \in \mathscr{M}_0$, $\mathfrak{M}_1 \in \mathscr{M}_1$. We conclude that $C(\mathfrak{Z}_0) \subseteq \mathfrak{M}$.

We next show that if $\mathfrak{M} \in \mathscr{M}_1$ and $C(\mathfrak{Z}_0)$ contains an element of $\hat{\boldsymbol{H}}(\mathfrak{M})$, then $C(\mathfrak{Z}_0) \subseteq \mathfrak{M}$. Here, as above, $\mathfrak{Z}_0$ is a subgroup of $\mathfrak{P}$ of order p. Let $\mathfrak{Q}_1$ be a $\mathfrak{P}$-invariant S_q-subgroup of $C(\mathfrak{Z}_0) \cap \mathfrak{M}$ with $\mathfrak{Q}_1 \cap \hat{\boldsymbol{H}}(\mathfrak{M}) \neq \emptyset$. From Lemma 26.7, we conclude that $C(\mathfrak{Z}_0) \cap \mathfrak{M}$ contains a S_q-subgroup $\mathfrak{Q}_2$ of $C(\mathfrak{Z}_0)$, and we can assume that $\mathfrak{Q}_1 = \mathfrak{Q}_2$.

Let $\mathfrak{M}_1 \in \mathscr{M}$, $C(\mathfrak{Z}_0) \subseteq \mathfrak{M}_1$. If $\mathfrak{M}_1 = \mathfrak{M}^G$, then $\mathfrak{M} \cap \mathfrak{M}_1 \supseteq \mathfrak{Q}_1$, so $\mathfrak{M} = \mathfrak{M}_1$. If $\mathfrak{M}_1'$ is nilpotent, then by Lemma 26.7, we see that $\mathfrak{M}_1 \cap \mathfrak{M}$ contains a S_q-subgroup $\mathfrak{Q}_3$ of $\mathfrak{M}_1$ which is $\mathfrak{Z}$-invariant. Since $\mathfrak{Z}\mathfrak{Q}_3$ is a Frobenius group, $\mathfrak{Q}_3 \lhd \mathfrak{M}_1$ and so $\mathfrak{M}_1 = \mathfrak{M}$. We can suppose that $\mathfrak{M}_1'$ is not nilpotent, and that $\mathfrak{M}_1 \neq \mathfrak{M}$. In particular, $\mathfrak{M}_1 \in \mathscr{M}_1$. It follows that $\mathfrak{P}$ is a S_p-subgroup of $\mathfrak{M}_1$, so that $\mathfrak{Z}\boldsymbol{H}(\mathfrak{M}_1)$ is a Frobenius group, and so $\mathfrak{Q}_2$ centralizes $\boldsymbol{H}(\mathfrak{M}_1)$, and $\mathfrak{M} = \mathfrak{M}_1$ follows.

Thus, $\mathfrak{M} = \mathfrak{M}_1$ in all cases.

Suppose now that $\mathfrak{P}$ contains two distinct subgroups $\mathfrak{Z}_0$, $\mathfrak{Z}_1$ such that $C(\mathfrak{Z}_0) \cap \hat{H}(\mathfrak{M}) \neq \emptyset$ and $C(\mathfrak{Z}_1) \cap \hat{H}(\mathfrak{M}) \neq \emptyset$. We can choose P in $\mathfrak{P}_0$ such that $\mathfrak{Z}_0 = \mathfrak{Z}_1^P$. If $\mathfrak{M} \in \mathscr{M}_1$, we get an easy contradiction. Namely, $C(\mathfrak{Z}_0) \subseteq \mathfrak{M} \cap \mathfrak{M}^P$, and so $\mathfrak{M} = \mathfrak{M}^P$ and $P \in \mathfrak{M} \cap \mathfrak{P}_0 = \mathfrak{P}$, so that $\mathfrak{Z}_0 = \mathfrak{Z}_1$, contrary to assumption.

If $\mathfrak{M} \in \mathscr{M}_0$, then $\mathfrak{M} \cap \mathfrak{M}^P$ contains $C(\mathfrak{Z}_0) \cap H(\mathfrak{M})$. If $H(\mathfrak{M})$ contains an abelian S_q-subgroup $\mathfrak{Q}$ with $C(\mathfrak{Z}_0) \cap \mathfrak{Q} \neq 1$, then $C(\mathfrak{Z}_0) \cap \mathfrak{Q} \lhd \langle \mathfrak{M}, \mathfrak{M}^P \rangle$, and $\mathfrak{M} = \mathfrak{M}^P$, which is the desired contradiction. Otherwise, if $\mathfrak{Q}$ is a S_q-subgroup of $H(\mathfrak{M})$ with $C(\mathfrak{Z}_0) \cap \mathfrak{Q} = \mathfrak{Q}_1 \neq 1$, then $N(\mathfrak{Q}_1) \cap \mathfrak{M}$ is of index q in $\mathfrak{M}$ and $N(\mathfrak{Q}_1) \cap \mathfrak{M}^P$ is of index q in $\mathfrak{M}^P$, while both $N(\mathfrak{Q}_1) \cap H(\mathfrak{M})$ and $N(\mathfrak{Q}_1) \cap H(\mathfrak{M}^P)$ are S-subgroups of $N(\mathfrak{Q}_1)$. Furthermore, since a $S_{p,q}$-subgroup $\mathfrak{L}_0$ of $N(\mathfrak{Q}_1)$ is q-closed, it follows that $\mathfrak{P}(N(\mathfrak{Q}_1) \cap H(\mathfrak{M}))$ and $\mathfrak{P}(N(\mathfrak{Q}_1) \cap H(\mathfrak{M}^P))$ are S-subgroups of $N(\mathfrak{Q}_1)$. Furthermore, $\mathfrak{P}$ has a normal complement in $N(\mathfrak{Q}_1)$, since $q \in \pi_2$, and no element of $\mathfrak{P}^\#$ centralizes $N(\mathfrak{Q}_1) \cap \mathfrak{Q}$. By the conjugacy of Sylow systems in $N(\mathfrak{Q}_1)$, we can therefore find $C \in C(\mathfrak{P}) \cap N(\mathfrak{Q}_1)$ such that $(N(\mathfrak{Q}_1) \cap H(\mathfrak{M}^P))^C = N(\mathfrak{Q}_1) \cap H(\mathfrak{M})$. Since $(N(\mathfrak{Q}_1) \cap H(\mathfrak{M}^P))^C = N(\mathfrak{Q}_1) \cap H(\mathfrak{M}^{PC})$, and $N(\mathfrak{Q}_1) \cap H(\mathfrak{M}) \lhd \mathfrak{M}$, we conclude that $\mathfrak{M} = \mathfrak{M}^{PC}$, so $PC \in \mathfrak{M}$, which is not the case, since C is in $\mathfrak{M}$ and P is not.

Hence, there is exactly one subgroup $\mathfrak{Z}_0$ of $\mathfrak{P}$ of order p which has a fixed point on $\hat{H}(\mathfrak{M})$, so $\mathfrak{Z}_0$ centralizes $H(\mathfrak{M})$. Since $\mathfrak{P} = \mathfrak{Z}_0 \times \mathfrak{P}^*$, where $\mathfrak{P}^* \supseteq \mathfrak{Z}$, the lemma follows.

Lemma 26.12 is quite important because, given $\mathfrak{M}$, (and the hypothesis of Lemma 26.12) it produces a unique factorization of $\Omega_1(\mathfrak{P})$. Namely, exactly one subgroup $\mathfrak{Z}$ of $\mathfrak{P}$ of order p is in the center of a S_p-subgroup of $\mathfrak{G}$, and exactly one subgroup $\mathfrak{Z}_0$ of $\mathfrak{P}$ of order p centralizes $H(\mathfrak{M})$, and $\mathfrak{Z} \neq \mathfrak{Z}_0$. This is a critical point in dealing with tamely imbedded subsets. Furthermore, Lemma 26.12 shows that $H(\mathfrak{M})$ is nilpotent, a useful fact.

LEMMA 26.13. *Suppose $\mathfrak{M} \in \mathscr{M}$ and $\mathfrak{P}$ is an abelian, non cyclic S_p-subgroup of $\mathfrak{M}$ for some prime p. Suppose further that a S_p-subgroup of $\mathfrak{G}$ is abelian. Then the following statements are true:*

(i) *$\mathfrak{P}$ is a S_p-subgroup of $\mathfrak{G}$.*

(ii) $C(\Omega_1(\mathfrak{P})) \subseteq \mathfrak{M}$.

(iii) *If P and P_1 are elements of $\mathfrak{P}$ which are conjugate in $\mathfrak{G}$ but are not conjugate in $\mathfrak{M}$, either $C(P) \cap H(\mathfrak{M}) = 1$ or $C(P_1) \cap H(\mathfrak{M}) = 1$.*

(iv) *Either $\mathfrak{M}$ dominates $\Omega_1(\mathfrak{P})$ or $C(\Omega_1(\mathfrak{P})) \cap H(\mathfrak{M}) = 1$.*

(v) *One of the following conditions holds:*

(a) $\mathfrak{P} \subseteq \mathfrak{M}'$.

(b) *$N(\mathfrak{P}_0) \subseteq \mathfrak{M}$ for every non identity subgroup $\mathfrak{P}_0$ of $\mathfrak{P}$ such* *that* $\boldsymbol{C}(\mathfrak{P}_0) \cap \boldsymbol{H}(\mathfrak{M}) \neq 1$.

Proof. If $p \in \pi_0$, then $\mathfrak{P} \in \mathscr{X}_1$ and all parts of the lemma follow immediately. We can suppose that $p \in \pi_2$.

In proving this lemma, appeal to Lemmas 8.5 and 8.16 will be made repeatedly.

If $\Omega_1(\mathfrak{P})$ centralizes $\boldsymbol{H}(\mathfrak{M})$, then $\mathfrak{M} = \boldsymbol{N}(\Omega_1(\mathfrak{P}))$ and all parts of the lemma follow immediately. We can suppose that $\Omega_1(\mathfrak{P})$ does not centralize $\boldsymbol{H}(\mathfrak{M})$. This implies that $\boldsymbol{H}(\mathfrak{M}) \cap \mathfrak{P} = 1$.

We first prove an auxiliary result: if $\mathfrak{R}$ is any p, q-subgroup of $\mathfrak{G}$ containing $\Omega_1(\mathfrak{P})$ and if $\mathfrak{R} \cap \boldsymbol{H}(\mathfrak{M}) \neq 1$, then $\mathfrak{R}$ is q-closed. To see this, let $\mathfrak{Q}$ be a S_q-subgroup of $\mathfrak{R} \cap \boldsymbol{H}(\mathfrak{M})$, and let $\mathfrak{P}_1$ be a S_p-subgroup of $\mathfrak{R} \cap \mathfrak{M}$ which contains $\Omega_1(\mathfrak{P})$. Let $\mathfrak{R}_q$ be a S_q-subgroup of $\mathfrak{R}$ containing $\mathfrak{Q}$ and let $\mathfrak{R}_p$ be a S_p-subgroup of $\mathfrak{R}$ containing $\mathfrak{P}_1$. If $\mathfrak{R}_q \in \mathscr{X}_1$, then $\mathfrak{R} \subseteq \mathfrak{M}^G$ for some G in $\mathfrak{G}$ and so $\mathfrak{R}_q \lhd \mathfrak{R}$. If $\mathfrak{R}_q \in \mathscr{X}_0$, then $\mathfrak{R}$ does not contain elementary subgroups of order p^3 or q^3, so either $\mathfrak{R}_q \lhd \mathfrak{R}$ or $\mathfrak{R}_p \lhd \mathfrak{R}$. If $\mathfrak{R}_p \lhd \mathfrak{R}$, and $\mathfrak{R}_q \ntriangleleft \mathfrak{R}$, then $p > q$. Suppose $q \in \pi_1 \cup \pi_2$. Then $\mathfrak{P}$ centralizes the S_q-subgroup $\mathfrak{Q}_1$ of $\mathfrak{M}$. There is no loss of generality in supposing that $\mathfrak{R}$ is a maximal p, q-subgroup of $\mathfrak{G}$. It follows from this normalization that $\boldsymbol{O}_q(\mathfrak{R})$ is a S_q-subgroup of $\mathfrak{G}$, and $\mathfrak{R} = \mathfrak{R}_p \times \mathfrak{R}_q$. Hence, we can suppose $q \in \pi_0$. Since $\mathfrak{R}_q \ntriangleright \mathfrak{R}$, $\mathfrak{R}_q \in \mathscr{X}_0$. If $\boldsymbol{O}_q(\mathfrak{R})$ is not of order q, then $\mathfrak{R}$ is contained in a conjugate of $\mathfrak{M}$, by Lemma 26.7, and we are done. Hence, we can suppose that $\mathfrak{Q} = \boldsymbol{O}_q(\mathfrak{R})$ is of order q. But now $\boldsymbol{N}(\mathfrak{Q}) \cap \mathfrak{M}$ contains S_q-subgroups of order exceeding q, so that $S_{p,q}$-subgroups of $\boldsymbol{N}(\mathfrak{Q})$ are q-closed. Since $\mathfrak{R} \subseteq \boldsymbol{N}(\mathfrak{Q})$, $\mathfrak{R}$ is q-closed

(i) is an immediate application of the preceding paragraph, since some element of $\mathfrak{P}^\#$ centralizes an element of $\boldsymbol{H}(\mathfrak{M})^\#$.

We turn next to (iv). Suppose $\boldsymbol{C}(\Omega_1(\mathfrak{P})) \cap \boldsymbol{H}(\mathfrak{M}) \neq 1$, and $\mathfrak{Q}_0$ is a non identity $\mathfrak{P}$-invariant S_q-subgroup of $\boldsymbol{C}(\Omega_1(\mathfrak{P})) \cap \boldsymbol{H}(\mathfrak{M})$. Let $\mathfrak{Q}_1$ be a S_q-subgroup of $\boldsymbol{N}(\Omega_1(\mathfrak{P}))$ permutable with $\mathfrak{P}$. By the first paragraph of the proof, $\mathfrak{P}$ normalizes $\mathfrak{Q}_1$, so by Sylow's theorem $\boldsymbol{N}(\mathfrak{Q}_1)$ dominates $\Omega_1(\mathfrak{P})$. Suppose for some $n \geqq 1$, $\mathfrak{P}$ normalizes $\mathfrak{Q}_n$ and $\mathfrak{Q}_n$ dominates $\Omega_1(\mathfrak{P})$. Let $\mathfrak{Q}_{n+1}$ be a S_q-subgroup of $\boldsymbol{N}(\mathfrak{Q}_n)$ permutable with $\mathfrak{P}$. Then $\mathfrak{P}$ normalizes $\mathfrak{Q}_{n+1}$ and so $\mathfrak{Q}_{n+1}$ dominates $\Omega_1(\mathfrak{P})$. Since $\mathfrak{Q}_0 \subseteqq \mathfrak{Q}_1 \subseteqq \cdots$, we see that some S_q-subgroup of $\mathfrak{G}$ dominates $\Omega_1(\mathfrak{P})$ and is normalized by $\mathfrak{P}$. It follows that the normalizer of every S_q-subgroup of $\mathfrak{M}$ dominates $\Omega_1(\mathfrak{P}^M)$ for some M in $\mathfrak{M}$, and so $\mathfrak{M}$ dominates $\Omega_1(\mathfrak{P})$. (iv) is proved.

Notice that if $\boldsymbol{C}(\Omega_1(\mathfrak{P})) \cap \boldsymbol{H}(\mathfrak{M}) \neq 1$, then by (iv), elements of $\mathfrak{P}$ are conjugate in $\mathfrak{C}$ if and only if they are conjugate in $\mathfrak{M}$. Thus, in the case, it only remains to prove (ii). We emphasize that in any case (i) and (iv) are proved.

Since $\mathfrak{P} \subseteq \mathfrak{M}'$, if $\mathfrak{M} \in \mathscr{M}_0$, then $\mathfrak{P} \triangleleft \mathfrak{M}$ and the lemma follows. We can suppose that $\mathfrak{M} \in \mathscr{M}_1$. Let $q \in \pi(\boldsymbol{H}_1(\mathfrak{M}))$ and let $\mathfrak{Q}$ be a $\mathfrak{P}$-invariant S_q-subgroup of $\mathfrak{M}$. If $\Omega_1(\mathfrak{P})$ centralizes $\boldsymbol{T}(\mathfrak{Q})$, then (ii) follows immediately. Thus, we can choose P in $\Omega_1(\mathfrak{P})^\#$ such that $\Omega_1(\mathfrak{P})$ does not centralize $\boldsymbol{T}(\mathfrak{Q}) \cap \boldsymbol{C}(\mathfrak{P}) = \mathfrak{Q}_1$. If $\mathfrak{Q}_1 \in \mathscr{X}_1$, then $\boldsymbol{C}(P) \subseteq \mathfrak{M}$, so that (ii) holds. If $\mathfrak{Q}_1 \in \mathscr{X}_0$, then $\mathfrak{Q}_1$ is cyclic, by Lemma 8.16, and the containment $\mathfrak{P} \subseteq \mathfrak{M}'$. Hence $\Omega_1(\mathfrak{P}) = \langle P \rangle \times \mathfrak{P}_0$, where $\mathfrak{P}_0\mathfrak{Q}_1$ is a Frobenius group.

Let $\mathfrak{C} = \boldsymbol{C}(P)$. If $\mathfrak{C}'$ is nilpotent, then $\mathfrak{Q}_1 \subseteq \boldsymbol{O}_q(\mathfrak{C})$, so by Lemma 26.7, $\mathfrak{C} \subseteq \mathfrak{M}$, and (ii) follows. Suppose $\mathfrak{C}'$ is not nilpotent. Hence, $\mathfrak{C}$ contains an elementary subgroup of order r^3 for some prime r. If $r \in \pi(\boldsymbol{H}_1(\mathfrak{M}))$ then $\mathfrak{C} \subseteq \mathfrak{M}^G$ for some G in $\mathfrak{G}$. Since $\mathfrak{M} \cap \mathfrak{M}^G \supseteq \mathfrak{Q}_1$, we have $\mathfrak{M} = \mathfrak{M}^G$ and (ii) follows. Suppose $r \notin \pi(\boldsymbol{H}_1(\mathfrak{M}))$. In this case, $\Omega_1(\mathfrak{P})\mathfrak{Q}_1$ normalizes a S_r-subgroup $\mathfrak{R}$ of $\mathfrak{C}$. Since P centralizes $\mathfrak{R}$ and $\mathfrak{P}_0\mathfrak{Q}_1$ is a Frobenius group, and since $q \not\sim r$, it follows that $\mathfrak{R} \cap \boldsymbol{C}(\Omega_1(\mathfrak{P})) \neq 1$. Let $\mathfrak{M}_1 = \boldsymbol{M}(\mathfrak{C})$. By (iv) applied to $\mathfrak{M}_1$, we get $\mathfrak{P} \subseteq \mathfrak{M}_1'$. Since $\mathfrak{Q}_1 \cap \boldsymbol{H}(\mathfrak{M}_1) = 1$, and since the derived group of $\mathfrak{M}_1/\boldsymbol{H}(\mathfrak{M}_1)$ is nilpotent, $\mathfrak{P}$ centralizes $\mathfrak{Q}_1$, which is a contradiction. Hence, $\boldsymbol{C}(P) \subseteq \mathfrak{M}$, and (ii) holds. The lemma is proved in case $\boldsymbol{C}(\Omega_1(\mathfrak{P})) \cap \boldsymbol{H}(\mathfrak{M}) \neq 1$, and (i) is proved in all cases.

Throughout the remainder of the proof, we assume

$$\boldsymbol{C}(\Omega_1(\mathfrak{P})) \cap \boldsymbol{H}(\mathfrak{M}) \quad 1 \tag{26.1}$$

Suppose $\mathfrak{P}_0$ is a non identity subgroup of $\mathfrak{P}$ and

$$\boldsymbol{C}(\mathfrak{P}_0) \cap \boldsymbol{H}(\mathfrak{M}) \neq 1 \,. \tag{26.2}$$

There are three cases:

(a) $\mathfrak{M} \in \mathscr{M}_1$ and $\boldsymbol{C}(\mathfrak{P}_0) \cap \hat{\boldsymbol{H}}(\mathfrak{M}) \neq \emptyset$.

(b) $\mathfrak{M} \in \mathscr{M}_1$ and $\boldsymbol{C}(\mathfrak{P}_0) \cap \hat{\boldsymbol{H}}(\mathfrak{M}) = \emptyset$

(c) $\mathfrak{M} \in \mathscr{M}_0$.

In each of these cases, we will show that

$$\boldsymbol{N}(\mathfrak{P}_0) \subseteq \mathfrak{M} \tag{26.3}$$

Case a_1. $\boldsymbol{N}(\mathfrak{P}_0)'$ is nilpotent.

Choose q so that $\boldsymbol{C}(\mathfrak{P}_0) \cap \hat{\boldsymbol{H}}(\mathfrak{M})$ contains an element of order q, and let $\mathfrak{Q}_0$ be a $\mathfrak{P}$-invariant S_q-subgroup of $\boldsymbol{C}(\mathfrak{P}_0) \cap \boldsymbol{H}(\mathfrak{M})$. By (26.1), $\mathfrak{Q}_0 \subseteq \boldsymbol{N}(\mathfrak{P}_0)'$, so $\mathfrak{Q}_0 \subseteq \boldsymbol{O}_q(\boldsymbol{N}(\mathfrak{P}_0))$. If $q \in \pi(\boldsymbol{H}_1(\mathfrak{M}))$, we conclude that $\boldsymbol{N}(\boldsymbol{O}_q(\boldsymbol{N}(\mathfrak{P}_0))) \subseteq \mathfrak{M}$, by Lemma 26.7. If $q \in \pi(\boldsymbol{E}_1(\mathfrak{M}))$, then $\boldsymbol{O}_q(\boldsymbol{N}(\mathfrak{P}_0))$ centralizes $\boldsymbol{H}_1(\mathfrak{M})^G$ for some G in $\mathfrak{G}$, and so $\boldsymbol{N}(\dot{\mathfrak{Q}}_0) \supseteq \langle \boldsymbol{H}_1(\mathfrak{M}), \boldsymbol{H}_1(\mathfrak{M})^G \rangle$, and $G \in \mathfrak{M}$ follows.

Case a_2. $\boldsymbol{N}(\mathfrak{P}_0)'$ is not nilpotent.

In this case, $\boldsymbol{N}(\mathfrak{P}_0)$ contains an elementary subgroup of order r^3

for some prime r. If $r \in \pi(\boldsymbol{H}(\mathfrak{M}))$, then $\boldsymbol{M}(\boldsymbol{N}(\mathfrak{P}_0)) = \mathfrak{M}^G$, for some G in $\mathfrak{G}$. Since $\mathfrak{M}^G \cap \hat{\boldsymbol{H}}(\mathfrak{M}) \neq \emptyset$, we have $\mathfrak{M} = \mathfrak{M}^G$. If $r \notin \pi(\boldsymbol{H}(\mathfrak{M}))$, let $\mathfrak{R}$ be a S_r-subgroup of $\boldsymbol{N}(\mathfrak{P}_0)$ normalized by $\Omega_1(\mathfrak{P})\mathfrak{Q}_0$, where $\mathfrak{Q}_0$ is a non identity S_q-subgroup of $\boldsymbol{C}(\mathfrak{P}_0) \cap \boldsymbol{H}(\mathfrak{M})$, as in Case a_1. Let $\Omega_1(\mathfrak{P}) = \Omega_1(\mathfrak{P}_0) \times \mathfrak{P}_1$ so that $\mathfrak{Q}_0\mathfrak{P}_1$ is a Frobenius group by (26.1). If $\mathfrak{P}_1\mathfrak{R}$ is a Frobenius group, then $\mathfrak{Q}_0$ centralizes $\mathfrak{R}$, and $\mathfrak{R} \subseteq \mathfrak{M}$. This is not the case, since $r \not\sim r_1$ for all $r_1 \in \pi(\boldsymbol{H}_1(\mathfrak{M}))$. Hence, $\mathfrak{P}_1$ has a fixed point on $\mathfrak{R}^\#$, so $\Omega_1(\mathfrak{P})$ has a fixed point on $\boldsymbol{H}(\boldsymbol{M}(\mathfrak{R}))$. By (iv) applied to $\boldsymbol{M}(\mathfrak{R})$, it follows that $\Omega_1(\mathfrak{P}) \subseteq \boldsymbol{M}(\mathfrak{R})'$, and so $\Omega_1(\mathfrak{P})$ centralizes $\mathfrak{Q}_0$, which is not the case. Thus (26.3) holds in case (a).

In analysing case (b), we use the fact that $\boldsymbol{E}_1(\mathfrak{M})^\# \subseteq \hat{\boldsymbol{H}}(\mathfrak{M})$, and that if $\mathfrak{B}$ is any subgroup of $\boldsymbol{H}(\mathfrak{M})$ which is disjoint from $\hat{\boldsymbol{H}}(\mathfrak{M})$, then $\mathfrak{B}$ is of square free order and $q \in \pi_0 \cap \pi^*$ for every q in $\pi(\mathfrak{B})$.

Let $\mathfrak{Q}$ be a non identity $\mathfrak{P}$-invariant S_q-subgroup of $\boldsymbol{C}(\mathfrak{P}_0) \cap \boldsymbol{H}(\mathfrak{M})$. so that $|\mathfrak{Q}| = q$. Suppose that (26.3) does not hold.

We will show that $\mathfrak{P}\mathfrak{Q}$ is contained in a maximal subgroup $\mathfrak{M}_1$ of $\mathfrak{G}$ such that $\mathfrak{M}_1'$ is not nilpotent, and such that $\mathfrak{M}_1$ is not conjugate to $\mathfrak{M}$.

Case b_1. $\boldsymbol{N}(\mathfrak{P}_0) \subseteq \mathfrak{M}^G$ for some G in $\mathfrak{G}$.

Consider $\boldsymbol{N}(\mathfrak{Q})$. Since $\boldsymbol{N}(\mathfrak{Q}) \cap \mathfrak{M}$ and $\boldsymbol{N}(\mathfrak{Q}) \cap \mathfrak{M}^G$ have non cyclic S_q-subgroups, and since $\mathfrak{M} \neq \mathfrak{M}^G$, it follows that $\boldsymbol{N}(\mathfrak{Q})$ is contained in no conjugate of $\mathfrak{M}$. Let $\mathfrak{Q}_1$ be a $\mathfrak{P}$-invariant S_q-subgroup of $\boldsymbol{N}(\mathfrak{Q}) \cap \boldsymbol{H}(\mathfrak{M})$. If $\boldsymbol{N}(\mathfrak{Q})'$ is nilpotent, then $\mathfrak{Q}_1 \subseteq \boldsymbol{O}_q(\boldsymbol{N}(\mathfrak{Q}))$, and so $\boldsymbol{N}(\mathfrak{Q}) \subseteq \mathfrak{M}$ by Lemma 26.7. This is not the case, since $\boldsymbol{N}(\mathfrak{Q}) \cap \mathfrak{M}^G$ has non cyclic S_q-subgroups. Hence, $\boldsymbol{N}(\mathfrak{Q})'$ is not nilpotent, so we take $\mathfrak{M}_1 = \boldsymbol{M}(\boldsymbol{N}(\mathfrak{Q}))$.

Case b_2. $\boldsymbol{N}(\mathfrak{P}_0)'$ is nilpotent, but $\boldsymbol{N}(\mathfrak{P}_0)$ is not contained in any conjugate of $\mathfrak{M}$.

Since $\mathfrak{Q} \subseteq \boldsymbol{N}(\mathfrak{P}_0)'$, $\mathfrak{Q} \subseteq \boldsymbol{O}_q(\boldsymbol{N}(\mathfrak{P}_0))$. If $\boldsymbol{O}_q(\boldsymbol{N}(\mathfrak{P}_0))$ is not of order q, then $\boldsymbol{N}(\mathfrak{P}_0) \subseteq \mathfrak{M}^G$ for some G in $\mathfrak{G}$. Suppose that $\mathfrak{Q} = \boldsymbol{O}_q(\boldsymbol{N}(\mathfrak{P}_0))$ is of order q. Let $\mathfrak{N}_1 = \boldsymbol{N}(\mathfrak{Q})$, so that $\mathfrak{N}_1 \cap \mathfrak{M}$ has non cyclic S_q-subgroups and $\boldsymbol{N}(\mathfrak{P}_0) \subseteq \mathfrak{N}_1$. Since $\boldsymbol{N}(\mathfrak{P}_0)$ is contained in no conjugate of $\mathfrak{M}$, neither is $\mathfrak{N}_1$. If $\mathfrak{N}_1'$ is nilpotent, then a S_q-subgroup of $\mathfrak{N}_1 \cap \mathfrak{M}$ is contained in $\boldsymbol{O}_q(\mathfrak{N}_1)$, by (26.1) and so $\mathfrak{N}_1 = \mathfrak{M}$, which is not the case.

We apply (iv) to $\mathfrak{M}_1$. If $\boldsymbol{C}(\Omega_1(\mathfrak{P})) \cap \boldsymbol{H}(\mathfrak{M}_1) \neq 1$, then $\mathfrak{P} \subseteq \mathfrak{M}_1'$, so that $\mathfrak{P}$ centralizes $\mathfrak{Q}$, which is not the case. Hence, (26.1) holds with $\mathfrak{M}_1$ replacing $\mathfrak{M}$. Let $\mathfrak{P}_1$ be any subgroup of $\mathfrak{P}$ of order p different from $\Omega_1(\mathfrak{P}_0)$. Then $\mathfrak{P}_1\mathfrak{Q}$ is a Frobenius group. Choose $r \in \pi(\boldsymbol{H}_1(\mathfrak{M}_1))$ and let $\mathfrak{R}$ be a S_r-subgroup of $\mathfrak{M}_1$ invariant under $\mathfrak{P}\mathfrak{Q}$. If $\mathfrak{Q}$ does not centralize $\boldsymbol{T}(\mathfrak{R})$, then $\boldsymbol{C}(\mathfrak{P}_1) \cap \boldsymbol{T}(\mathfrak{R}) \neq 1$, so that case (a) holds with $\mathfrak{M}_1$ replacing $\mathfrak{M}$, $\mathfrak{P}_1$ replacing $\mathfrak{P}_0$.

Suppose then that $\mathfrak{Q}$ centralizes $\boldsymbol{T}(\mathfrak{R})$. Then $\boldsymbol{N}(\mathfrak{Q}) \subseteq \mathfrak{M}_1$, so a S_q-subgroup $\mathfrak{Q}_1$ of $\boldsymbol{N}(\mathfrak{Q}) \cap \mathfrak{M}$ is contained in $\mathfrak{M}_1$. We suppose without loss of generality that $\mathfrak{Q}_1$ normalizes $\mathfrak{R}$. If now $\mathfrak{P}_2$ is any subgroup of $\mathfrak{P}$ of order p which does not centralize $\mathfrak{Q}_1/\mathfrak{Q}$, then since $\mathfrak{Q}_1$ does not centralize $\boldsymbol{T}(\mathfrak{R})$, we conclude that $\boldsymbol{C}(\mathfrak{P}_2) \cap \boldsymbol{T}(\mathfrak{R}) \neq 1$.

Thus, in all cases, if $\mathfrak{P}_1^*, \mathfrak{P}_2^*, \cdots, \mathfrak{P}_n^*$ are the distinct subgroups of $\mathfrak{P}$ of order p which have fixed points on $\hat{\boldsymbol{H}}(\mathfrak{M}_1)$, then $n \geqq p$, so that $n = p$ or $p + 1$.

Choose $N \in \boldsymbol{N}(\Omega_1(\mathfrak{P}))$. Then there are indices i, j, not necessarily distinct, such that $\mathfrak{P}_i^* = \mathfrak{P}_j^{*N}$. If $i = j$, then $N \in \mathfrak{M}_1$, by (a). If $i \neq j$, then $\boldsymbol{N}(\mathfrak{P}_i^*) \subseteqq \mathfrak{M}_1 \cap \mathfrak{M}_1^N$, so that $\hat{\boldsymbol{H}}(\mathfrak{M}_1) \cap \mathfrak{M}_1^N \neq \emptyset$ and $\mathfrak{M}_1 = \mathfrak{M}_1^N$. Hence, $\boldsymbol{N}(\Omega_1(\mathfrak{P})) \subseteqq \mathfrak{M}_1$, so $\Omega_1(\mathfrak{P}) \subseteqq \mathfrak{M}_1'$, and $\Omega_1(\mathfrak{P})$ centralizes $\mathfrak{Q}$, which is not the case. Hence, (b) implies (26.3).

We will now complete the proof of this lemma in case $\mathfrak{M} \in \mathscr{M}_1$.

Since some element of $\Omega_1(\mathfrak{P})^\#$ has a fixed point on $\hat{\boldsymbol{H}}(\mathfrak{M})$, (ii) holds by (26.3). Also, by (26.3), alternative (v)b holds. It remains to prove (iii). Suppose P_1, P_2 are elements of $\mathfrak{P}$ which are conjugate in $\mathfrak{G}$, but are not conjugate in $\mathfrak{M}$, and that $\boldsymbol{C}(P_i) \cap \boldsymbol{H}(\mathfrak{M}) \neq 1$, $i = 1, 2$. Theorem 17.1 is violated.

We next verify (26.3) under hypothesis (c).

Suppose by way of contradiction that (26.3) does not hold. Let $\mathfrak{Q}$ be a non identity $\mathfrak{P}$-invariant S_q-subgroup of $\boldsymbol{C}(\mathfrak{P}_0) \cap \boldsymbol{H}(\mathfrak{M})$. We will produce a subgroup $\mathfrak{K}$ of $\mathfrak{G}$ such that $\mathfrak{K}'$ is not nilpotent, and such that $\mathfrak{Q}\mathfrak{P} \subseteqq \mathfrak{K}$. Once this is done, then it will follow as in case b_2 that p of the $p + 1$ subgroups of $\mathfrak{P}$ of order p have fixed points on $\boldsymbol{H}(\boldsymbol{M}(\mathfrak{K}))^\#$, and (26.3) will follow.

Suppose $\mathfrak{M}_1$ is a maximal subgroup of $\mathfrak{G}$ containing $\boldsymbol{N}(\mathfrak{P}_0)$. If $\mathfrak{M}_1'$ is nilpotent, then $\mathfrak{Q} \subseteqq \boldsymbol{O}_q(\mathfrak{M}_1)$. If $\boldsymbol{O}_q(\mathfrak{M}_1)$ is non abelian, then $\mathfrak{M}_1 = \mathfrak{M}^G$ for some G in $\mathfrak{G}$. Furthermore, from (26.1) and the fact that $\mathfrak{Q}$ is not a S_q-subgroup of $\mathfrak{G}$, we conclude that $\mathfrak{Q} = \boldsymbol{O}_q(\mathfrak{M}_1) \cap \boldsymbol{C}(\mathfrak{P}_0)$. Hence, $\boldsymbol{N}(\mathfrak{Q})$ contains $\boldsymbol{C}(\mathfrak{P}_0)$. Let $\mathfrak{M}_2$ be a maximal subgroup of $\mathfrak{G}$ containing $\boldsymbol{N}(\mathfrak{Q})$. If $\mathfrak{M}_2'$ is nilpotent, then $\mathfrak{M}_2 = \mathfrak{M}$ and (26.3) holds. Hence, $\mathfrak{M}_2'$ is not nilpotent, so we can take $\mathfrak{K} = \mathfrak{M}_2$. If $\boldsymbol{O}_q(\mathfrak{M}_1)$ is abelian, then $\mathfrak{M} = \mathfrak{M}_1$ and (26.3) holds. Thus, (26.3) holds in all cases.

The completion of the proof that (26.3) implies this lemma is a straightforward application of Theorem 17.1.

LEMMA 26.14. *Suppose* $\mathfrak{M} \in \mathscr{M}$ *and* $\mathfrak{P}$ *is a non abelian* S_p*-subgroup of* $\mathfrak{M}$. *Then* $\boldsymbol{N}(\Omega_1(\boldsymbol{Z}(\mathfrak{P}))) \subseteqq \mathfrak{M}$. *Furthermore, one of the following conditions is true*:

(a) $\Omega_1(\boldsymbol{Z}(\mathfrak{P}))$ *centralizes* $\boldsymbol{H}(\mathfrak{M})$.

(b) $\boldsymbol{N}(\mathfrak{P}_0) \subseteqq \mathfrak{M}$ *for every non identity subgroup* $\mathfrak{P}_0$ *of* $\mathfrak{P}$.

(c) $\mathfrak{P} \subseteqq \boldsymbol{H}(\mathfrak{M})$.

Proof. Suppose $p \in \pi_0$. If $\mathfrak{P} \in \mathscr{X}_1$, then $\mathfrak{M} = \boldsymbol{M}(\mathfrak{P})$, and so

$N(\Omega_1(\boldsymbol{Z}(\mathfrak{P}))) \subseteq \mathfrak{M}$. Since $\mathfrak{P} \subseteq \boldsymbol{H}(\mathfrak{M})$, the lemma is proved. If $\mathfrak{P} \in \mathscr{X}_0$ then $\mathfrak{P}$ contains a cyclic subgroup of index p. Since $\mathfrak{P}$ is assumed to be non abelian, $\mathfrak{P}$ is a non abelian metacyclic group, so $\mathfrak{P} \not\subseteq \mathfrak{M}'$, by 3.8. Lemma 26.10 is violated.

Through the remainder of the proof, we assume $p \in \pi_2$.

Let $\mathfrak{Z} = \Omega_1(\boldsymbol{Z}(\mathfrak{P}))$, so that $\mathfrak{Z}$ is of order p, by Lemma 26.2 and Lemma 26.10.

If $\mathfrak{M}'$ is nilpotent, then $\mathfrak{Z} \lhd \mathfrak{M}$, and all parts of the lemma follow. We can suppose that $\mathfrak{M}'$ is not nilpotent. In particular, $\mathfrak{M} \in \mathscr{M}_1$. We can further assume that $p \notin \pi(\boldsymbol{H}(\mathfrak{M}))$.

Since $\mathfrak{P}$ is non abelian, $\mathfrak{Z}$ centralizes $\boldsymbol{E}_1(\mathfrak{M})$.

Choose $q \in \pi(\boldsymbol{H}_1(\mathfrak{M}))$ and let $\mathfrak{Q}$ be a $\mathfrak{P}$-invariant S_q-subgroup of $\mathfrak{M}$. If $q \in \pi^*$, then $\mathfrak{Z}$ centralizes $\mathfrak{Q}$.

Thus, if $\tilde{\pi} = \pi(\boldsymbol{E}_1(\mathfrak{M})) \cup (\pi^* \cap \pi(\boldsymbol{H}_1(\mathfrak{M})))$, then $\mathfrak{Z}$ centralizes a $S_{\tilde{\pi}}$-subgroup of $\mathfrak{M}$. If $\tilde{\pi} = \pi(\boldsymbol{H}(\mathfrak{M}))$, all parts of the lemma follow.

Let $r \in \pi(\boldsymbol{H}(\mathfrak{M})) - \tilde{\pi}$ and let $\mathfrak{R}$ be a S_r-subgroup of $\mathfrak{M}$ normalized by $\mathfrak{P}$, and such that $\mathfrak{Z}$ does not centralize $\mathfrak{R}$. If there are no such primes r, we are done.

Let $\mathfrak{P}_1$ be any subgroup of $\mathfrak{P}$ of order p different from $\mathfrak{Z}$. We will show that $N(\mathfrak{P}_1) \subseteq \mathfrak{M}$.

Since $\mathfrak{Z}$ does not centralize $\mathfrak{R}$, $\mathfrak{R} \cap \boldsymbol{C}(\mathfrak{P}_1) \not\subseteq \boldsymbol{C}(\mathfrak{Z})$. Set $\mathfrak{R}_1 = \mathfrak{R} \cap \boldsymbol{C}(\mathfrak{P}_1)$. If $\mathfrak{R}_1 \in \mathscr{X}_1$, then $N(\mathfrak{P}_1) \subseteq \mathfrak{M}$. Otherwise, $\mathfrak{R}_1$ is a non trivial cyclic subgroup of $\mathfrak{R}$, and $\mathfrak{Z}\mathfrak{R}_1$ is a Frobenius group.

Let $\mathfrak{M}_1$ be a maximal subgroup of $\mathfrak{G}$ containing $N(\mathfrak{P}_1)$. If $\mathfrak{M}_1'$ is nilpotent, then $\mathfrak{R}_1 \subseteq \boldsymbol{O}_r(\mathfrak{M}_1)$, so $\mathfrak{M}_1 \subseteq \mathfrak{M}$, by Lemma 26.6. We can suppose that $\mathfrak{M}_1'$ is not nilpotent and that $\mathfrak{M}_1$ is not conjugate to $\mathfrak{M}$. If a S_p-subgroup of $\mathfrak{M}_1$ is non abelian, then $\mathfrak{Z}$ centralizes $\mathfrak{R}_1$, which is not the case. Hence, a S_p-subgroup of $\mathfrak{M}_1$ is abelian and non cyclic. We can apply Lemma 26.12 to $\mathfrak{M}_1$ and a S_p-subgroup $\mathfrak{P}^*$ of $\mathfrak{M}_1$ which contains $\mathfrak{P}_1\mathfrak{Z}$. We conclude that $\mathfrak{Z}\boldsymbol{H}(\mathfrak{M}_1)$ is a Frobenius group. Since $\mathfrak{Z}\mathfrak{R}_1$ is a Frobenius group, $\mathfrak{R}_1$ centralizes $\boldsymbol{H}(\mathfrak{M}_1)$, and so $\mathfrak{M} = \mathfrak{M}_1$. We conclude that $\mathfrak{M}$ contains $N(\mathfrak{P}_1)$ in all cases.

Now let $\mathfrak{P}_1, \cdots, \mathfrak{P}_n$ be the distinct subgroups of $\mathfrak{P}$ of order p different from $\mathfrak{Z}$. Here $n = p^2 + p$. Let $\mathfrak{L}$ be any proper subgroup of $\mathfrak{G}$ containing $\Omega_1(\mathfrak{P})$. Let $\mathfrak{L}_1 = \boldsymbol{O}_{p'}(\mathfrak{L})$. Since $\mathfrak{L}_1$ is generated by its subgroups $\boldsymbol{C}(\mathfrak{P}_i) \cap \mathfrak{L}_1$, $1 \leq i \leq n$, we have $\mathfrak{L}_1 \subseteq \mathfrak{M}$. Let $\mathfrak{L}_2 = \mathfrak{L} \cap N(\Omega_1(\mathfrak{P}))$, and choose L in $\mathfrak{L}_2$. We can then find indices i, j, not necessarily distinct, such that $\mathfrak{P}_i^L = \mathfrak{P}_j$. Hence, $N(\mathfrak{P}_j) \subseteq \mathfrak{M} \cap \mathfrak{M}^L$. Since $N(\mathfrak{P}_j)$ contains an element of $\mathfrak{R}^* \subseteq \hat{\boldsymbol{H}}(\mathfrak{M})$, we have $\mathfrak{M} = \mathfrak{M}^L$. Hence, $\mathfrak{L} \subseteq \mathfrak{M}$, so in particular, $N(\mathfrak{Z}) \subseteq \mathfrak{M}$.

Let $\mathfrak{P}_0$ be any non identity subgroup of $\mathfrak{P}$. If $\mathfrak{P}_0$ is non cyclic, then $N(\mathfrak{P}_0) \subseteq N(\mathfrak{Z}) \subseteq \mathfrak{M}$. If $\mathfrak{P}_0$ is cyclic, then $N(\Omega_1(\mathfrak{P}_0)) \subseteq \mathfrak{M}$. The proof is complete.

LEMMA 26.15. *Suppose* $\mathfrak{M} \in \mathscr{M}$, $\mathfrak{A}$ *is a cyclic S-subgroup of* $\mathfrak{M}$ *and* $\mathfrak{A} \cap \mathfrak{M}' = 1$. *Then* $\mathfrak{A}$ *is prime on* $\boldsymbol{H}(\mathfrak{M})$, *and* $C(\mathfrak{A}) \cap \boldsymbol{H}(\mathfrak{M})$ *is a Z-group.*

Proof. Suppose $\mathfrak{A}$ is prime on $\boldsymbol{H}(\mathfrak{M})$, but that $\mathfrak{Q}$ is a non cyclic S_q-subgroup of $C(\mathfrak{A}) \cap \boldsymbol{H}(\mathfrak{M})$. Choose $p \in \pi(\mathfrak{A})$ and let $\mathfrak{A}_p$ be the S_p-subgroup of $\mathfrak{A}$. Since $N(\mathfrak{A}_p) \not\subseteq \mathfrak{M}$, it follows that $\mathfrak{Q} \in \mathscr{X}_0$. Thus, if $q \in \pi_2$, $\mathfrak{Q}$ is a S_q-subgroup of $\mathfrak{G}$, while if $q \in \pi_0$, $\mathfrak{Q}$ is also a S_q-subgroup of $\mathfrak{G}$, by Lemma 8.12. Since $\mathfrak{Q} \in \mathscr{X}_0$, we have $q \in \pi_2$, so that $\mathfrak{M} = N(\mathfrak{Q})$.

Let $\mathfrak{M}_1$ be a maximal subgroup of $\mathfrak{G}$ containing $N(\mathfrak{A}_p)$. If a S_p-subgroup of $\mathfrak{G}$ is cyclic, then $\mathfrak{M} = N(\mathfrak{Q})$ dominates $\mathfrak{A}_p$, which is not the case, since $\mathfrak{A}_p \cap \mathfrak{M}' = 1$. Hence, $p \in \pi_0 \cup \pi_2$. Let $\mathfrak{A}_p^*$ be a S_p-subgroup of $\mathfrak{M}_1$ permutable with $\mathfrak{Q}$. If $\mathfrak{A}_p^*$ is a S_p-subgroup of $\mathfrak{G}$, then $\mathfrak{Q}$ normalizes $\mathfrak{A}_p^*$. Otherwise, $\mathfrak{Q}$ normalizes $\mathfrak{A}_p^*$ since $\mathfrak{A}_p \subset \mathfrak{A}_p^*$, and Lemma 8.5 applies to $\mathfrak{Q}\mathfrak{A}_p^*$.

Let $\mathfrak{K}$ be a maximal p, q-subgroup of $\mathfrak{G}$ containing $\mathfrak{Q}\mathfrak{A}_p^*$, and let $\mathfrak{K}_p$ be a S_p-subgroup of $\mathfrak{K}$. Then $\mathfrak{K}_p \lhd \mathfrak{K}$, so that $\mathfrak{K}_p$ is a S_p-subgroup of $\mathfrak{G}$. Let $\mathfrak{M}_2$ be a maximal subgroup of $\mathfrak{G}$ containing $N(\mathfrak{K}_p)$.

If $\mathfrak{Q}$ were non abelian, then $\mathfrak{M} \subseteq \mathfrak{M}_2$ by Lemma 26.14, which is not the case. Hence, $\mathfrak{Q}$ is abelian. If $p \in \pi_0$, then by Lemma 26.13, we have $N(\Omega_1(\mathfrak{Q})) \subseteq \mathfrak{M}_2$ since $\mathfrak{Q}$ centralizes $\mathfrak{A}_p \neq 1$. Since this is impossible, we see that $p \in \pi_2$.

If $\mathfrak{A}_p \not\subseteq \mathfrak{K}_p'$, then by Lemma 26.1, together with the fact that $N(\mathfrak{Q})$ covers $N(\mathfrak{K}_p)/\mathfrak{K}_p C(\mathfrak{K}_p)$, we see that $\mathfrak{A}_p \cap \mathfrak{M}' \neq 1$, contrary to hypothesis. Hence, $\mathfrak{A}_p \subseteq \mathfrak{K}_p'$. Since $\mathfrak{A}_p = C(\mathfrak{Q}) \cap \mathfrak{K}_p$, this implies that $\mathfrak{K}_p$ is a non abelian group of order p^3 and exponent p.

Since some element of $\mathfrak{Q}^\#$ has a non identity fixed point on $\boldsymbol{H}(\mathfrak{M}_2)^\#$, and since $\mathfrak{M}'$ centralizes $\mathfrak{Q}$, we see that $\mathfrak{M}' \subseteq \mathfrak{M}_2$, by Lemma 26.13. Since $N(\mathfrak{A}_p) \subseteq \mathfrak{M}_2$ and since $\mathfrak{A}_p \cap \mathfrak{M}' = 1$, it follows that $\mathfrak{M} \subseteq \mathfrak{M}_2$, the desired contradiction.

Thus, in proving this lemma, it suffices to show that $\mathfrak{A}$ is prime on $\boldsymbol{H}(\mathfrak{M})$.

First, suppose that $\mathfrak{A}$ is a p-group for some prime p. We can clearly suppose that $|\mathfrak{A}| \geqq p^2$, and that $C(\Omega_1(\mathfrak{A})) \cap \boldsymbol{H}(\mathfrak{M}) \neq 1$.

Case 1. $p \in \pi_0$. Let $q \in \pi(\boldsymbol{E}_1(\mathfrak{M}))$, so that $q \in \pi_1 \cup \pi_2$. Lemma 26.9 applies. Let $q \in \pi(\boldsymbol{H}_1(\mathfrak{M}))$. Then $p \not\sim q$ since $\mathfrak{A} \cap \mathfrak{M}' = 1$. Lemma 26.10 applies. If $\mathfrak{M} \in \mathscr{M}_0$, Lemma 26.9 applies.

Case 2. $p \in \pi_2$ and a S_p-subgroup of $\mathfrak{G}$ is abelian.

If $q \in \pi(\boldsymbol{E}_1(\mathfrak{M}))$, or $q \in \pi(\boldsymbol{H}(\mathfrak{M}))$ and $\mathfrak{M} \in \mathscr{M}_0$, Lemma 26.7 applies.

Let $q \in \pi(\boldsymbol{H}_1(\mathfrak{M}))$, and let $\mathfrak{Q}$ be an $\mathfrak{A}$-invariant S_q-subgroup of $\mathfrak{M}$. If $\mathfrak{A}$ centralizes $\mathfrak{Q}$, we have an immediate contradiction. Hence, $\mathfrak{A}$ does not centralize $\mathfrak{Q}$.

We can suppose by way of contradiction that $[C(\Omega_1(\mathfrak{A})) \cap \mathfrak{Q}, \mathfrak{A}] \neq 1$. If $C(\Omega_1(\mathfrak{A})) \cap \mathfrak{Q} \in \mathscr{X}_1$, $\mathfrak{M}$ contains a S_p-subgroup of $\mathfrak{G}$, which is not the case. Otherwise, $q > p$, so every p, q-subgroup of $\mathfrak{G}$ is q-closed, and $\mathfrak{M}$ contains a S_p-subgroup of $\mathfrak{G}$, which is not the case.

Case 3. $p \in \pi_2$ and a S_p-subgroup of $\mathfrak{G}$ is non abelian.

Here, $\mathfrak{A} \subseteq N(\Omega_1(\mathfrak{A}))'$, by Lemma 26.2. Since $C(\Omega_1(\mathfrak{A})) \cap H(\mathfrak{M}) \in \mathscr{H}_0$, the lemma follows.

Case 4. $p \in \pi_1$. In this case, also, we have $\mathfrak{A} \subseteq N(\Omega_1(\mathfrak{A}))'$, and the lemma follows.

Next, suppose that $\mathfrak{A} = \mathfrak{A}_1 \times \mathfrak{A}_2$, where $\mathfrak{A}_i$ is a non identity p_i-group, $i = 1, 2$. Suppose by way of contradiction that $\mathfrak{Q}$ is an $\mathfrak{A}$-invariant S_q-subgroup of $H(\mathfrak{M})$ and that $\mathfrak{A}$ is not prime on $\mathfrak{Q}$. We can suppose that $\mathfrak{A}_2$ does not centralize $\mathfrak{Q} \cap C(\Omega_1(\mathfrak{A}_1)) = \mathfrak{Q} \cap C(\mathfrak{A}_1) = \mathfrak{Q}_1$.

Let $\mathfrak{M}_1$ be a maximal subgroup of $\mathfrak{G}$ containing $N(\Omega_1(\mathfrak{A}_1))$. Then $\mathfrak{M}_1$ is not conjugate to $\mathfrak{M}$, either because $\mathfrak{A}_1$ is not a S-subgroup of $\mathfrak{M}_1$, or because $\mathfrak{A}_1 \subseteq \mathfrak{M}_1'$. Let $\mathfrak{Q}_2$ be a S_q-subgroup of $\mathfrak{M} \cap \mathfrak{M}_1$ which contains $\mathfrak{Q}_1$ and is $\mathfrak{A}$-invariant.

Suppose $\mathfrak{Q}_1 \subset \mathfrak{Q}_2$. Then $\mathfrak{A}_1 \not\subseteq H(\mathfrak{M}_1)$, since $[\mathfrak{Q}_2, \mathfrak{A}_1] \neq 1$, and $q \notin \pi(H(\mathfrak{M}_1))$. Furthermore, $\mathfrak{Q}_2$ is non cyclic. Suppose $q \in \pi_2$. In this case, $q > p_1$, so a S_{p_1}-subgroup $\mathfrak{A}_1^*$ of $\mathfrak{M}_1$ normalizes some S_q-subgroup of $\mathfrak{M}_1$, and it follows that $\mathfrak{A}_1^*$ normalizes some S_q-subgroup of $\mathfrak{G}$. This implies that $\mathfrak{A}_1$ is a S_{p_1}-subgroup of $\mathfrak{G}$. But in this case $\mathfrak{A}_1 \subseteq N(\Omega_1(\mathfrak{A}_1))'$ so that $\mathfrak{A}_1$ centralizes $\mathfrak{Q}_2$ and so $\mathfrak{Q}_1 = \mathfrak{Q}_2$. Suppose $q \in \pi_0$. If $\mathfrak{Q}_2 \in \mathscr{H}_1$, then $N(\Omega_1(\mathfrak{A}_1)) \subseteq \mathfrak{M}$, which is not the case. Hence, $\mathfrak{Q}_2 \in \mathscr{H}_0$ so that $q > p_1$. Once again we get that $\mathfrak{Q}_2 = \mathfrak{Q}_1$. Hence, we necessarily have $\mathfrak{Q}_2 = \mathfrak{Q}_1$ in all cases.

Since $\mathfrak{A}_1$ is prime on $H(\mathfrak{M})$, from the first part of the lemma, we conclude that $\mathfrak{Q}_1$ is cyclic.

We next assume that $\mathfrak{M}_1'$ is nilpotent.

Suppose $\Omega_1(O_q(\mathfrak{M}_1)) = \Omega_1(\mathfrak{Q}_1)$. Since $\mathfrak{Q}_1$ is a S_q-subgroup of $\mathfrak{M}_1 \cap \mathfrak{M}$, it follows that $q \in \pi_1$ and $\mathfrak{Q}_1$ is a S_q-subgroup of $\mathfrak{G}$, so that $\mathfrak{M} = \mathfrak{M}_1$. Since $\mathfrak{Q}_1 = [\mathfrak{Q}_1, \mathfrak{A}_2] \subseteq O_q(\mathfrak{M}_1)$, we can suppose that $O_q(\mathfrak{M}_1)$ is non cyclic. In this case, however, $O_q(\mathfrak{M}_1)$ is a S_q-subgroup of $\mathfrak{G}$ and $\mathfrak{M}_1$ is conjugate to $\mathfrak{M}$, which is not the case.

We can now suppose that $\mathfrak{M}_1'$ is not nilpotent.

Suppose $p_1 \notin \pi(H_1(\mathfrak{M}_1))$. Let $\mathfrak{E}$ be a complement for $H_1(\mathfrak{M}_1)$ in $\mathfrak{M}_1$ which contains $\mathfrak{Q}_1\mathfrak{A}$. Then $\mathfrak{E}'$ is nilpotent and so $[\mathfrak{Q}_1, \mathfrak{A}_2] \subseteq O_q(\mathfrak{E})$.

Case 1. $q \in \pi_1$. In this case, $\mathfrak{A}_1$ is a S_{p_1}-subgroup of $\mathfrak{G}$, and $\mathfrak{Q}_1$ dominates $\mathfrak{A}_1$. This violates $\mathfrak{A}_1 \cap \mathfrak{M}' = 1$.

Case 2. $q \in \pi_2$, and a S_q-subgroup of $\mathfrak{G}$ is abelian. In this case, $\Omega_1([\mathfrak{Q}_1, \mathfrak{A}_2]) = \Omega_1(O_q(\mathfrak{E}))$, so once again $\mathfrak{A}_1$ is a S_{p_1}-subgroup of $\mathfrak{G}$ and $\mathfrak{M}$ dominates $\mathfrak{A}_1$.

Case 3. $q \in \pi_2$ and a S_q-subgroup of $\mathfrak{G}$ is non abelian. Since $\mathfrak{Q}_1$ is cyclic, we have $q > p_1$, so some S_{p_1}-subgroup $\mathfrak{E}_{p_1}$ of $\mathfrak{E}$ normalizes some S_q-subgroup of $\mathfrak{G}$. But now $\mathfrak{M}$ dominates $\mathfrak{A}_1$ since every p_1, q-

subgroup of $\mathfrak{G}$ is q-closed, and $\mathfrak{E}$ dominates $\mathfrak{A}_1$.

Case 4. $q \in \pi_0$. If $q \in \pi^*$, then every p_1, q-subgroup of $\mathfrak{G}$ which contains a $S_{p_1,q}$-subgroup of $\mathfrak{M}_1$ is q-closed, so once again $\mathfrak{M}$ dominates $\mathfrak{A}_1$ and $\mathfrak{A}_1$ is a S_{p_1}-subgroup of $\mathfrak{G}$. Hence, $q \notin \pi^*$. Since $\mathfrak{M}_1$ is not conjugate to $\mathfrak{M}$, it follows that if $\mathfrak{Q}_3$ is a S_q-subgroup of $\mathfrak{E}$ containing $\mathfrak{Q}_1$, then $\mathfrak{Q}_3 \in \mathscr{X}_0$, which implies that $\mathfrak{Q}_3$ is cyclic, and $\mathfrak{Q}_3 \subseteq \mathfrak{M}$. Hence, $\mathfrak{Q}_1 = \mathfrak{Q}_3$, since $\mathfrak{A}_1$ centralizes $\mathfrak{Q}_3$. But now $\mathfrak{Q}_1 = [\mathfrak{Q}_1, \mathfrak{A}_2] \lhd \mathfrak{E}$, so $\mathfrak{E} \subseteq \mathfrak{M}$. Thus, once again $\mathfrak{A}_1$ is a S_{p_1}-subgroup of $\mathfrak{G}$ and $\mathfrak{M}$ dominates $\mathfrak{A}_1$.

All these possibilities have led to a contradiction. We now get to the heart of the matter. Suppose $p_1 \in \pi(\boldsymbol{H}_1(\mathfrak{M}_1))$.

We will show that $p_1 \notin \pi^*$.

Let $\mathfrak{P}_1$ be a S_{p_1}-subgroup of $\boldsymbol{H}_1(\mathfrak{M}_1)$ containing $\mathfrak{A}_1$ and invariant under $\mathfrak{A}_2\mathfrak{Q}_1$. Suppose that

$$N([\mathfrak{A}_2, \mathfrak{Q}_1]) \subseteq \mathfrak{M}_1 \tag{26.4}$$

We will derive a contradiction from the assumption that (26.4) holds.

If $q \in \pi_1$, (26.4) is an absurdity, since $N([\mathfrak{A}_2, \mathfrak{Q}_1]) = \mathfrak{M}$. If $q \in \pi_2 \cup \pi_0$, then a S_q-subgroup of $N([\mathfrak{A}_2, \mathfrak{Q}_1]) \cap \mathfrak{M}$ is non cyclic, so $q \in \pi_2$, as already remarked. If $q < p_1$, then $\mathfrak{A}_1$ centralizes a S_q-subgroup of $\mathfrak{M}$, so $\mathfrak{Q}_1$ is a S_q-subgroup of $\mathfrak{G}$. In this case, however, $[\mathfrak{A}_2, \mathfrak{Q}_1] \lhd \mathfrak{M}$, an absurdity, by (26.4). Thus, if (26.4) holds, then $q \in \pi_2$ and $q > p_1$.

Since (26.4) is assumed to hold, it follows that $\mathfrak{Q}_1$ is a S_q-subgroup of $\mathfrak{M} \cap N([\mathfrak{A}_2, \mathfrak{Q}_1])$. Hence, $\mathfrak{Q}_1$ is non cyclic. We have already shown that $\mathfrak{Q}_1$ is cyclic. We conclude that (26.4) does not hold.

If $p_1 \in \pi^*$, then $[\mathfrak{A}_2, \mathfrak{Q}_1]$ centralizes $\mathfrak{P}_1$, by Lemma 8.16 (ii), so (26.4) holds. Hence, $p_1 \notin \pi^*$.

Since (26.4) does not hold, and since $p_1 \notin \pi^*$, $C([\mathfrak{A}_2, \mathfrak{Q}_1]) \cap \mathfrak{P}_1$ is cyclic. It follows that $C(\mathfrak{A}_2) \cap \mathfrak{P}_1$ is non cyclic. This implies that $N(\mathfrak{A}_2) \subseteq \mathfrak{M}_1$, since $C(\mathfrak{A}_2) \cap \mathfrak{P}_1 \in \mathscr{X}_1$. Since $p_2 \notin \pi(\boldsymbol{H}(\mathfrak{M}_1))$, and since $q > p_2$, it follows that a $S_{p_2,q}$-subgroup of $\mathfrak{M}_1/\boldsymbol{H}(\mathfrak{M}_1)$ is q-closed. This in turn implies that some S_{p_2}-subgroup of $\mathfrak{M}_1$ normalizes some S_q-subgroup of $\mathfrak{G}$. Since $\mathfrak{A}_2$ is a S_{p_2}-subgroup of $\mathfrak{M}$, $\mathfrak{A}_2$ is forced to be a S_{p_2}-subgroup of $\mathfrak{G}$. But $N(\mathfrak{A}_2) \subseteq \mathfrak{M}_1$, and $\mathfrak{A}_2 \subseteq N(\mathfrak{A}_2)'$, so $\mathfrak{A}_2$ centralizes $\mathfrak{Q}_1$. The proof of the lemma is complete in case $\pi(\mathfrak{A}) = \{p_1, p_2\}$.

If $|\pi(\mathfrak{A})| \geqq 3$, the lemma follows immediately by applying the preceding result to all pairs of elements of $\pi(\mathfrak{A})$.

LEMMA 26.16. *Suppose $\mathfrak{M} \in \mathscr{M}$ and $\boldsymbol{H}(\mathfrak{M})$ is not nilpotent. Then $|\mathfrak{M} : \mathfrak{M}'|$ is a prime and $\mathfrak{M}'$ is a S-subgroup of $\mathfrak{M}$.*

Proof. Let $p \in \pi(\mathfrak{M}/\mathfrak{M}')$ and let $\mathfrak{A}_p$ be a S_p-subgroup of $\mathfrak{M}$. By Lemma 26.11, $\mathfrak{A}_p$ is abelian. Suppose $\mathfrak{A}_p$ is non cyclic. If a S_p-subgroup of $\mathfrak{G}$ is non abelian, then $\boldsymbol{H}(\mathfrak{M})$ is nilpotent, by Lemma 26.12. Hence, we can suppose that a S_p-subgroup of $\mathfrak{G}$ is abelian. By Lemma 26.13 $\mathfrak{A}_p$ is a S_p-subgroup of $\mathfrak{G}$. By Grün's theorem, the simplicity of $\mathfrak{G}$, and Lemma 26.15, $\mathfrak{A}_p$ contains elements A_1, A_2 which are conjugate in $\mathfrak{G}$ but are not conjugate in $\mathfrak{M}$. If $\Omega_1(\langle A_1\rangle) = \Omega_1(\langle A_2\rangle)$ and if $\Omega_1(\langle A_1\rangle)$ has a fixed point on $\boldsymbol{H}(\mathfrak{M})^{\sharp}$, then $N(\Omega_1(\langle A_1\rangle)) \subseteq \mathfrak{M}$, so that A_1 and A_2 are conjugate in $\mathfrak{M}$. Since this is not the case, $\Omega_1(\langle A_1\rangle)\boldsymbol{H}(\mathfrak{M})$ is a Frobenius group, and so $\boldsymbol{H}(\mathfrak{M})$ is nilpotent, contrary to assumption. Hence, $\Omega_1(\langle A_1\rangle) \neq \Omega_1(\langle A_2\rangle)$. By Lemma 26.13, either $\Omega_1(\langle A_1\rangle)\boldsymbol{H}(\mathfrak{M})$ or $\Omega_1(\langle A_2\rangle)\boldsymbol{H}(\mathfrak{M})$ is a Frobenius group, which is not the case. Hence, $\mathfrak{A}_p$ is cyclic.

Let $\mathfrak{A}$ be a complement to $\mathfrak{M}'$ in $\mathfrak{M}$, so that $\mathfrak{A}$ is a cyclic S-subgroup of $\mathfrak{M}$.

By Lemma 26.15, $\mathfrak{A}$ is prime on $\boldsymbol{H}(\mathfrak{M})$ and $\boldsymbol{C}(\mathfrak{A}) \cap \boldsymbol{H}(\mathfrak{M})$ is a Z-group.

Let $\mathfrak{K} = [\mathfrak{A}, \boldsymbol{H}(\mathfrak{M})]$ and suppose that $|\mathfrak{A}|$ is not a prime. By Lemma 26.3, $\mathfrak{K}$ is nilpotent. By 3.7, $\mathfrak{K} \triangleleft \boldsymbol{H}(\mathfrak{M})$. Hence $\boldsymbol{F}(\boldsymbol{H}(\mathfrak{M})) \supseteq \mathfrak{K}$, so that $\boldsymbol{H}(\mathfrak{M})/\boldsymbol{F}(\boldsymbol{H}(\mathfrak{M}))$ is a Z-group. It follows that $\boldsymbol{H}(\mathfrak{M}) \not\subseteq \mathfrak{M}'$, the desired contradiction.

LEMMA 26.17. *Suppose* $\mathfrak{M} \in \mathscr{M}$ *and* $\tau_1 = \pi(\boldsymbol{H}(\mathfrak{M})) \cap \pi^*, \tau_2 = \pi(\mathfrak{M}/\boldsymbol{H}(\mathfrak{M})) \cap \pi^*$. *Let* $\tau_1 = \{p_1, \cdots, p_n\}, p_1 > p_2 > \cdots > p_n$, *and* $\tau_2 = \{q_1, \cdots, q_m\}, q_1 > \cdots > q_m$. *Set* $\tau = \tau_1 \cup \tau_2$. *Then a* S_τ*-subgroup of* $\mathfrak{M}$ *has a Sylow series of complexion* $(p_1, \cdots, p_n, q_1, \cdots, q_m)$. *Furthermore, if* $r \in \tau$, $\mathfrak{M}$ *has r-length* 1.

Proof. We first show that $\mathfrak{M}$ has r-length 1 for each r in τ. If $r \notin \pi(\boldsymbol{H}_1(\mathfrak{M}))$, this is clear, so suppose $r \in \pi(\boldsymbol{H}_1(\mathfrak{M}))$. Let $\mathfrak{R}$ be a S_r-subgroup of $\mathfrak{M}$ and let $\mathfrak{A}$ be a subgroup of $\mathfrak{R}$ of order r such that $C_{\mathfrak{R}}(\mathfrak{A}) = \mathfrak{A} \times \mathfrak{B}$ where $\mathfrak{B}$ is cyclic.

Let $\mathfrak{R}_1 = \mathfrak{R} \cap \boldsymbol{O}_{r',r}(\mathfrak{M})$, and $\mathfrak{M}_1 = N(\mathfrak{R}_1)$. It suffices to show that $\mathfrak{M}_1$ has r-length one, since $\mathfrak{M} = \mathfrak{M}_1\boldsymbol{O}_{r'}(\mathfrak{M})$. Let $\mathfrak{B}$ be a subgroup of $\mathfrak{R}_1$ chosen in accordance with Lemma 8.2, and set $\mathfrak{W} = \Omega_1(\mathfrak{B})$. Then $\ker(\mathfrak{M}_1 \rightarrow \operatorname{Aut}\mathfrak{W}) \subseteq \mathfrak{M}_1 \cap \boldsymbol{O}_{r',r}(\mathfrak{M})$. If $\mathfrak{A} \subseteq \mathfrak{R}_1$, then $m(\mathfrak{W}) \leq 2$, and we are done. We can suppose that $\mathfrak{A} \not\subseteq \mathfrak{R}_1$. This implies that $m(\mathfrak{W}) \leq r$, since $\boldsymbol{C}(\mathfrak{A}) \cap \mathfrak{W}$ has order r and $\mathfrak{W}$ is of exponent r. We are assuming by way of contradiction that $\mathfrak{M}$ has r-length ≥ 2, so by (B), we have $m(\mathfrak{W}) \geq r$. Hence, $m(\mathfrak{W}) = r$.

Set $\mathfrak{W}_1 = \mathfrak{W}/\boldsymbol{D}(\mathfrak{W})$ and let $\mathfrak{M}_2 = \mathfrak{M}_1/\ker(\mathfrak{M}_1 \rightarrow \operatorname{Aut}\mathfrak{W}_1)$. Then $\mathfrak{A}$ maps onto a S_r-subgroup of $\mathfrak{M}_2$. Hence $\mathfrak{M}_2$ has a normal series $1 \subset \mathfrak{C}_1 \subset \mathfrak{C}_2 \subseteq \mathfrak{M}_2$, where $\mathfrak{C}_1$ and $\mathfrak{M}_2/\mathfrak{C}_2$ are r'-groups and $|\mathfrak{C}_2 : \mathfrak{C}_1| = r$.

Since $m(\mathfrak{W}) = r$, $\mathfrak{C}_1$ is abelian. Also $\mathfrak{M}_2/\mathfrak{C}_2$ is faithfully represented on $\mathfrak{C}_2/\mathfrak{C}_1$ and since $r \in \pi(\boldsymbol{H}(\mathfrak{M}))$, $\mathfrak{C}_2 \subset \mathfrak{M}_2$.

By Lemma 26.16, $|\mathfrak{M} : \mathfrak{M}'| = q$ is a prime, and $\mathfrak{M}'$ is a S-subgroup of $\mathfrak{M}$. We let $\mathfrak{Q}$ be a S_q-subgroup of $\mathfrak{M}_1$, so that $\mathfrak{Q}$ is of order q. Since $|\mathfrak{M} : \mathfrak{M}'| = |\mathfrak{M}_1 : \mathfrak{M}_1'|$, it follows that $\mathfrak{Q}$ maps onto $\mathfrak{M}_2/\mathfrak{C}_2$. Let

$\bar{\mathfrak{A}}$ denote the image of $\mathfrak{A}$ in $\mathfrak{M}_2$ and let $\bar{\mathfrak{Q}}$ denote the image of $\mathfrak{Q}$ in $\mathfrak{M}_2$. Since $\mathfrak{E}_1$ is a r'-group and a q'-group, we assume without loss of generality that $\bar{\mathfrak{Q}}$ normalizes $\bar{\mathfrak{A}}$.

Let α be the linear character of $\bar{\mathfrak{Q}}$ on $\bar{\mathfrak{A}}$, so that $\alpha \neq 1$. Let β be the linear character of $\bar{\mathfrak{Q}}$ on $\mathfrak{B}_1/\gamma\mathfrak{B}_1\mathfrak{A}$. Since q divides $(r-1)/2$, $C_{\mathfrak{B}_1}(\bar{\mathfrak{Q}})$ is non cyclic. Hence, $C(\mathfrak{Q}) \cap \boldsymbol{H}(\mathfrak{M})$ is not a Z-group, contrary to Lemma 26.15.

Thus, $\mathfrak{M}$ has r-length one for each $r \in \tau$. Since a S_{τ_2}-subgroup of $\mathfrak{M}$ has a Sylow series of complexion $(q_1, \cdots, q_m)$ and since a S_τ-subgroup of $\mathfrak{M}$ is τ_1-closed, it suffices to show that a S_{τ_1}-subgroup of $\mathfrak{M}$ has a Sylow series of complexion $(p_1, \cdots, p_n)$.

Let $\mathfrak{R}$ be a S_{p_i,p_j}-subgroup of $\mathfrak{M}$ with Sylow system $\mathfrak{R}_i, \mathfrak{R}_j$ where $p_i > p_j$. By Lemma 8.16, $\mathfrak{R}_i \cap N(\mathfrak{R}_j)$ centralizes $\mathfrak{R}_j$. Hence $\mathfrak{R}$ is p_i-closed, since $\mathfrak{R}$ has p_j-length one. The lemma follows.

LEMMA 26.18. *Let $\mathfrak{M} \in \mathscr{M}$ and let $\mathfrak{E}$ be a complement for $\boldsymbol{H}(\mathfrak{M})$ in $\mathfrak{M}$. Then there is at most one prime p in $\pi(\mathfrak{E})$ with the following properties*:

(i) *A S_p-subgroup of $\mathfrak{E}$ is a non cyclic abelian group.*

(ii) *A S_p-subgroup of $\mathfrak{G}$ is non abelian.*

Furthermore, if $\pi(\mathfrak{E})$ contains a prime p satisfying (i) *and* (ii), *then a $S_{p'}$-subgroup of $\mathfrak{E}$ is a Z-group.*

Proof. Suppose $p_1, p_2 \in \pi(\mathfrak{E})$, $p_1 \neq p_2$ and both p_1 and p_2 satisfy (i) and (ii). Let $\mathfrak{E}_1$ be a S_{p_1}-subgroup of $\mathfrak{E}$ and let $\mathfrak{E}_2$ be a S_{p_2}-subgroup of $\mathfrak{E}$ permutable with $\mathfrak{E}_1$.

Let $\mathfrak{E}_i = \mathfrak{A}_i \times \mathfrak{B}_i$, where $|\mathfrak{A}_i| = p_i$, $\mathfrak{A}_i$ centralizes $\boldsymbol{H}(\mathfrak{M})$, $\mathfrak{B}_i\boldsymbol{H}(\mathfrak{M})$ is a Frobenius group and $\Omega_1(\mathfrak{B}_i) \subseteq \boldsymbol{Z}(\mathfrak{P}_i)$ for some S_{p_i}-subgroup $\mathfrak{P}_i$ of $\mathfrak{G}$, $i = 1, 2$. Assume without loss of generality that $p_1 > p_2$. Then $\mathfrak{E}_2$ normalizes $\mathfrak{E}_1$. It follows that $\Omega_1(\mathfrak{E}_2)$ centralizes $\mathfrak{E}_1/\mathfrak{A}_1$, and this implies that $\Omega_1(\mathfrak{E}_2)$ centralizes $\Omega_1(\mathfrak{B}_1)$. It follows that $\mathfrak{G}$ satisfies E_{p_1,p_2}.

By Lemma 26.17, $N(\mathfrak{P}_1)$ contains a S_{p_2}-subgroup $\mathfrak{P}_2^*$ of $\mathfrak{G}$. By Lemma 8.16, $\mathfrak{P}_2^{*\prime}$ centralizes $\mathfrak{P}_1$, so centralizes $\mathfrak{E}_1$. Since $C(\mathfrak{A}_1) \subseteq \mathfrak{M}$, we see that $p_2 \in \pi_2$. By Lemma 26.2, and Lemma 26.10, $\mathfrak{P}_2$ now centralizes $\mathfrak{P}_1$. This is a contradiction, proving the first assertion.

Now suppose $p \in \pi(\mathfrak{E})$ satisfies (i) and (ii), $\mathfrak{E}_p$ is a S_p-subgroup of $\mathfrak{E}$ and $\mathfrak{E}_q$ is a non cyclic S_q-subgroup of $\mathfrak{E}$ permutable with $\mathfrak{E}_p$, $q \in \pi(\mathfrak{E})$, $q \neq p$.

Case 1. $\mathfrak{E}_q$ is non abelian.

In this case, $\mathfrak{E}_q$ is a S_q-subgroup of $\mathfrak{G}$ and $q \in \pi_2$, by Lemma 26.14. Since $\mathfrak{E}_q \subseteq \mathfrak{M}'$, $\mathfrak{E}_p$ normalizes $\mathfrak{E}_q$. Write $\mathfrak{E}_p = \mathfrak{A} \times \mathfrak{B}$, where $\mathfrak{A}$ centralizes $\boldsymbol{H}(\mathfrak{M})$, $\mathfrak{B}\boldsymbol{H}(\mathfrak{M})$ is a Frobenius group, and $\Omega_1(\mathfrak{B}) \subseteq \boldsymbol{Z}(\mathfrak{P})$ for some S_p-subgroup of $\mathfrak{P}$ of $\mathfrak{G}$ with $\mathfrak{E}_p \subseteq \mathfrak{P}$. Then $\Omega_1(\mathfrak{E}_p)$ centralizes $\mathfrak{E}_q/\mathfrak{E}_q \cap C(\boldsymbol{H}(\mathfrak{M}))$. If $\mathfrak{E}_p$ centralizes $\mathfrak{E}_q$, then $\mathfrak{G}$ satisfies $E_{p,q}$ as can be seen by considering

$N(\mathfrak{E}_p)$.

We now show that $\mathfrak{G}$ does not satisfy $\mathfrak{E}_{p,q}$. Otherwise, since $N(\mathfrak{E}_q) \subseteq \mathfrak{M}$, we see that $\mathfrak{E}_q$ normalizes some S_p-subgroup $\mathfrak{P}^*$ of $\mathfrak{G}$. Then $\mathfrak{E}_q$ centralizes $\mathfrak{P}^*$ by Lemma 26.2, Lemma 26.14, and Lemma 8.16. This is not possible since $\mathfrak{E}_p$ is abelian.

Hence, $\mathfrak{G}$ does not satisfy $E_{p,q}$, so $\Omega_1(\mathfrak{E}_p)$ does not centralize $\mathfrak{E}_q$ and $q > p$. This implies that $|\mathfrak{E}_q : \mathfrak{E}_q \cap C(\boldsymbol{H}(\mathfrak{M}))| = q$. Hence $\mathfrak{E}_q \cap C(\Omega_1(\mathfrak{E}_p)) = \mathfrak{E}_q^*$ is of order q.

Consider $N(\Omega_1(\mathfrak{E}_p)) = \mathfrak{N}$. Since a S_p-subgroup of $\mathfrak{N}$ has order $p|\mathfrak{E}_p|$, it follows that a $S_{p,q}$-subgroup of $\mathfrak{N}$ is q-closed. Let $\mathfrak{F}_q$ be a S_q-subgroup of $\mathfrak{N}$ containing $\mathfrak{E}_q^*$. If $\mathfrak{F}_q$ is not of order q, then $N(\Omega_1(\mathfrak{F}_q))$ contains a S_q-subgroup of $\mathfrak{G}$, a $S_{p,q}$-subgroup of $N(\Omega_1(\mathfrak{F}_q))$ is q-closed, and a S_p-subgroup of $N(\Omega_1(\mathfrak{F}_q))$ has larger order than $\mathfrak{E}_p$. As $N(\mathfrak{E}_q) \subseteq \mathfrak{M}$, this is not possible. Hence $\mathfrak{F}_q = \mathfrak{E}_q^*$ has order q. But now a S_q-subgroup of $N(\mathfrak{F}_q)$ contains $\mathfrak{E}_p$ and $\boldsymbol{Z}(\mathfrak{E}_q)$, so a $S_{p,q}$-subgroup of $N(\mathfrak{F}_q)$ is q-closed. This in turn implies that a S_p-subgroup of $N(\mathfrak{E}_q)$ has order larger than $|\mathfrak{E}_p|$, which is a contradiction.

Case 2. $\mathfrak{E}_q$ is a non cyclic abelian group.

By the first part of the proof, and by Lemma 26.13, $\mathfrak{E}_q$ is a S_q-subgroup of $\mathfrak{G}$. Since $\Omega_1(\mathfrak{E}_p)$ centralizes $\mathfrak{E}_q/\mathfrak{E}_q \cap C(\boldsymbol{H}(\mathfrak{M}))$, and since $\mathfrak{E}_q \not\subseteq C(\boldsymbol{H}(\mathfrak{M}))$, it follows that $\mathfrak{G}$ satisfies $\mathfrak{E}_{p,q}$. This implies that a $S_{p,q}$-subgroup of $\mathfrak{G}$ is p-closed, by Lemma 26.2. Hence, $\mathfrak{E}_q$ centralizes the center $\mathfrak{Z}$ of some S_p-subgroup of $\mathfrak{G}$, since $\Omega_1(\mathfrak{E}_q)$ centralizes $\Omega_1(\mathfrak{B})$, (where $\mathfrak{E}_p = \mathfrak{A} \times \mathfrak{B}$, as in Case 1). To obtain the relation $[\Omega_1(\mathfrak{E}_q), \Omega_1(\mathfrak{B})] = 1$, we have used Lemma 26.13 to conclude that there are at least 2 subgroups of $\mathfrak{E}_q$ of order q which have no fixed points on $\boldsymbol{H}(\mathfrak{M})$, or else $\mathfrak{E}_q \subseteq \mathfrak{M}'$ in which case $\mathfrak{E}_p$ normalizes $\mathfrak{E}_q$ and so $\Omega_1(\mathfrak{B})$ centralizes $\mathfrak{E}_q$.

But now $N(\Omega_1(\mathfrak{B}))$ dominates $\mathfrak{E}_q$, so $\mathfrak{E}_q$ centralizes some S_p-subgroup of $\mathfrak{G}$, contrary to $C(\mathfrak{E}_q) \subseteq \mathfrak{M}$. The proof is complete.

LEMMA 26.19. *Let $\mathfrak{M} \in \mathscr{M}$. Suppose $\mathfrak{M}/\boldsymbol{H}(\mathfrak{M})$ is abelian. Suppose further that either $\boldsymbol{H}(\mathfrak{M})$ is nilpotent or $|\mathfrak{M} : \boldsymbol{H}(\mathfrak{M})|$ is not a prime. Then $\mathfrak{M}$ is of type* I *or* V.

Proof. Let $\mathfrak{E}$ be a complement for $\boldsymbol{H}(\mathfrak{M})$. Since $\boldsymbol{H}(\mathfrak{M}) = \mathfrak{M}'$ by hypothesis (we always have $\boldsymbol{H}(\mathfrak{M}) \subseteq \mathfrak{M}'$), $\mathfrak{E} \cong \mathfrak{M}/\mathfrak{M}'$ is abelian.

Case 1. $\mathfrak{E}$ is cyclic.

We wish to show that $\boldsymbol{H}(\mathfrak{M})$ is nilpotent, so suppose $|\mathfrak{E}|$ is not a prime. Since $|\mathfrak{E}|$ is not a prime, since $\mathfrak{E}$ is prime on $\boldsymbol{H}(\mathfrak{M})$, since $\mathfrak{E}$ has no fixed points on $\boldsymbol{H}(\mathfrak{M})/\boldsymbol{H}(\mathfrak{M})'$, and since $C(\mathfrak{E}) \cap \boldsymbol{H}(\mathfrak{M})$ is a Z-group, it follows from Lemma 26.3 that $\boldsymbol{H}(\mathfrak{M})$ is nilpotent, so that $C(\mathfrak{E}) \cap \boldsymbol{H}(\mathfrak{M}) = \mathfrak{E}_1$ is cyclic.

Case 1a. $\mathfrak{E}_1 = 1$.

In this case, $\mathfrak{M}$ is a Frobenius group with Frobenius kernel

$\boldsymbol{H}(\mathfrak{M}) = \mathfrak{M}'$, so condition (i) in type I holds. If $\boldsymbol{H}(\mathfrak{M})$ is a T.I. set in $\mathfrak{G}$, then $\mathfrak{M}$ is of type I, since (ii) (a) holds, so suppose $\boldsymbol{H}(\mathfrak{M})$ is not a T.I. set in $\mathfrak{G}$. Let $\boldsymbol{H}(\mathfrak{M}) = \mathfrak{P}_1 \times \cdots \times \mathfrak{P}_n$, where $\mathfrak{P}_i$ is the S_{p_i}-subgroup of $\mathfrak{M}$ and $\{p_1, \cdots, p_n\} = \pi(\boldsymbol{H}(\mathfrak{M}))$. If $p_i \in \pi_1$, then clearly $p_i \in \pi_1^*$. If $p_i \in \pi_0 \cap \pi^*$; then also $p_i \in \pi_1^*$, since $\mathfrak{E}\boldsymbol{Z}(\mathfrak{P}_i)$ is a Frobenius group. Similarly, if $p_i \in \pi_2$ and $\mathfrak{P}_i$ is non abelian, then $p_i \in \pi_1^*$.

Suppose $p_i \notin \pi_1^*$. Then either $p_i \in \pi_2$ and $\mathfrak{P}_i$ is abelian, or $p_i \in \pi_0 - \pi^*$. We will show that the second possibility cannot occur.

Choose G in $\mathfrak{G} - \mathfrak{M}$ such that $\mathfrak{D} = \boldsymbol{H}(\mathfrak{M}) \cap \boldsymbol{H}(\mathfrak{M})^G \neq 1$, and let H be an element of $\mathfrak{D}$ of prime order p. If $p_i \in \pi_0 - \pi^*$, and $p \neq p_i$, then $C(H) \supseteq \langle \mathfrak{P}_i, \mathfrak{P}_i^G \rangle$, and $\mathfrak{M} = \mathfrak{M}^G$, contrary to assumption. Hence, $p = p_i$. In this case, $C(H) \supseteq \langle C(H) \cap \mathfrak{P}_i, C(H) \cap \mathfrak{P}_i^G \rangle$, and since $p_i \in \pi_0 - \pi^*$, both $C(H) \cap \mathfrak{P}_i$ and $C(H) \cap \mathfrak{P}_i^G$ are in $\mathscr{X}_1$, so $\mathfrak{M} = \mathfrak{M}^G$. Hence, $(\pi_0 - \pi^*) \cap \pi(\boldsymbol{H}(\mathfrak{M})) = \varnothing$.

Thus, if $\pi(\boldsymbol{H}(\mathfrak{M})) \not\subseteq \pi_1^*$, then $\pi(\boldsymbol{H}(\mathfrak{M}))$ contains a prime q such that the S_q-subgroup $\mathfrak{Q}$ of $\mathfrak{M}$ is abelian and $q \in \pi_2$. Since $|\mathfrak{E}|$ does not divide $q-1$ or $q+1$, but $|\mathfrak{E}|$ does divide q^2-1, we can find $r_1, r_2 \in \pi(\mathfrak{E})$ such that $r_1 \mid q-1$ and $r_2 \mid q+1$. Let $\mathfrak{E}_{r_1}$ be the S_{r_1}-subgroup of $\mathfrak{E}$. Then $\mathfrak{Q} = \mathfrak{Q}_1 \times \mathfrak{Q}_2$, where $\mathfrak{Q}_i$ is normalized by $\mathfrak{E}_{r_1}$ and $\mathfrak{Q}_i$ is cyclic, $i = 1, 2$. Since $r_2 \mid q+1$, it follows that $\mathfrak{Q}_1$ and $\mathfrak{Q}_2$ are isomorphic $\mathfrak{E}_{r_1}$-modules. Hence, $\mathfrak{E}_{r_1}$ normalizes every subgroup of $\mathfrak{Q}$.

Once again, choose G in $\mathfrak{G} - \mathfrak{M}$ so that $\mathfrak{D} = \boldsymbol{H}(\mathfrak{M}) \cap \boldsymbol{H}(\mathfrak{M})^G \neq 1$. Then $C(\mathfrak{D}) \supseteq \langle \mathfrak{Q}, \mathfrak{Q}^G \rangle$, so $C(\mathfrak{D})$ is not contained in any conjugate of $\mathfrak{M}$. Let $C(\mathfrak{D}) \subseteq \mathfrak{M}_1 \in \mathscr{M}$. We apply Lemma 26.13 to $\mathfrak{M}_1$ and $\mathfrak{Q}$. Since $C(\Omega_1(\mathfrak{Q})) = \boldsymbol{H}(\mathfrak{M})$, we have $\boldsymbol{H}(\mathfrak{M}) \subseteq \mathfrak{M}_1$.

Suppose $\boldsymbol{H}(\mathfrak{M})$ were not abelian. Let $\mathfrak{R}$ be a non abelian S_r-subgroup of $\boldsymbol{H}(\mathfrak{M})$. Apply Lemma 26.16 to $\mathfrak{M}_1$ and $\mathfrak{R}$, and conclude that $N(\Omega_1(\boldsymbol{Z}(\mathfrak{R}))) \subseteq \mathfrak{M}_1$, and so $\mathfrak{M} \subseteq \mathfrak{M}_1$, which is not the case. Thus, alternative (ii) (c) in the definition of type I holds, so $\mathfrak{M}$ is of type I. (Since $\boldsymbol{H}(\mathfrak{M}) \in \mathscr{X}_0$, $\boldsymbol{H}(\mathfrak{M})$ is generated by two elements.)

Case 1b. $\mathfrak{E}_1 \neq 1$.

Since $\boldsymbol{H}(\mathfrak{M}) = \mathfrak{M}'$, we have $\mathfrak{E}_1 \subseteq \boldsymbol{H}(\mathfrak{M})' \subseteq \hat{\boldsymbol{H}}(\mathfrak{M}) \cup \{1\}$. It follows that $N(\mathfrak{E}_0) \subseteq \mathfrak{M}$ for every non empty subset $\mathfrak{E}_0$ of $\mathfrak{E}_1^\sharp$. Let $\hat{\mathfrak{E}} = \mathfrak{E}\mathfrak{E}_1 - \mathfrak{E} - \mathfrak{E}_1$. If $\hat{\mathfrak{E}}_0$ is any non empty subset of $\hat{\mathfrak{E}}$, then each element of $\hat{\mathfrak{E}}_0$ is of the form EE_1, $E \in \mathfrak{E}^\sharp$, $E_1 \in \mathfrak{E}_1^\sharp$. Thus, if $\hat{\mathfrak{F}}_0 = \{E_0^{|\mathfrak{E}|} \mid E_0 \in \hat{\mathfrak{E}}_0\}$, then $N(\hat{\mathfrak{E}}_0) \subseteq N(\hat{\mathfrak{F}}_0) \subseteq \mathfrak{M}$. Since $\mathfrak{M} \cap N(\hat{\mathfrak{E}}_0) = \mathfrak{E}\mathfrak{E}_1$, $\mathfrak{M}$ is a three step group with $\mathfrak{E}$ in the role of $\mathfrak{Q}^*$, $\boldsymbol{H}(\mathfrak{M})$ in the role of $\mathfrak{H}$, $\mathfrak{E}_1$ in the role of $\mathfrak{H}^*$. Since $\boldsymbol{H}(\mathfrak{M}) = \mathfrak{M}'$, we take $\mathfrak{U} = 1$, so that (i) in the definition of type V holds. If (ii) (a) holds, then $\mathfrak{M}$ is of type V, so suppose (ii) (a) does not hold.

Since $\mathfrak{E}_1 \subseteq \boldsymbol{H}(\mathfrak{M})'$, $\boldsymbol{H}(\mathfrak{M})$ is non abelian. Let $\boldsymbol{H}(\mathfrak{M}) = \mathfrak{P} \times \mathfrak{S}_0$, where $\mathfrak{P}$ is a non abelian S_p-subgroup of $\boldsymbol{H}(\mathfrak{M})$ (there may be several).

We will show that $\mathfrak{S}_0$ is a T.I. set in $\mathfrak{G}$. Suppose $G \in \mathfrak{G} - \mathfrak{M}$ and $\mathfrak{S}_0 \cap \mathfrak{S}_0^G = \mathfrak{D}$ is a maximal intersection, so that $N(\mathfrak{D})$ is contained in no conjugate of $\mathfrak{M}$. Let $\mathfrak{M}_1 \in \mathscr{M}$ with $N(\mathfrak{D}) \subseteq \mathfrak{M}_1$. Apply Lemma 26.14 to $\mathfrak{M}_1$ and $\mathfrak{P}$ and conclude that $\mathfrak{M} \subseteq \mathfrak{M}_1$, a contradiction. Hence, $\mathfrak{S}_0$ is a T.I. set in $\mathfrak{G}$.

Since $\boldsymbol{H}(\mathfrak{M})$ is not a T.I. set in $\mathfrak{G}$, choose $G \in \mathfrak{G} - \mathfrak{M}$ so that $1 \neq \boldsymbol{H}(\mathfrak{M}) \cap \boldsymbol{H}(\mathfrak{M})^G$ is a maximal intersection. Since $\mathfrak{S}_0$ is a T.I. set in $\mathfrak{G}$, we see that $\boldsymbol{H}(\mathfrak{M}) \cap \boldsymbol{H}(\mathfrak{M})^G = \mathfrak{D}_1 = \mathfrak{P} \cap \mathfrak{P}^G$, and $N(\mathfrak{D}_1)$ is contained in no conjugate of $\mathfrak{M}$, while $N(\mathfrak{D}_1) \supseteq \mathfrak{S}_0$. Since $\mathfrak{S}_0$ is a T.I. set in $N(\mathfrak{D}_1)$, and since $N(\mathfrak{D}_1) \not\subseteq \mathfrak{M}$, $\mathfrak{S}_0$ is cyclic. By construction, $\mathfrak{P}$ is non abelian, so $p \in \pi^*$. It only remains to show that $p \in \pi_1^*$.

Apply Lemma 8.16 to $\mathfrak{P}$ and $\mathfrak{E}$. If $\mathfrak{E}$ does not centralize $\boldsymbol{Z}(\mathfrak{P})$, then $|\mathfrak{E}|$ divides $p-1$ and we are done. Suppose that $\mathfrak{E}$ centralizes $\boldsymbol{Z}(\mathfrak{P})$. Then $\mathfrak{E}$ is faithfully represented on $\Omega_1(\boldsymbol{Z}_2(\mathfrak{P}))/\Omega_1(\boldsymbol{Z}(\mathfrak{P}))$, so if $|\Omega_1(\boldsymbol{Z}_2(\mathfrak{P})) : \Omega_1(\boldsymbol{Z}(\mathfrak{P}))| = p$, we are done. Otherwise, we let P_0 be an element of $\mathfrak{P}$ of order p such that $C_{\mathfrak{P}}(P_0) = \langle P_0 \rangle \times \mathfrak{A}$, where $\mathfrak{A}$ is cyclic. Since $|\Omega_1(\boldsymbol{Z}_2(\mathfrak{P})) : \Omega_1(\boldsymbol{Z}(\mathfrak{P}))| \geqq p^2$, we have $P_0 \in \Omega_1(\boldsymbol{Z}_2(\mathfrak{P}))$, so $\langle P_0, \Omega_1(\boldsymbol{Z}(\mathfrak{P})) \rangle \lhd \mathfrak{P}$. By Lemma 8.9, $\mathscr{S}\mathscr{C}\mathscr{N}_3(\mathfrak{P})$ is empty. By Lemma 26.2, $\mathfrak{P}$ is a central product of a cyclic group and $\Omega_1(\mathfrak{P})$, with $|\Omega_1(\mathfrak{P})| = p^3$. Since $\mathfrak{P} \subseteq \mathfrak{M}'$ and since $\mathfrak{E}$ centralizes $\boldsymbol{Z}(\mathfrak{P})$, we have $|\mathfrak{P}| = p^3$. $\mathfrak{E}$ is faithfully represented on $\mathfrak{P}/\mathfrak{P}'$, and since $\mathfrak{E}$ centralizes $\mathfrak{P}'$, each element of $\mathfrak{E}$ induces a linear transformation of $\mathfrak{P}/\mathfrak{P}'$ of determinant 1. Thus, $|\mathfrak{E}|$ divides either $p-1$ or $p+1$, since $\mathfrak{E}$ is isomorphic to a cyclic p'-subgroup of $SL(2, \mathrm{p})$. Hence, $p \in \pi_1^*$, and $\mathfrak{M}$ is of type V.

Case 2. $\mathfrak{E}$ is non cyclic.

Case 2a. There is an element $p \in \pi(\mathfrak{E})$ such that the S_p-subgroup $\mathfrak{E}_p$ of $\mathfrak{E}$ is non cyclic and a S_p-subgroup of $\mathfrak{G}$ is non abelian. In this case, Lemma 26.18 implies that $\mathfrak{E} = \mathfrak{E}_p \times \mathfrak{F}$ where $\mathfrak{F}$ is cyclic.

Let $\mathfrak{E}_p = \mathfrak{E}_{p0} \times \mathfrak{E}_{p1}$, with $|\mathfrak{E}_{p0}| = p$, $\mathfrak{E}_{p0} \subseteq \boldsymbol{Z}(\mathfrak{M})$, and with $\mathfrak{E}_{p1}\boldsymbol{H}(\mathfrak{M})$ a Frobenius group. Also $\mathfrak{F}$ is a cyclic S-subgroup of $\mathfrak{M}$.

We will show that $\mathfrak{E}_{p1}\mathfrak{F}\boldsymbol{H}(\mathfrak{M})$ is a Frobenius group. If $\mathfrak{F} = 1$, this is the case, so suppose $\mathfrak{F} \neq 1$. By Lemma 26.16, $\mathfrak{F}$ is prime on $\boldsymbol{H}(\mathfrak{M})$. Let $\mathfrak{H}^* = \boldsymbol{C}(\mathfrak{F}) \cap \boldsymbol{H}(\mathfrak{M})$, and suppose $\mathfrak{H}^* \neq 1$. Then $\mathfrak{E}_{p1}\mathfrak{H}^*$ is a Frobenius group. Let $\mathfrak{M}_1$ be a maximal subgroup of $\mathfrak{G}$ containing $N(\mathfrak{F}_1)$, $\mathfrak{F}_1$ being a fixed subgroup of $\mathfrak{F}$ of prime order. Then $\mathfrak{M}_1$ is not conjugate to $\mathfrak{M}$. Hence, $\mathfrak{M} \cap \mathfrak{M}_1 \in \mathscr{X}_0$. Since $\mathfrak{E}_{p1}\mathfrak{H}^*$ is a Frobenius group, $\mathfrak{E}_{p1} \cap \mathfrak{M}_1' = 1$, so a S_p-subgroup of $\mathfrak{M}_1$ is abelian. By Lemma 26.12, $\mathfrak{E}_{p1}\boldsymbol{H}(\mathfrak{M}_1)$ is a Frobenius group, so $\boldsymbol{H}(\mathfrak{M}) \cap \mathfrak{M}_1$ centralizes $\boldsymbol{H}(\mathfrak{M}_1)$. Since $1 \subset \mathfrak{H}^* \subseteq \boldsymbol{H}(\mathfrak{M}) \cap \mathfrak{M}_1$, we see that $\mathfrak{M} \subseteq \mathfrak{M}_1$, which is not the case. Hence, $\mathfrak{H}^* = 1$, so $\mathfrak{F}\boldsymbol{H}(\mathfrak{M})$ is a Frobenius group, as is $\mathfrak{E}_{p1}\mathfrak{F}\boldsymbol{H}(\mathfrak{M})$. $\mathfrak{M}$ itself is a group of Frobenius type.

Suppose $\mathfrak{M}'$ is not a T.I set in $\mathfrak{G}$ and $\pi(\mathfrak{M}') \not\subseteq \pi_1^*$. It follows readily that $\mathfrak{M}'$ is abelian and is generated by two elements. $\mathfrak{M}$ is of type I.

Case 2b. Whenever a S_p-subgroup of $\mathfrak{E}$ is non cyclic, a S_p-subgroup of $\mathfrak{G}$ is abelian.

Let $\tilde{\pi}$ be the set of primes p in $\pi(\mathfrak{E})$ such that a S_p-subgroup of $\mathfrak{E}$ is non cyclic. Let $\mathfrak{E} = \mathfrak{E}_1 \times \mathfrak{E}_2$, where $\mathfrak{E}_1$ is the $S_{\tilde{\pi}}$-subgroup of $\mathfrak{E}$. Thus $\mathfrak{E}_2$ is a cyclic S-subgroup of $\mathfrak{M}$, and $\tilde{\pi} \neq \emptyset$. By Lemma 26.13, $\mathfrak{E}_1$ is a S-subgroup of $\mathfrak{G}$.

We first show that if $p \in \tilde{\pi}$ and $\mathfrak{E}_p$ is the S_p-subgroup of $\mathfrak{E}_1$, then

$$C(\Omega_1(\mathfrak{E}_p)) \cap \boldsymbol{H}(\mathfrak{M}) = 1 \tag{26.6}$$

This is an immediate consequence of Lemma 26.13 (iv) and Grün's theorem, since $\mathfrak{M}' \cap \mathfrak{E}_p = 1$.

We next show that either $\mathfrak{E}_2 = 1$ or $\mathfrak{E}_2\boldsymbol{H}(\mathfrak{M})$ is a Frobenius group. Suppose $\mathfrak{E}_2 \neq 1$. By Lemma 26.15 $\mathfrak{E}_2$ is prime on $\boldsymbol{H}(\mathfrak{M})$. Suppose $\mathfrak{H}^* = C(\mathfrak{E}_2) \cap \boldsymbol{H}(\mathfrak{M}) \neq 1$. Let $\mathfrak{E}_q$ be the S_q-subgroup of $\mathfrak{E}_2$ for some $q \in \pi(\mathfrak{E}_2)$, and let $\mathfrak{M}_1$ be a maximal subgroup of $\mathfrak{G}$ containing $N(\mathfrak{E}_q)$. Then $\mathfrak{M}_1$ is not conjugate to $\mathfrak{M}$.

By Lemma 26.13 (ii), together with $\mathfrak{M}' \cap \mathfrak{E}_p = 1$, there is some element of $\Omega_1(\mathfrak{E}_p)^\#$ which has no fixed points on $\boldsymbol{H}(\mathfrak{M})^\#$, so $\mathfrak{H}^*$ is cyclic. By construction $\langle \mathfrak{E}, \mathfrak{H}^* \rangle \subseteq \mathfrak{M}_1$. Suppose $\mathfrak{M}_1 \cap \boldsymbol{H}(\mathfrak{M}^G)$ is non cyclic for some G in $\mathfrak{G}$. Let $\mathfrak{R}$ be a non cyclic S_r-subgroup of $\mathfrak{M}_1 \cap \boldsymbol{H}(\mathfrak{M}^G)$. If a S_r-subgroup of $\mathfrak{G}$ is abelian, then $\boldsymbol{H}(\mathfrak{M}^G) \subseteq \mathfrak{M}_1$ by Lemma 26.13 (i) and (ii). Since $\mathfrak{E} \subseteq \mathfrak{M}_1$, we have $\mathfrak{M}^G = \mathfrak{M}_1$, which is not the case. Hence, a S_r-subgroup of $\mathfrak{G}$ is non abelian. If $\mathfrak{R}$ were non abelian, then $\mathfrak{M}_1 = \mathfrak{M}^{G_1}$ for some G_1 in $\mathfrak{G}$, by Lemma 26.14 with $\mathfrak{R}$ in the role of $\mathfrak{P}$. Hence, $\mathfrak{R}$ is abelian. By Lemma 26.13, $\mathfrak{R} = \mathfrak{R}_0 \times \mathfrak{R}_1$, $|\mathfrak{R}_0| = r$, $\mathfrak{R}_0$ centralizes $\boldsymbol{H}(\mathfrak{M}_1)$ and $\mathfrak{R}_1\boldsymbol{H}(\mathfrak{M}_1)$ is a Frobenius group. By (26.6), $\mathfrak{R} \subseteq \mathfrak{M}_1'$, so $\mathfrak{R}_0 \lhd \mathfrak{M}_1$. Since $\mathfrak{R}\boldsymbol{H}(\mathfrak{M}_1) \lhd \mathfrak{M}_1$, we can find a S_r-subgroup $\mathfrak{R}^*$ of $\mathfrak{M}_1$ which is normalized by $\mathfrak{E}_p$. Since $\mathfrak{M}$ and $\mathfrak{M}_1$ are not conjugate, $\pi(\boldsymbol{H}(\mathfrak{M})) \cap \pi(\boldsymbol{H}(\mathfrak{M}_1)) = \emptyset$, so $\mathfrak{R}^*$ does not lie in $\boldsymbol{H}(\mathfrak{M}_1)$, and $\mathfrak{R}^*$ does not centralize $\boldsymbol{H}(\mathfrak{M}_1)$. There are at least p subgroups $\mathfrak{P}_0$ of $\Omega_1(\mathfrak{E}_p)$ with the property that $\mathfrak{P}_0\mathfrak{R}^*/\mathfrak{R}_0$ is a Frobenius group, by (26.6). Each of these has a fixed point on $\boldsymbol{H}(\mathfrak{M}_1)^\#$. It follows from Lemma 26.13 (iii) that $\mathfrak{M}_1$ dominates $\mathfrak{E}_p$. This is absurd, by (26.6) and Lemma 8.13. Hence, $\mathfrak{M}_1 \cap \boldsymbol{H}(\mathfrak{M}^G)$ is cyclic for all G in $\mathfrak{G}$. In particular, $\mathfrak{M}_1 \cap \boldsymbol{H}(\mathfrak{M})$ is cyclic. This implies that $\mathfrak{M}_1 \cap \boldsymbol{H}(\mathfrak{M})$ is faithfully represented on $\boldsymbol{H}(\mathfrak{M}_1)$, so $\mathfrak{H}^*$ is faithfully represented on $\boldsymbol{H}(\mathfrak{M}_1)$. By (26.6), at least p subgroups of $\mathfrak{E}_p$ or order p have fixed points on $\boldsymbol{H}(\mathfrak{M}_1)$, so $\mathfrak{M}_1$ dominates $\mathfrak{E}_p$, which violates (26.6), by Lemma 8.13. $\mathfrak{E}_2\boldsymbol{H}(\mathfrak{M})$ is a Frobenius group. Thus, in the definition of a group of Frobenius type, the primes in $\pi(\mathfrak{E}_2)$ are taken care of. Let $\mathfrak{E}_p = \mathfrak{E}_{p1} \times \mathfrak{E}_{p2}$, with $|\mathfrak{E}_{p1}| \leq |\mathfrak{E}_{p2}|$, $p \in \tilde{\pi}$, and where $\mathfrak{E}_{pi}$ is cyclic, $i = 1, 2$. If $|\mathfrak{E}_{p1}| < |\mathfrak{E}_{p2}|$, then $\Omega_1(\mathfrak{E}_{p2})$ char $\mathfrak{E}_p$. By Lemma 26.14

(v), it follows that $\mathfrak{E}_{p_2}\boldsymbol{H}(\mathfrak{M})$ is a Frobenius group. If $|\mathfrak{E}_{p_1}| = |\mathfrak{E}_{p_2}|$, then by Lemma 26.14 (iii), there is some element P of order p in $\mathfrak{E}_p$ such that $\langle P\rangle\boldsymbol{H}(\mathfrak{M})$ is a Frobenius group. Thus, $\mathfrak{E}$ contains a subgroup $\mathfrak{E}^*$ of the same exponent as $\mathfrak{E}$ with the property that $\mathfrak{E}^*\boldsymbol{H}(\mathfrak{M})$ is a Frobenius group. $\mathfrak{M}$ is of Frobenius type.

If $\boldsymbol{H}(\mathfrak{M})$ is not a T.I. set in $\mathfrak{G}$, and $\pi(\boldsymbol{H}(\mathfrak{M})) \not\subseteq \pi_1^*$, it follows readily that $\boldsymbol{H}(\mathfrak{M})$ is abelian and is generated by two elements. The proof is complete.

LEMMA 26.20. *Let $\mathfrak{M} \in \mathscr{M}$ and let $\tilde{\pi}$ be the subset of primes p in $\pi(\mathfrak{M}/\boldsymbol{H}(\mathfrak{M}))$ such that a S_p-subgroup of $\mathfrak{M}$ is a non cyclic abelian group and a S_p-subgroup of $\mathfrak{G}$ is abelian. Let $\mathfrak{E}$ be a complement for $\boldsymbol{H}(\mathfrak{M})$ in $\mathfrak{M}$. Then a $S_{\tilde{\pi}}$-subgroup $\mathfrak{P}$ of $\mathfrak{E}$ is a normal abelian subgroup of $\mathfrak{E}$ and $\mathfrak{P} \cap \mathfrak{E}' = 1$ or $\mathfrak{P}$.*

Proof. We can suppose $\mathfrak{P} \neq 1$. Let $p \in \tilde{\pi}$ and let $\mathfrak{E}_p$ be a S_p-subgroup of $\mathfrak{E}$. We first show that $\mathfrak{E}_p \lhd \mathfrak{E}$. Let $q \in \pi(\mathfrak{E})$ and let $\mathfrak{E}_q$ be a S_q-subgroup of $\mathfrak{E}$ permutable with $\mathfrak{E}_p$. If $\mathfrak{E}_q$ is non abelian, then $N(\Omega_1(\boldsymbol{Z}(\mathfrak{E}_q))) \subseteq \mathfrak{M}$, by Lemma 26.14. If $\Omega_1(\boldsymbol{Z}(\mathfrak{E}_q))$ centralizes $\Omega_1(\mathfrak{E}_p)$, then $\mathfrak{E}_p \subseteq \mathfrak{M}'$ so that $\mathfrak{E}_p$ centralizes $\mathfrak{E}_q$. We can suppose that $\Omega_1(\boldsymbol{Z}(\mathfrak{E}_q))$ does not centralize $\Omega_1(\mathfrak{E}_p)$. Since $\Omega_1(\mathfrak{E}_p)$ centralizes $\mathfrak{E}_q/\mathfrak{E}_q \cap C(\boldsymbol{H}(\mathfrak{M}))$, and since $\mathfrak{E}_q \not\subseteq C(\boldsymbol{H}(\mathfrak{M}))$, it follows that $\mathfrak{E}_p \subseteq \mathfrak{M}'$ so that $\mathfrak{E}_p$ centralizes $\mathfrak{E}_q$.

If $\mathfrak{E}_q$ is a non cyclic abelian group, then $q \in \tilde{\pi}$ by Lemma 26.18. If $\mathfrak{E}_p \ntriangleleft \mathfrak{E}_p\mathfrak{E}_q$, then $\mathfrak{E}_p$ normalizes $\mathfrak{E}_q$ and $\Omega_1(\mathfrak{E}_p)$ centralizes $\mathfrak{E}_q/\mathfrak{E}_q \cap C(\boldsymbol{H}(\mathfrak{M}))$. If $\mathfrak{E}_q \cap C(\boldsymbol{H}(\mathfrak{M})) = 1$, then $N(\mathfrak{E}_q)$ dominates $\Omega_1(\mathfrak{E}_p)$, so $\mathfrak{E}_p$ centralizes $\mathfrak{E}_q$. If $\mathfrak{E}_q \cap C(\boldsymbol{H}(\mathfrak{M})) \neq 1$, then $\mathfrak{E}_q \cap C(\Omega_1(\mathfrak{E}_p))$ dominates $\mathfrak{E}_p$, so that $\mathfrak{E}_q$ dominates $\mathfrak{E}_p$ and once again $\mathfrak{E}_p$ centralizes $\mathfrak{E}_q$.

Suppose $\mathfrak{E}_q$ is cyclic. We can suppose that $\mathfrak{E}_p$ normalizes $\mathfrak{E}_q$. Then $\Omega_1(\mathfrak{E}_p)$ centralizes $\mathfrak{E}_q$. If $q \in \pi_1 \cup \pi_2$, then $\mathfrak{E}_p$ centralizes $\mathfrak{E}_q$, since $\mathfrak{E}_p \subseteq N(\Omega_1(\mathfrak{E}_q))'$. We can suppose $q \in \pi_0$ and that a S_q-subgroup $\mathfrak{Q}$ of $C(\Omega_1(\mathfrak{E}_p))$ is in $\mathscr{H}_1$. In this case, however, $C(P) \subseteq \boldsymbol{M}(\mathfrak{Q})$ for all $P \in \mathfrak{E}_p^\sharp$, so $\mathfrak{M} = \boldsymbol{M}(\mathfrak{Q})$ which is absurd. Hence, $\mathfrak{E}_p \lhd \mathfrak{E}$, so that $\mathfrak{P}$ is a normal abelian subgroup of $\mathfrak{E}$.

Suppose $\mathfrak{E}$ contains a non abelian S_q-subgroup $\mathfrak{E}_q$ for some prime q. Then $N(\Omega_1(\boldsymbol{Z}(\mathfrak{E}_q))) \subseteq \mathfrak{M}$, which implies that $\mathfrak{P} \subseteq \mathfrak{M}'$, since $N(\mathfrak{E}_q)$ dominates each Sylow subgroup of $\mathfrak{P}$.

Thus, in showing that $\mathfrak{P} \cap \mathfrak{M}' = 1$ or $\mathfrak{P}$, we can suppose that every Sylow subgroup of $\mathfrak{E}$ is abelian. By Lemma 26.18 and the definition of $\tilde{\pi}$, this implies that a $S_{\tilde{\pi}'}$-subgroup $\mathfrak{F}$ of $\mathfrak{E}$ is a Z-group. This in turn implies that $\mathfrak{F} \cap \mathfrak{M}'$ is a S-subgroup of $\mathfrak{M}$. Let $\mathfrak{F}_0$ be a complement for $\mathfrak{F} \cap \mathfrak{M}'$ in $\mathfrak{F}$. Then $\mathfrak{F}_0$ is cyclic. If $\mathfrak{F}_0 = 1$, then $\mathfrak{E}$ is abelian and we are done. We can suppose $\mathfrak{F}_0 \neq 1$.

Suppose $\mathfrak{F}_0$ is not of prime order. Let $\mathfrak{T} = [\mathfrak{F}_0, \mathfrak{P}\boldsymbol{H}(\mathfrak{M})]$. By

Lemma 26.3, and Lemma 26.16 $\mathfrak{T}$ is nilpotent. If $[\mathfrak{F}_0, \mathfrak{P}] \neq 1$, then $[\mathfrak{F}_0, \mathfrak{E}_p] \neq 1$, for some S_p-subgroup $\mathfrak{E}_p$ of $\mathfrak{P}$. Hence, $N([\mathfrak{F}_0, \mathfrak{E}_p])$ dominates every Sylow subgroup of $\mathfrak{P}$. Since $[\mathfrak{F}_0, \boldsymbol{H}(\mathfrak{M})]$ can be assumed non cyclic, $\mathfrak{P} \subseteq \mathfrak{M}'$, and we are done. If $[\mathfrak{F}_0, \mathfrak{P}] = 1$, then $\mathfrak{P} \cap \mathfrak{M}' = 1$, and we are done.

We can now suppose that $\mathfrak{F}_0$ is of prime order r. We can now write $\mathfrak{P} = \mathfrak{P}_0 \times \mathfrak{P}_1$, where $\mathfrak{P}_0 = \mathfrak{P} \cap C(\mathfrak{F}_0)$ and $\mathfrak{P}_1 = [\mathfrak{P}, \mathfrak{F}_0]$, and we suppose by way of contradiction that $\mathfrak{P}_i \neq 1, i = 0, 1$.

Choose p so that $\mathfrak{E}_p \cap \mathfrak{P}_0 \neq 1$, where $\mathfrak{E}_p$ is the S_p-subgroup of $\mathfrak{P}$.

If $\mathfrak{P}_0 \cap \mathfrak{E}_p$ centralizes $\boldsymbol{H}(\mathfrak{M}) \cap C(\mathfrak{F}_0)$, then $N(\mathfrak{P}_0) \subseteq \mathfrak{M}$, by Lemma 26.13, since $\boldsymbol{H}(\mathfrak{M}) \cap C(\mathfrak{F}_0) \neq 1$. Since $\mathfrak{P}_0 \cap \mathfrak{E}_p \lhd N(\mathfrak{E}_p)$, $\mathfrak{P}_0 \cap \mathfrak{E}_p \subseteq \mathfrak{M}'$, contrary to construction. Hence we can assume that $(\mathfrak{P}_0 \cap \mathfrak{E}_p)\mathfrak{H}^*$ is a Frobenius group, where $\mathfrak{H}^* = \boldsymbol{H}(\mathfrak{M}) \cap C(\mathfrak{F}_0)$. Let $\mathfrak{M}_1$ be a maximal subgroup of $\mathfrak{G}$ containing $N(\mathfrak{F}_0)$. Since $\mathfrak{P}_0 \cap \mathfrak{E}_p \lhd N(\mathfrak{E}_p)$, it follows that $\mathfrak{P}_0 \cap \mathfrak{E}_p \subseteq \mathfrak{M}_1'$, since $N(\mathfrak{F}_0)$ dominates $\mathfrak{E}_p$. Since $\mathfrak{M}_1$ is not conjugate to $\mathfrak{M}$, it follows that $\pi(\boldsymbol{H}(\mathfrak{M}_1)) \cap \pi(\boldsymbol{H}(\mathfrak{M})) = \emptyset$, so that $\mathfrak{H}^* \cap \boldsymbol{H}(\mathfrak{M}_1) = 1$. Since $[\mathfrak{H}^*, \mathfrak{P}_0 \cap \mathfrak{E}_p] \neq 1$, both $\mathfrak{P}_0 \cap \mathfrak{E}_p$ and $[\mathfrak{H}^*, \mathfrak{P}_0 \cap \mathfrak{E}_p]$ are in $\mathfrak{M}_1'$, so commute elementwise. Thus $[\mathfrak{H}^*, \mathfrak{P}_0 \cap \mathfrak{E}_p] = 1$, contrary to the above argument. The lemma is proved.

LEMMA 26.21. *Let $\mathfrak{M} \in \mathscr{M}$ and suppose $\pi(\mathfrak{M}/\mathfrak{M}')$ contains a prime p such that a S_p-subgroup of $\mathfrak{M}$ is non cyclic. Then $\mathfrak{M}$ is of type I.*

Proof.

Case 1. A S_p-subgroup of $\mathfrak{G}$ is abelian.

Case 2. A S_p-subgroup of $\mathfrak{G}$ is non abelian.

In Case 1, let $\tilde{\pi}$ be the subset of those q in $\pi(\mathfrak{M}/\boldsymbol{H}(\mathfrak{M}))$ such that a S_q-subgroup of $\mathfrak{M}$ is an abelian non cyclic S_q-subgroup of $\mathfrak{G}$. Then $p \in \tilde{\pi}$, and if $\mathfrak{E}$ is a complement for $\boldsymbol{H}(\mathfrak{M})$ in $\mathfrak{M}$, then a $S_{\tilde{\pi}}$-subgroup $\mathfrak{P}$ of $\mathfrak{E}$ is an abelian direct factor of $\mathfrak{E}$ by Lemma 26.20. Let $\mathfrak{E} = \mathfrak{P} \times \mathfrak{F}$. If $\mathfrak{F}$ were not a Z-group, then some Sylow subgroup $\mathfrak{F}_r$ of $\mathfrak{F}$ would be non abelian, by Lemma 26.18 and the definition of $\tilde{\pi}$. But then $N(\mathfrak{F}_r) \subseteq \mathfrak{M}$, by Lemma 26.14. Since $N(\mathfrak{F}_r)$ dominates every Sylow subgroup of $\mathfrak{P}$, we would find $\mathfrak{P} \subseteq \mathfrak{M}'$, which is not the case. Hence, $\mathfrak{F}$ is a Z-group.

Let $\mathfrak{F}_0$ be a complement for $\mathfrak{F}'$ in $\mathfrak{F}$, and let $\mathfrak{F}_r$ be the S_r-subgroup of $\mathfrak{F}_0$. Let $\mathfrak{H}^* = \boldsymbol{H}(\mathfrak{M}) \cap C(\Omega_1(\mathfrak{F}_r))$. Since $\mathfrak{H}^*$ is a Z-group, and since $N(\Omega_1(\mathfrak{F}_r))$ dominates every Sylow subgroup of $\mathfrak{P}$, $\mathfrak{P}$ centralizes $\mathfrak{H}^*$. By Lemma 26.13, $\mathfrak{H}^* = 1$. Hence $\mathfrak{F}_0\boldsymbol{H}(\mathfrak{M})$ is a Frobenius group.

Let $\mathfrak{F}_s$ be the S_s-subgroup of $\mathfrak{F}'$, and let $\mathfrak{H}^* = \boldsymbol{H}(\mathfrak{M}) \cap C(\Omega_1(\mathfrak{F}_s))$. If $\mathfrak{H}^*$ is a Z-group, then $\mathfrak{H}^* = 1$ as in the preceding paragraph. If

$\mathfrak{H}^*$ is not a Z-group, then since $N(\Omega_1(\mathfrak{F}_s))$ dominates every Sylow subgroup of $\mathfrak{P}$, we find $\mathfrak{P} \subseteq \mathfrak{M}'$, which is not the case. Hence, $\mathfrak{F}\boldsymbol{H}(\mathfrak{M})$ is a Frobenius group.

If $\mathfrak{F}$ is non abelian, then $m(\boldsymbol{Z}(\mathfrak{H}_r)) \geqq 3$ for every S_r-subgroup $\mathfrak{H}_r$ of $\boldsymbol{H}(\mathfrak{M})$, so that $\boldsymbol{H}(\mathfrak{M})$ is a T.I. set in $\mathfrak{G}$. By Lemma 26.13, $\mathfrak{M}$ is of Frobenius type, so $\mathfrak{M}$ is of type I. If $\mathfrak{F}$ is abelian, $\mathfrak{E}$ is abelian, so $\mathfrak{M}$ is of type I by Lemma 26.19.

In Case 2, let $\mathfrak{E}$ be a complement for $\boldsymbol{H}(\mathfrak{M})$ in $\mathfrak{M}$, let $\mathfrak{E}_p$be a S_p-subgroup of $\mathfrak{E}$, and let $\mathfrak{F}$ be a $S_{p'}$-subgroup of $\mathfrak{E}$. Let $\mathfrak{F}_0$ be a complement for $\mathfrak{F} \cap \mathfrak{M}'$ in $\mathfrak{F}$. Then $\mathfrak{F}_0$ is a S-subgroup of $\mathfrak{M}$, and $\mathfrak{F}_0 = 1$ is a possibility. We can suppose $\mathfrak{F}_0$ is permutable with $\mathfrak{E}_p$, so that $\mathfrak{F}_0$ normalizes $\mathfrak{E}_p$, since by Lemma 26.18, $\mathfrak{F}$ is a Z-group, and $\mathfrak{F}_0 \cap \mathfrak{M}' = 1$.

Let $\mathfrak{E}_p = \mathfrak{A} \times \mathfrak{B}$, where $\mathfrak{A}$ centralizes $\boldsymbol{H}(\mathfrak{M})$, $\mathfrak{B}\boldsymbol{H}(\mathfrak{M})$ is a Frobenius group, $\mathfrak{F}_0$ normalizes both $\mathfrak{A}$ and $\mathfrak{B}$, and $\Omega_1(\mathfrak{B}) \subseteq \boldsymbol{Z}(\mathfrak{P})$ for some S_p-subgroup $\mathfrak{P}$ of $\mathfrak{G}$. By hypothesis, $[\mathfrak{F}_0, \mathfrak{E}_p] \subset \mathfrak{E}_p$.

Suppose $\mathfrak{F}_0 \neq 1$. Let $\mathfrak{F}^* = \mathfrak{F}_0 \cap \boldsymbol{C}(\mathfrak{B})$, and suppose that $1 \subset \mathfrak{F}^* \subset \mathfrak{F}_0$. Let $\mathfrak{F}_0^*$ be a fixed subgroup of $\mathfrak{F}^*$ of prime order. Then $\mathfrak{H}^* = \boldsymbol{H}(\mathfrak{M}) \cap \boldsymbol{C}(\mathfrak{F}_0^*) = \boldsymbol{H}(\mathfrak{M}) \cap \boldsymbol{C}(\mathfrak{F}_0)$ is a Z-group normalized by $\mathfrak{F}_0\mathfrak{B}$. Since $\mathfrak{F}_0\mathfrak{B}$ is non abelian, $\mathfrak{H}^* = 1$. Hence $\mathfrak{F}^*\mathfrak{B}\boldsymbol{H}(\mathfrak{M})$ is a Frobenius group. Since $\mathfrak{F}_0$ is prime on $\boldsymbol{H}(\mathfrak{M})$, $\mathfrak{F}_0\boldsymbol{H}(\mathfrak{M})$ is a Frobenius group. In particular, every subgroup of $\mathfrak{F}_0$ of prime order centralizes $\mathfrak{B}$.

Let $\mathfrak{F}_1 = \mathfrak{F} \cap \mathfrak{M}'$, and suppose that $\mathfrak{F}_1 \neq 1$, so that our running assumptions are: $\mathfrak{F}_0 \neq 1$, $1 \subset \mathfrak{F}^* \subset \mathfrak{F}_0$, $\mathfrak{F}_1 \neq 1$. Suppose $\mathfrak{F}_1\boldsymbol{H}(\mathfrak{M})$ is not a Frobenius group, and let $\mathfrak{F}_1^*$ be a subgroup of prime order such that $\mathfrak{H}^* = \boldsymbol{H}(\mathfrak{M}) \cap \boldsymbol{C}(\mathfrak{F}_1^*) \neq 1$. It follows that $N(\mathfrak{F}_1^*) \subseteq \mathfrak{M}$. But $\mathfrak{F}_1^*$ centralizes $\Omega_1(\mathfrak{E}_p)$, so $\mathfrak{E}_p$ is not a S_p-subgroup of $N(\mathfrak{F}_1^*)$. Hence $\mathfrak{B}\mathfrak{F}\boldsymbol{H}(\mathfrak{M})$ is a Frobenius group, in case $1 \subset \mathfrak{F}^* \subset \mathfrak{F}_0$. Hence, $\mathfrak{M}$ is of Frobenius type in this case. If $\mathfrak{B}\mathfrak{F}$ is non abelian, then $m(\boldsymbol{Z}(\mathfrak{H}_r)) \geqq 3$ for every S_r-subgroup $\mathfrak{H}_r$ of $\boldsymbol{H}(\mathfrak{M})$, $r \in \pi(\boldsymbol{H}(\mathfrak{M}))$, so $\boldsymbol{H}(\mathfrak{M})$ is a T.I. set in $\mathfrak{G}$ and $\mathfrak{M}$ is of type I. If $\mathfrak{B}\mathfrak{F}$ is abelian, and $\boldsymbol{H}(\mathfrak{M})$ is not a T.I set in $\mathfrak{G}$, and $\pi(\boldsymbol{H}(\mathfrak{M})) \not\subseteq \pi_1^*$, then $m(\boldsymbol{H}(\mathfrak{M})) = 2$ and $\boldsymbol{H}(\mathfrak{M})$ is abelian. $\mathfrak{M}$ is of type I in this case.

Suppose now that $\mathfrak{F}_0 = \mathfrak{F}^* \neq 1$. In this case $\mathfrak{A}\mathfrak{F} \lhd \mathfrak{E}$. Since $\mathfrak{B}\boldsymbol{H}(\mathfrak{M})$ is a Frobenius group and $\mathfrak{A}$ centralizes $\boldsymbol{H}(\mathfrak{M})$, it follows readily that $\mathfrak{F}\boldsymbol{H}(\mathfrak{M})$ is a Frobenius group, and that $\mathfrak{M}$ is of type I.

Next suppose $\mathfrak{F}^* = 1$, $\mathfrak{F}_0 \neq 1$. Since $\mathfrak{F}_0$ is prime on $\boldsymbol{H}(\mathfrak{M})$, $\mathfrak{F}_0$ is of prime order. Since $\mathfrak{F}_0$ does not centralize $\mathfrak{B}$, $\mathfrak{F}_0$ does centralize $\mathfrak{A}$. Let $\mathfrak{H}^* = \boldsymbol{H}(\mathfrak{M}) \cap \boldsymbol{C}(\mathfrak{F}_0)$, so that $\mathfrak{H}^* \neq 1$. Since $\mathfrak{A}$ centralizes $\boldsymbol{H}(\mathfrak{M})$, $\mathfrak{A}$ centralizes $\mathfrak{H}^*$. Since $\mathfrak{B}\mathfrak{F}_0$ is non abelian and $\mathfrak{B}\boldsymbol{H}(\mathfrak{M})$ is a Frobenius group, it follows that $\boldsymbol{H}(\mathfrak{M})$ is a T.I. set in $\mathfrak{G}$ and that $\mathfrak{H}^*$ is cyclic.

Let $\mathfrak{M}_1$ be a maximal subgroup of $\mathfrak{G}$ containing $N(\mathfrak{F}_0)$. Then $\mathfrak{M}_1$ is not conjugate to $\mathfrak{M}$. Let $\mathfrak{E}_1$ be a complement to $\boldsymbol{H}(\mathfrak{M}_1)$ which contains $\mathfrak{H}^*$. If $\mathfrak{A} \subseteq \boldsymbol{H}(\mathfrak{M}_1)$, then since $C(\mathfrak{A}) \subseteq \mathfrak{M}$, $\mathfrak{H}^*$ centralizes a non cyclic p-group, which is not the case. Hence, $\mathfrak{A} \not\subseteq \boldsymbol{H}(\mathfrak{M})$, and we can suppose that $\mathfrak{A} \subseteq \mathfrak{E}_1$.

Since $N(\mathfrak{D}) \subseteq \mathfrak{M}$ for every non empty subset $\mathfrak{D}$ of $(\mathfrak{A}\mathfrak{H}^*)^\#$, it follows that $\mathfrak{A}\mathfrak{H}^*$ is prime on $\boldsymbol{H}(\mathfrak{M}_1)$. Let $\mathfrak{H}_0^* = \boldsymbol{H}(\mathfrak{M}) \cap \mathfrak{M}_1$, so that $\mathfrak{H} \subseteq \mathfrak{H}_0^*$, and $\mathfrak{H}_0^*$ is prime on $\boldsymbol{H}(\mathfrak{M}_1)$. Since $N(\mathfrak{F}_0) \subseteq \mathfrak{M}_1$, it follows that $\mathfrak{H}_0^* = \mathfrak{H}^*$.

If $\mathfrak{A}$ is not a S_p-subgroup of $\mathfrak{M}_1$, then $\Omega_1(\mathfrak{B})^M \subseteq \mathfrak{M}_1$ for some M in $\mathfrak{M}$. But then $\Omega_1(\mathfrak{B})^M \boldsymbol{H}(\mathfrak{M}_1)$ is a Frobenius group, as is $\Omega_1(\mathfrak{B})^M \mathfrak{H}^*$, so that $\mathfrak{H}^*$ centralizes $\boldsymbol{H}(\mathfrak{M}_1)$, which is absurd. Hence $\mathfrak{A}$ is a S_p-subgroup of $\mathfrak{M}_1$.

If $\mathfrak{F}_0 \not\subseteq \boldsymbol{H}_1(\mathfrak{M}_1)$, then either $|\mathfrak{F}_0| \in \pi_1$ or a $S_{|\mathfrak{F}_0|}$-subgroup of $\mathfrak{M}_1$ is abelian. But in the first case, $\mathfrak{H}^*$ dominates $\mathfrak{F}_0$, contrary to $\mathfrak{F}_0 \cap \mathfrak{M}' = 1$, while in the second case $\mathfrak{H}^*\mathfrak{A}$ normalizes some $S_{|\mathfrak{F}_0|}$-subgroup $\mathfrak{K}$ of $\mathfrak{M}_1$ with $\mathfrak{F}_0 \subseteq \mathfrak{K}$, and $[\mathfrak{K}, \mathfrak{H}^*\mathfrak{A}]\mathfrak{H}^*\mathfrak{A}$ is a Frobenius group. As $\mathfrak{H}^*\mathfrak{A}$ is prime on $\boldsymbol{H}(\mathfrak{M}_1)$ and $|\mathfrak{H}^*\mathfrak{A}|$ is not a prime, it follows that $[\mathfrak{H}^*\mathfrak{A}, \mathfrak{K}]$ centralizes $\boldsymbol{H}(\mathfrak{M}_1)$. If $\mathfrak{K}$ is a $S_{|\mathfrak{F}_0|}$-subgroup of $\mathfrak{G}$, then $\mathfrak{H}^*\mathfrak{A}$ dominates $\mathfrak{K}$, so $\mathfrak{F}_0 \subseteq \mathfrak{M}'$, which is not the case. Otherwise, a $S_{|\mathfrak{F}_0|}$-subgroup of $\mathfrak{G}$ is non abelian, and $\Omega_1([\mathfrak{K}, \mathfrak{H}^*\mathfrak{A}])$ is contained in the center of some $S_{|\mathfrak{F}_0|}$-subgroup of $\mathfrak{G}$. But $N([\mathfrak{H}^*\mathfrak{A}, \mathfrak{K}]) \subseteq \mathfrak{M}_1$, and a $S_{|\mathfrak{F}_0|}$-subgroup of $\mathfrak{M}_1$ is abelian. Hence, $\mathfrak{F}_0 \subseteq \boldsymbol{H}_1(\mathfrak{M}_1)$.

We next show that $\mathfrak{H}^*\mathfrak{A}$ is a complement to $\boldsymbol{H}(\mathfrak{M}_1)$ in $\mathfrak{M}_1$. Namely, turning back to the definition of $\mathfrak{F}_0$, we have $\mathfrak{F} = \mathfrak{F}_0(\mathfrak{F} \cap \mathfrak{M}')$. But $\mathfrak{B} \subseteq \mathfrak{M}'$, and $\mathfrak{A}$ centralizes $\boldsymbol{H}(\mathfrak{M})$. Hence, $\mathfrak{F} = \mathfrak{F}_0$ or $\mathfrak{F}$ is a Frobenius group with Frobenius kernel $\mathfrak{F} \cap \mathfrak{M}'$. Now, since $\mathfrak{F}_0 \subseteq \boldsymbol{H}_1(\mathfrak{M}_1)$, it follows that $\mathfrak{M}_1 \cap \mathfrak{M} \subseteq \mathfrak{H}^*\mathfrak{A}\boldsymbol{H}(\mathfrak{M}_1)$. This implies that $\mathfrak{H}^*\mathfrak{A}$ has a normal complement in $\mathfrak{E}_1$. If $\mathfrak{H}^*\mathfrak{A} \neq \mathfrak{E}_1$, then $\mathfrak{E}_1$ is a Frobenius group with Frobenius kernel $\mathfrak{E}_1'$ and $\mathfrak{E}_1 = \mathfrak{E}_1'\mathfrak{H}^*\mathfrak{A}$. This is absurd since $\mathfrak{H}^*\mathfrak{A}$ is prime on $\boldsymbol{H}(\mathfrak{M}_1)$, and $|\mathfrak{H}^*\mathfrak{A}|$ is not a prime. Thus $\mathfrak{H}^*\mathfrak{A}$ is a complement to $\boldsymbol{H}(\mathfrak{M}_1)$ in $\mathfrak{M}_1$. Now, however, $\boldsymbol{H}(\mathfrak{M}_1)$ is nilpotent. Since $\mathfrak{F}_0$ has no fixed points on $(\mathfrak{E} \cap \mathfrak{M}')^\#$, it follows that $\mathfrak{M} \cap \mathfrak{M}_1 = \mathfrak{F}_0\mathfrak{H}^*\mathfrak{A}$.

Since $\mathfrak{H}^*\mathfrak{A}$ centralizes $\mathfrak{F}_0$, it follows that $\mathfrak{F}_0 \subseteq \boldsymbol{H}(\mathfrak{M}_1)'$. We next show that $\boldsymbol{H}(\mathfrak{M}_1)$ is a T.I. set in $\mathfrak{G}$. Namely, $|\mathfrak{F}_0|$ divides $p-1$, since $[\mathfrak{B}, \mathfrak{F}_0] = \mathfrak{B}$. Hence $p > |\mathfrak{F}_0|$; since $|\mathfrak{F}_0|$ is a prime, $|\mathfrak{F}_0| \in \pi_0 - \pi^*$, so $\boldsymbol{H}(\mathfrak{M}_1)$ is a T.I. set in $\mathfrak{G}$.

We now turn to $N(\mathfrak{E}_p)$. Let $\mathfrak{M}_2$ be a maximal subgroup of $\mathfrak{G}$ which contains $N(\Omega_1(\mathfrak{B}))$. Then $\mathfrak{M}_2$ is not conjugate to either $\mathfrak{M}$ or $\mathfrak{M}_1$, since the S_p-subgroups of these three maximal subgroups are pairwise non isomorphic. Let $\mathfrak{P}$ be a S_p-subgroup of $\mathfrak{M}_2$ containing $\mathfrak{E}_p$ and normalized by $\mathfrak{F}_0$. If $p \in \pi_2$, then $\mathfrak{F}_0$ does not map onto $N(\mathfrak{P})/\mathfrak{P}C(\mathfrak{P})$, since $\mathfrak{F}_0$ centralizes $\mathfrak{A}$. But then $N(\mathfrak{F}_0)$ covers $N(\mathfrak{P})/\mathfrak{P}C(\mathfrak{P})$. This is not the case since $N(\mathfrak{F}_0) \subseteq \mathfrak{M}_1$, and $\mathfrak{A} \not\subseteq \mathfrak{M}_1'$. Hence, $p \notin \pi_2$, so $p \in \pi_0$, and $\mathfrak{P} \subseteq \boldsymbol{H}_1(\mathfrak{M}_2)$.

Since $C(\mathfrak{F}_0) \cap \boldsymbol{H}_1(\mathfrak{M}_2) \subseteq \mathfrak{M}_1$, and since

$$1 = (|\boldsymbol{H}_1(\mathfrak{M}_2)|, |\boldsymbol{H}(\mathfrak{M}_1)| \cdot |\boldsymbol{H}(\mathfrak{M})|) ,$$

it follows that $C(\mathfrak{F}_0) \cap \boldsymbol{H}_1(\mathfrak{M}_2) = \mathfrak{A}$. Hence, $N(\mathfrak{F}_0) \cap \mathfrak{M}_2$ normalizes $\mathfrak{A}$. But $N(\mathfrak{F}_0) \cap N(\mathfrak{A}) = \mathfrak{F}_0\mathfrak{A}\mathfrak{H}^*$. (This turns the tide.) Suppose $N(\mathfrak{F}_0) \cap \mathfrak{M}_2 \supset \mathfrak{A}\mathfrak{F}_0$. Then $\mathfrak{M}_2$ contains a non identity subgroup $\mathfrak{H}^{**}$ of $\mathfrak{H}^*$. But $\boldsymbol{H}(\mathfrak{M}_2)$ contains $\mathfrak{B}$, and we find that $[\mathfrak{H}^{**}, \mathfrak{B}] = \mathfrak{H}^{**} \subseteq \boldsymbol{H}(\mathfrak{M}_2)$, which is not the case. Hence $N(\mathfrak{F}_0) \cap \mathfrak{M}_2 = \mathfrak{A}\mathfrak{F}_0$.

By Lemma 26.17, $\mathfrak{M}_2$ has p-length one. Let $\mathfrak{K}_2 = \boldsymbol{O}_{p'}(\mathfrak{M}_2)$, so that $\mathfrak{P}\mathfrak{K}_2/\mathfrak{K}_2 = \bar{\mathfrak{P}} \lhd \bar{\mathfrak{M}}_2 = \mathfrak{M}_2/\mathfrak{K}_2$. Then $\mathfrak{M}_2/\mathfrak{P}\mathfrak{K}_2$ is a Frobenius group whose Frobenius kernel is of index $|\mathfrak{F}_0|$, or else $\mathfrak{M}_2 = \mathfrak{P}\mathfrak{K}_2\mathfrak{F}_0$. In any case, by Lemma 8.16, $\mathfrak{M}_2'$ centralizes $\bar{\mathfrak{P}}/\bar{\mathfrak{P}}'$. But now $\mathfrak{A} \not\subseteq \mathfrak{M}_2'$, which is a contradiction to $\boldsymbol{H}(\mathfrak{M}_2) \subseteq \mathfrak{M}_2'$.

We have now exhausted all possibilities under the assumption that $\mathfrak{F}_0 \neq 1$.

Suppose $\mathfrak{F}_0 = 1$. In this case, $\mathfrak{F} \subseteq \mathfrak{M}'$, $\mathfrak{F}$ is cyclic and $\mathfrak{F}$ is normalized by $\mathfrak{E}_p$. Since $\mathfrak{B}\boldsymbol{H}(\mathfrak{M})$ is a Frobenius group, $\Omega_1(\mathfrak{B})$ centralizes $\mathfrak{F}$, so $\Omega_1(\mathfrak{E}_p)$ centralizes $\mathfrak{F}$. This implies that $\mathfrak{F}\boldsymbol{H}(\mathfrak{M})$ is a Frobenius group, or $\mathfrak{F} = 1$. In both cases, $\mathfrak{M}$ is of Frobenius type. If $\mathfrak{F} \neq 1$, then $\mathfrak{B}\mathfrak{F}$ is non abelian, so $m(\boldsymbol{Z}(\mathfrak{H}_r)) \geqq 3$ for every S_r-subgroup $\mathfrak{H}_r$ of $\boldsymbol{H}(\mathfrak{M})$, $r \in \pi(\boldsymbol{H}(\mathfrak{M}))$, and $\boldsymbol{H}(\mathfrak{M})$ is a T.I. set in $\mathfrak{G}$. If $\mathfrak{F} = 1$, then $\mathfrak{E} = \mathfrak{E}_p$ is abelian, and the lemma follows from Lemma 26.19.

LEMMA 26.22. *Let $\mathscr{X}$ be the set of Z-subgroups $\mathfrak{Z}$ of $\mathfrak{G}$ with the following properties:*

(i) *If p, q are primes, every subgroup of $\mathfrak{Z}$ of order pq is cyclic.*

(ii) *$\mathfrak{Z} = \mathfrak{Z}_1 \times \mathfrak{Z}_2$, $|\mathfrak{Z}_i| = z_i \neq 1$, $i = 1, 2$ and for any non empty subset $\mathfrak{Z}_0$ of $\mathfrak{Z} - \mathfrak{Z}_1 - \mathfrak{Z}_2$, $N(\mathfrak{Z}_0) \subseteq \mathfrak{Z}$.*

Then $\mathscr{X}$ is empty or consists of a unique conjugate class of subgroups.

Proof. If $\mathfrak{Z} \in \mathscr{X}$, and $\mathfrak{Z} = \mathfrak{Z}_1 \times \mathfrak{Z}_2$ satisfies (i) and (ii), then $\hat{\mathfrak{Z}} = \mathfrak{Z} - \mathfrak{Z}_1 - \mathfrak{Z}_2$ contains $(z_1 - 1)(z_2 - 1)$ elements. Since $\mathfrak{Z}$ is a Z-group, $(z_1, z_2) = 1$. $\hat{\mathfrak{Z}}$ is clearly a normal subset of $\mathfrak{Z}$, so $N(\hat{\mathfrak{Z}}) = \mathfrak{Z}$. Suppose $G \in \mathfrak{G}$ and $Z \in \hat{\mathfrak{Z}} \cap \hat{\mathfrak{Z}}^G$. Then there is a power of Z, say $Z_1 = Z^k$ such that $Z_1 \in \hat{\mathfrak{Z}} \cap \hat{\mathfrak{Z}}^G$ and such that Z_1 has order p_1p_2 where p_i is a prime divisor of $|\mathfrak{Z}_i| = z_i$. Then $\langle \mathfrak{Z}_1 \rangle \lhd \langle \mathfrak{Z}, \mathfrak{Z}^G \rangle$ and so $\mathfrak{Z} = \mathfrak{Z}^G$, $G \in \mathfrak{Z}$. Thus, the number of elements of $\mathfrak{G}$ which are conjugate to an element of $\hat{\mathfrak{Z}}$ is

$$\frac{|\mathfrak{G}|}{|\mathfrak{Z}|}(z_1 - 1)(z_2 - 1) > \frac{|\mathfrak{G}|}{2} . \tag{26.7}$$

Suppose $\mathfrak{Z}^*$ is another subgroup of $\mathscr{Z}$ and $\mathfrak{Z}^* = \mathfrak{Z}_1^* \times \mathfrak{Z}_2^*$ satisfies (i) and (ii). Set $\hat{\mathfrak{Z}}^* = \mathfrak{Z}^* - \mathfrak{Z}_1^* - \mathfrak{Z}_2^*$. We can assume that $\hat{\mathfrak{Z}}^* \cap \hat{\mathfrak{Z}} \neq \varnothing$, by (26.7), and it follows that $\mathfrak{Z}^* = \mathfrak{Z}$. The proof is complete.

LEMMA 26.23. *Let $\mathfrak{M} \in \mathscr{M}$, and suppose $\mathfrak{M}'$ is a S-subgroup of $\mathfrak{M}$, $|\mathfrak{M} : \mathfrak{M}'|$ is not a prime, and $\mathfrak{M}/\mathfrak{M}'$ is cyclic. Then $\mathfrak{M}$ is of type I or V, or $\mathfrak{M}$ has the following properties:*

(i) *$\boldsymbol{H}(\mathfrak{M})$ is a nilpotent T.I. set in $\mathfrak{G}$.*

(ii) *If $\mathfrak{E}$ is a complement for $\boldsymbol{H}(\mathfrak{M})$ in $\mathfrak{M}$ then*

  (a) *$\mathfrak{E}$ is a non abelian Z-group and every subgroup of $\mathfrak{E}$ of order pq is cyclic, p, q primes.*

  (b) *$\mathfrak{E}$ is prime on $\boldsymbol{H}(\mathfrak{M})$, and $\mathfrak{E}_1 = \boldsymbol{H}(\mathfrak{M}) \cap \boldsymbol{C}(\mathfrak{E})$ is a non identity cyclic group.*

(iii) *$\mathfrak{E}\mathfrak{E}_1 = \mathfrak{Z}$ satisfies the hypotheses of Lemma 26.22.*

Proof. If $\mathfrak{M}' = \boldsymbol{H}(\mathfrak{M})$, the lemma follows from Lemma 26.19. We can therefore suppose that $\boldsymbol{H}(\mathfrak{M}) \subset \mathfrak{M}'$. Let $\mathfrak{E}$ be a complement for $\boldsymbol{H}(\mathfrak{M})$ in $\mathfrak{M}$, let $\mathfrak{F}$ be a complement to $\mathfrak{E}_0 = \mathfrak{E} \cap \mathfrak{M}'$ in $\mathfrak{E}$. Then $\mathfrak{F}$ is a cyclic S-subgroup of $\mathfrak{M}$, and $|\mathfrak{F}|$ is not a prime.

If $\mathfrak{M}$ is a Frobenius group, then $m(\boldsymbol{Z}(\mathfrak{H}_r)) \geqq 3$ for every non identity S_r-subgroup $\mathfrak{H}_r$ of $\boldsymbol{H}(\mathfrak{M})$, so $\boldsymbol{H}(\mathfrak{M})$ is a T.I. set in $\mathfrak{G}$, and we are done. We can suppose that $\mathfrak{M}$ is not a Frobenius group.

Suppose $\mathfrak{F}\boldsymbol{H}(\mathfrak{M})$ is a Frobenius group with Frobenius kernel $\boldsymbol{H}(\mathfrak{M})$. With this hypothesis, we will show that $\mathfrak{M}$ is of type I.

Let $\mathfrak{E}_p$ be a cyclic S_p-subgroup of $\mathfrak{E}_0$. Suppose $\mathfrak{H}^* = \boldsymbol{H}(\mathfrak{M}) \cap \boldsymbol{C}(\Omega_1(\mathfrak{E}_p)) \neq 1$. Then $\mathfrak{E}_p\mathfrak{F}_0$ normalizes $\mathfrak{H}^*$. Consider $\boldsymbol{N}(\Omega_1(\mathfrak{E}_p)) \supseteq \langle \mathfrak{H}^*, \mathfrak{E}_p, \mathfrak{F} \rangle$. Since $|\mathfrak{F}|$ is not a prime and $\mathfrak{F}\mathfrak{H}^*$ is a Frobenius group, it follows that $\boldsymbol{N}(\Omega_1(\mathfrak{E}_0)) \subseteq \mathfrak{M}$. Hence, $\mathfrak{E}_p$ is a S_p-subgroup of $\mathfrak{G}$. Since $\mathfrak{E}_p$ does not centralize $\boldsymbol{H}(\mathfrak{M})$, it follows that every subgroup of $\mathfrak{F}$ of prime order centralizes $\mathfrak{E}_p$. Since $\mathfrak{E}_p \subseteq \mathfrak{M}'$, $|\mathfrak{F}|$ is not square free, and $\mathfrak{F}$ contains a S_q-subgroup $\mathfrak{F}_q$ such that $[\mathfrak{E}_p, \mathfrak{F}_q] \neq 1$. Consider $\boldsymbol{N}(\Omega_1(\mathfrak{F}_q))$. If $q \in \pi_0$, then $[\mathfrak{F}_q, \mathfrak{E}_p] = 1$. If $q \in \pi_1$ or $q \in \pi_2$ and a S_q-subgroup of $\mathfrak{G}$ is non abelian, then $\mathfrak{F}_q \subseteq \boldsymbol{N}(\Omega_1(\mathfrak{F}_q))'$, so once again $[\mathfrak{F}_q, \mathfrak{E}_p] = 1$. If $q \in \pi_2$ and a S_q-subgroup of $\mathfrak{G}$ is abelian, then $\boldsymbol{N}(\Omega_1(\mathfrak{E}_p))$ contains a S_q-subgroup of $\mathfrak{G}$, contrary to $\boldsymbol{N}(\Omega_1(\mathfrak{E}_p)) \subseteq \mathfrak{M}$. Hence $\mathfrak{H}^* = 1$ and $\mathfrak{E}_p\boldsymbol{H}(\mathfrak{M})$ is a Frobenius group.

Since $\mathfrak{M}$ is not a Frobenius group, $\mathfrak{E}_0$ contains a non cyclic S_p-subgroup $\mathfrak{E}_p$ for some prime p. If $\mathfrak{E}_p$ is abelian, and a S_p-subgroup of $\mathfrak{G}$ is non abelian, then $\mathfrak{E} = \mathfrak{E}_p \cdot \mathfrak{E}_{p'}$, and $\mathfrak{E}_{p'}$ is a Z-group. In this case, $\mathfrak{E}_{p'}\boldsymbol{H}(\mathfrak{M})$ is a Frobenius group, and so $\mathfrak{M}$ is of type I. If $\mathfrak{E}_p$ is abelian, and a S_p-subgroup of $\mathfrak{G}$ is abelian, then $\mathfrak{E}_p$ is a S_p-subgroup of $\mathfrak{G}$. In this case, every subgroup of $\mathfrak{F}$ of prime order centralizes $\mathfrak{E}_p/\mathfrak{E}_p \cap \boldsymbol{C}(\boldsymbol{H}(\mathfrak{M}))$, so centralizes $\mathfrak{E}_p^*$ for some non identity subgroup of $\mathfrak{E}_p$. Since $p \in \pi_2$ and a S_p-subgroup of $\mathfrak{G}$ is abelian, it follows that if $\mathfrak{F}_q$ is a S_q-subgroup of $\mathfrak{F}$ which does not centralize $\mathfrak{E}_p^*$, then $q \in \pi_2$, a S_q-subgroup of $\mathfrak{G}$ is abelian, and $\mathfrak{E}_p$ is normalized by a S_q-subgroup $\mathfrak{Q}$ of $\mathfrak{G}$ with $\mathfrak{F}_q \subset \mathfrak{Q}$. Since $C(\Omega_1(\mathfrak{E}_p)) \subset \mathfrak{M}$, $C(\mathfrak{E}_p) \cap \mathfrak{Q} \subseteq \mathfrak{F}_q$. Since $\mathfrak{Q}$ is of type (q^a, q^b), $ab > 0$, there is a direct factor of $\mathfrak{Q}$ which normalizes every subgroup of $\mathfrak{E}_p$. Hence, $\mathfrak{F}_q$ is this direct factor. Hence, q divides $p - 1$, so we have $\mathfrak{E}_p = \mathfrak{E}_{p1} \times \mathfrak{E}_{p2}$, where $\mathfrak{E}_{pi}$ is normalized by $\mathfrak{Q}$. It follows that $\mathfrak{E}_{pi}\boldsymbol{H}(\mathfrak{M})$ is a Frobenius group for $i = 1, 2$.

Suppose every Sylow subgroup of $\mathfrak{E}$ is abelian. Let $\tilde{\pi}$ be the subset of p in $\pi(\mathfrak{E})$ such that a S_p-subgroup of $\mathfrak{E}$ is non cyclic, and let $\mathfrak{P}$ be a $S_{\tilde{\pi}}$-subgroup of $\mathfrak{E}$. By Lemma 26.18 and the preceding paragraph, $\mathfrak{P}$ is a normal abelian subgroup of $\mathfrak{E}$. Hence, $\mathfrak{M}$ is of Frobenius type. Since $\mathfrak{E}$ is non abelian, $\boldsymbol{H}(\mathfrak{M})$ is a T.I. set in $\mathfrak{G}$, so $\mathfrak{M}$ is of type I.

Thus, if $\mathfrak{F}\boldsymbol{H}(\mathfrak{M})$ is a Frobenius group and every Sylow subgroup of $\mathfrak{E}$ is abelian, then $\mathfrak{M}$ is of type I.

Suppose $\mathfrak{F}\boldsymbol{H}(\mathfrak{M})$ is a Frobenius group, and $\mathfrak{E}_p$ is a non abelian S_p-subgroup of $\mathfrak{E}$. Then $\mathfrak{E}_p$ is a S_p-subgroup of $\mathfrak{G}$ and $p \in \pi_2$. Since every subgroup of $\mathfrak{F}$ of prime order centralizes $\mathfrak{E}_p/\mathfrak{E}_p \cap C(\boldsymbol{H}(\mathfrak{M}))$, and since $\mathfrak{E}_p \not\subseteq C(\boldsymbol{H}(\mathfrak{M}))$, Lemma 26.9 implies that $\mathfrak{F}$ centralizes $\mathfrak{E}_p/\mathfrak{E}_p \cap C(\boldsymbol{H}(\mathfrak{M}))$. This violates the containment $\mathfrak{E}_p \subseteq \mathfrak{M}'$. Hence, if $\mathfrak{F}\boldsymbol{H}(\mathfrak{M})$ is a Frobenius group, $\mathfrak{M}$ is of type I.

Suppose now that $\mathfrak{F}\boldsymbol{H}(\mathfrak{M})$ is not a Frobenius group. Let $\mathfrak{E}_1 = C(\mathfrak{F}) \cap \boldsymbol{H}(\mathfrak{M})$. By Lemma 26.15, $\mathfrak{E}_1$ is a Z-group. By Lemma 26.3, $\boldsymbol{H}(\mathfrak{M})$ is nilpotent so $\mathfrak{E}_1$ is cyclic. Since every subgroup of $\mathfrak{F}$ of prime order centralizes $\mathfrak{E}'/\mathfrak{E}' \cap C(\boldsymbol{H}(\mathfrak{M}))$, it follows that $\mathfrak{E}$ normalizes $\mathfrak{E}_1$, so centralizes $\mathfrak{E}_1$ since Aut $\mathfrak{E}_1$ is abelian. Hence, $\mathfrak{E}_1 \subseteq \boldsymbol{H}(\mathfrak{M})'$.

Since every subgroup of $\mathfrak{F}$ of prime order centralizes $\mathfrak{E}'/\mathfrak{E}' \cap C(\boldsymbol{H}(\mathfrak{M}))$, it follows that $\mathfrak{E}'$ is abelian. Suppose $\mathfrak{E}'$ were non cyclic. Let $\mathfrak{E}_p$ be a non cyclic S_p-subgroup of $\mathfrak{E}'$. By Lemma 26.12, together with $\mathfrak{E}_1 \neq 1$, $\mathfrak{E}_p$ is a S_p-subgroup of $\mathfrak{G}$.

Let $\mathfrak{F}_q$ be a S_q-subgroup of $\mathfrak{F}$ which does not centralize $\mathfrak{E}_p/\mathfrak{E}_p \cap C(\boldsymbol{H}(\mathfrak{M}))$, and let $\mathfrak{E}_p^* = \mathfrak{E}_p \cap C(\Omega_1(\mathfrak{F}_q)) \neq 1$. Then $\mathfrak{N} = N(\Omega_1(\mathfrak{F}_q)) \supseteq \langle \mathfrak{F}, \mathfrak{E}_p^*, \mathfrak{E}_1 \rangle$. It follows now from $\mathfrak{E}_1 \subseteq \boldsymbol{H}(\mathfrak{M})' \subseteq \hat{\boldsymbol{H}}(\mathfrak{M}) \cup \{1\}$ that either $\mathfrak{F}_q$ is not a S_q-subgroup of $\mathfrak{G}$ or $\mathfrak{F}_q \subseteq \mathfrak{M}'$, both of which are false. Hence, $\mathfrak{E}'$ is cyclic. This yields that every subgroup of $\mathfrak{E}$ of order pq is cyclic, p, q being primes.

We next show that $\mathfrak{E}$ is prime on $\boldsymbol{H}(\mathfrak{M})$. Since $C(E) \supseteq C(\mathfrak{F}) \cap \boldsymbol{H}(\mathfrak{M}) = \mathfrak{E}_1$, for all $E \in \mathfrak{E}$, it suffices to show that $\mathfrak{E}_1 = C(E) \cap \boldsymbol{H}(\mathfrak{M})$ for all $E \in \mathfrak{E}^\sharp$. Suppose false and $\mathfrak{E}_p$ is a S_p-subgroup of $\mathfrak{E}$ such that $C(\Omega_1(\mathfrak{E}_p)) \cap \boldsymbol{H}(\mathfrak{M}) = \mathfrak{E}_2 \supset \mathfrak{E}_1$. Since $\mathfrak{F}\mathfrak{E}_2/\mathfrak{E}_1$ is a Frobenius group, it follows that $\mathfrak{E}_p$ is a S_p-subgroup of $\mathfrak{G}$ and $N(\mathfrak{E}_p) \subseteq \mathfrak{M}$. In this case, let $\mathfrak{F}_q$ be a S_q-subgroup of $\mathfrak{F}$ which does not centralize $\mathfrak{E}_p$ and consider $N(\Omega_1(\mathfrak{F}_q)) \supseteq \langle \mathfrak{E}_p, \mathfrak{F}\rangle$. If $q \in \pi_0$, Lemma 26.9 is violated; if $q \in \pi_1$, then $\mathfrak{F}_q \subseteq N(\Omega_1(\mathfrak{F}_q))'$ so $[\mathfrak{F}_q, \mathfrak{E}_p] = 1$; if $q \in \pi_2$, $\mathfrak{F}_q$ is not a S_q-subgroup of $N(\mathfrak{E}_p)$, contrary to $N(\mathfrak{E}_p) \subseteq \mathfrak{M}$. Hence, $\mathfrak{E}$ is prime on $H(\mathfrak{M})$, and so $\mathfrak{E}_1 = C(E) \cap H(\mathfrak{M})$ for all $E \in \mathfrak{E}^{\#}$. Since $\mathfrak{E}$ is non abelian, $H(\mathfrak{M})$ is a T.I. set in $\mathfrak{G}$.

Let $\mathfrak{Z} = \mathfrak{E}\mathfrak{E}_1$, and let $\hat{\mathfrak{Z}} = \mathfrak{E}\mathfrak{E}_1 - \mathfrak{E} - \mathfrak{E}_1$. By construction, $\mathfrak{E} \neq 1$, $\mathfrak{E}_1 \neq 1$, and $N(\hat{\mathfrak{Z}}) \cap \mathfrak{M} = \mathfrak{Z}$. Since $\mathfrak{E}_1 \subseteq H(\mathfrak{M})' \subseteq \hat{H}(\mathfrak{M}) \cup \{1\}$, $N(\mathfrak{Z}_0) \subseteq \mathfrak{M}$ for every non empty subset $\mathfrak{Z}_0$ of $\mathfrak{E}_1^{\#}$. Since $(|\mathfrak{E}|, |\mathfrak{E}_1|) = 1$, this implies that $N(\hat{\mathfrak{Z}}) = \mathfrak{Z}$ and $N(\hat{\mathfrak{Z}}_0) \subseteq \mathfrak{Z}$ for every non empty subset $\hat{\mathfrak{Z}}_0$ of $\hat{\mathfrak{Z}}$. Thus, $\mathfrak{Z}$ satisfies the hypotheses of Lemma 26.22. The proof is complete.

LEMMA 26.24. *Suppose $\mathfrak{M} \in \mathscr{M}$ and $\mathfrak{M}$ is of type* V. *Then $\mathfrak{M}'$ is tamely imbedded in $\mathfrak{G}$.*

Proof. We can suppose that $\mathfrak{M}'$ is not a T.I. set in $\mathfrak{G}$. Let $\mathfrak{E}_1 = \mathfrak{M}' \cap C(\mathfrak{E})$, where $\mathfrak{E}$ is a complement to $\mathfrak{M}'$ in $\mathfrak{M}$. Then $\mathfrak{E}_1 \neq 1$, and $\mathfrak{E}_1 \subseteq \mathfrak{M}''$. Hence, $\mathfrak{M}'$ is non abelian. Let $\mathfrak{M}' = \mathfrak{P} \times \mathfrak{S}_0$, where $\mathfrak{P}$ is a non abelian S_p-subgroup of $\mathfrak{M}'$, and $\mathfrak{S}_0$ is the $S_{p'}$-subgroup of $\mathfrak{M}'$ for some prime p (there may be several).

We show that $\mathfrak{S}_0$ is a T.I. set in $\mathfrak{G}$. If $\mathfrak{S}_0 = 1$, this is the case. Suppose $\mathfrak{S}_0 \neq 1$, and $S \in \mathfrak{S}_0 \cap \mathfrak{S}_0^G$, $S \neq 1$. Then $C(S) \supseteq \langle \mathfrak{P}, \mathfrak{P}^G\rangle$. Let $\mathfrak{M}_1$ be a maximal subgroup of $\mathfrak{G}$ containing $C(S)$. By Lemma 26.14, $N(\Omega_1(Z(\mathfrak{P}))) \subseteq \mathfrak{M}_1$. Hence $\mathfrak{M} \subseteq \mathfrak{M}_1$, so $\mathfrak{M} = \mathfrak{M}_1 \supseteq \mathfrak{P}^G$ and so $\mathfrak{P} = \mathfrak{P}^G$ and $G \in \mathfrak{M}$.

Since $\mathfrak{M}'$ is not a T.I. set in $\mathfrak{G}$, it follows that $\mathfrak{S}_0$ is cyclic.

Suppose $M \in \mathfrak{M}'$, $M \neq 1$, and $C(M) \not\subseteq \mathfrak{M}$. Since every subgroup of $\mathfrak{S}_0$ is normal in $\mathfrak{M}$, it follows that $M \in \mathfrak{P}$. Furthermore, $\langle M\rangle \cap T(\mathfrak{P}) = \langle 1\rangle$, so M is of order p, and $C_{\mathfrak{M}}(M) = \langle M\rangle \times \mathfrak{B} \times \mathfrak{S}_0$, where $\mathfrak{B}$ is a non identity cyclic subgroup of $\mathfrak{P}$, and $\mathfrak{B} \supseteq \Omega_1(Z(\mathfrak{P}))$. (Notice that since $M \notin \mathfrak{M}''$, $C_{\mathfrak{M}}(M) \subseteq \mathfrak{M}'$.)

Let $\mathfrak{M}_1$ be a maximal subgroup of $\mathfrak{G}$ containing $C(M)$. Then a S_p-subgroup of $\mathfrak{M}_1$ is abelian, by Lemma 26.14, so $\langle M\rangle \times \mathfrak{B}$ is a S_p-subgroup of $\mathfrak{M}_1$, by Lemma 26.6. By Lemma 26.12 $\mathfrak{B}H(\mathfrak{M}_1)$ is a Frobenius group.

Let $\mathfrak{L}$ be a complement to $H(\mathfrak{M}_1)$ in $\mathfrak{M}_1$ which contains $C_{\mathfrak{M}}(M)$. Since $\mathfrak{B}H(\mathfrak{M}_1)$ is a Frobenius group, it follows that $\langle M\rangle \times \Omega_1(\mathfrak{B}) \lhd \mathfrak{L}$. This implies that $\mathfrak{L} \subseteq \mathfrak{M}$, so that $\mathfrak{L} = \mathfrak{M} \cap \mathfrak{M}_1$.

We next show that $(|\mathfrak{M}|, |H(\mathfrak{M}_1)|) = 1$. This is equivalent to showing that $(|\mathfrak{E}|, |H(\mathfrak{M}_1)|) = 1$. Suppose false and q is a prime divisor of $(|\mathfrak{E}|, |H(\mathfrak{M}_1)|)$. Since $p \in \pi^*$, q divides $p+1$ or $p-1$. Since p divides $|\mathfrak{M}_1 : H(\mathfrak{M}_1)|$, and $\mathfrak{B}H(\mathfrak{M}_1)$ is a Frobenius group, $q \in \pi_0 - \pi^*$. Thus, if Q is any element of $\mathfrak{G}$ of order q, then $C(Q)$

is contained in a unique maximal subgroup of $\mathfrak{G}$. Let Q be an element of $\mathfrak{E}$ of order q, and let $\mathfrak{M}_2 = \boldsymbol{M}(C(Q))$. Then $\mathfrak{E}\mathfrak{E}_1 \subseteq \mathfrak{M}_2$. Since $q \in \pi_0 - \pi^*$, $\mathfrak{M}_2$ is conjugate to $\mathfrak{M}_1$ in $\mathfrak{G}$. Since $\mathfrak{E}\mathfrak{S}_0$ is a Frobenius group or $\mathfrak{S}_0 = 1$, $\mathfrak{E}_1$ is a p-group. We can thus find G in $\mathfrak{G}$ such that $\mathfrak{M}_2^G = \mathfrak{M}_1$, and we can suppose that $\langle \mathfrak{E}_1^G, M, \mathfrak{B}\rangle$ is a p-group. This implies that $\mathfrak{E}_1^G \subseteq \mathfrak{M}$, so that $G \in \mathfrak{M}$. Since $\langle M, \mathfrak{B}\rangle$ is a S_p-subgroup of $\mathfrak{M}_1$, we have $\mathfrak{E}_1^G \subseteq \langle M, \mathfrak{B}\rangle$. Since $\mathfrak{E}_1 \subseteq \mathfrak{M}''$ and $G \in \mathfrak{M}$, $\mathfrak{E}_1^G \subseteq \mathfrak{M}'' \cap \langle M, \mathfrak{B}\rangle$, and so $\Omega_1(\mathfrak{E}_1^G) = \Omega_1(\mathfrak{B})$. But now $[\Omega_1(\mathfrak{E}_1^G), \mathfrak{E}^G] = 1$, contrary to $Q^G \in \boldsymbol{H}(\mathfrak{M}_1)$ and $\Omega_1(\mathfrak{B})\boldsymbol{H}(\mathfrak{M}_1)$ a Frobenius group. Hence, $(|\mathfrak{M}|, |\boldsymbol{H}(\mathfrak{M}_1)|) = 1$.

By construction, $C(M) \subseteq \mathfrak{M}_1$. We next show that $N_{\mathfrak{M}}(\langle M\rangle)$ is a complement to $\boldsymbol{H}(\mathfrak{M}_1)$ in $\mathfrak{M}_1$. Since $\mathfrak{L} = \mathfrak{M} \cap \mathfrak{M}_1$, it follows that $\langle M\rangle \lhd \mathfrak{L}$, since $\langle M, \mathfrak{B}\rangle \lhd \mathfrak{L}$ and $\langle M\rangle \subseteq C(\boldsymbol{H}(\mathfrak{M}_1))$. Thus, $\mathfrak{L} = N_{\mathfrak{M}}(\langle M\rangle)$.

We next show that two elements of $\mathfrak{M}'$ are conjugate in $\mathfrak{G}$ if and only if they are conjugate in $\mathfrak{M}$. Let M, $M_1 \in \mathfrak{M}'^{\sharp}$, and $M = M_1^G$, $G \in \mathfrak{G}$. Since $\mathfrak{S}_0$ is a T.I. set in $\mathfrak{G}$, we can suppose $M, M_1 \in \mathfrak{P}$. If $M \in \hat{\boldsymbol{H}}(\mathfrak{M})$, then $C(M) \subseteq \mathfrak{M}$, so $\mathfrak{P}^G \cap \mathfrak{M}$ is non cyclic, and so $G \in \mathfrak{M}$. We can suppose $M \notin \hat{\boldsymbol{H}}(\mathfrak{M})$. In this case $C_{\mathfrak{P}}(M)$ is a S_p-subgroup of $C(M)$. Now $C(M) \supseteq \langle \Omega_1(\boldsymbol{Z}(\mathfrak{P})), \Omega_1(\boldsymbol{Z}(\mathfrak{P}^G))\rangle$, so we can find $C \in C(M)$ so that $\Omega_1(\boldsymbol{Z}(\mathfrak{P}^G))^C \subseteq C_{\mathfrak{P}}(M)$. As observed earlier, this implies that $\Omega_1(\boldsymbol{Z}(\mathfrak{P}^G))^C = \Omega_1(\boldsymbol{Z}(\mathfrak{P}))$. Since $\Omega_1(\boldsymbol{Z}(\mathfrak{P}^G))^C = \Omega_1(\boldsymbol{Z}(\mathfrak{P}))^{GC}$, and $\mathfrak{M} = N(\Omega_1(\boldsymbol{Z}(\mathfrak{P})))$, we have $GC \in \mathfrak{M}$. Then $M_1^{GC} = M^C$, so M and M_1 are conjugate in $\mathfrak{M}$, namely, by GC, since $C \in C(M)$.

Let $M_1, \cdots, M_m$ be a set of representatives for the conjugate classes $\mathfrak{C}_1, \cdots, \mathfrak{C}_m$ of elements in $\mathfrak{M}$ which are in $\mathfrak{M}'^{\sharp}$ and satisfy $C(M_i) \not\subseteq \mathfrak{M}$, $1 \leq i \leq m$. As we saw in the preceding paragraph, $C(M_i)$ is contained in a unique maximal subgroup of $\mathfrak{G}$, for each i, in fact, $N(\langle M_i\rangle)$ is the unique maximal subgroup of $\mathfrak{G}$ which contains $C(\langle M_i\rangle)$. Let $\mathfrak{N}_i = N(\langle M_i\rangle)$, $1 \leq i \leq m$, and suppose notation is chosen so that $\mathfrak{N}_1, \cdots, \mathfrak{N}_n$ are non conjugate in $\mathfrak{G}$, while $\mathfrak{N}_j$ is conjugate to some $\mathfrak{N}_i$ with $1 \leq i \leq n$, if $n + 1 \leq j \leq m$. Set $\mathfrak{H}_i = \boldsymbol{H}(\mathfrak{N}_i)$, $1 \leq i \leq n$, so that $(|\mathfrak{H}_i|, |\mathfrak{H}_j|) = 1$ if $1 \leq i, j \leq n, i \neq j$.

Let

$$\hat{\mathfrak{N}}_i = \bigcup_{H \in \mathfrak{H}_i^{\sharp}} C_{\mathfrak{N}_i}(H) - \mathfrak{H}_i^{\sharp} .$$

Since $M_i\mathfrak{H}_i \subseteq \hat{\mathfrak{N}}_i$, it follows that $N(\hat{\mathfrak{N}}_i) = \mathfrak{N}_i$. Also, $\mathfrak{N}_i = \mathfrak{H}_i(\mathfrak{N}_i \cap \mathfrak{M})$ and $\mathfrak{H}_i \cap \mathfrak{M} = 1$. If $\mathfrak{N}_i \cap \mathfrak{M} \subseteq \mathfrak{M}'$, then $\mathfrak{N}_i \cap \mathfrak{M}$ is abelian, and in fact $\mathfrak{N}_i \cap \mathfrak{M} = \langle M_i\rangle \times \mathfrak{B}_i \times \mathfrak{S}_0$, where $\mathfrak{B}_i$ is a cyclic subgroup of $\mathfrak{P}$. Since $(\mathfrak{B}_i \times \mathfrak{S}_0)\mathfrak{H}_i$ is a Frobenius group,

$$\hat{\mathfrak{N}}_i = \bigcup_{M \in \langle M_i\rangle^{\sharp}} M\mathfrak{H}_i \cup \{1\} , \tag{26.8}$$

so is a T.I. set in $\mathfrak{G}$.

Suppose $\mathfrak{N}_i \cap \mathfrak{M} \not\subseteq \mathfrak{M}'$. Then $\mathfrak{N}_i \cap \mathfrak{M}' \lhd \mathfrak{N}_i \cap \mathfrak{M}$, and $\mathfrak{N}_i \cap \mathfrak{M} = (\mathfrak{N}_i \cap \mathfrak{M}')\cdot\mathfrak{F}$, where $\mathfrak{F} \cap \mathfrak{M}' = 1$, and $\mathfrak{F}\langle M_i\rangle$ is a Frobenius group so that $|\mathfrak{F}|$ divides $p-1$. Now $\mathfrak{F}$ normalizes $\mathfrak{B}_i \times \mathfrak{S}_0$. ($\mathfrak{B}_i$ can be so chosen.) If $\mathfrak{F}\mathfrak{B}_i\mathfrak{S}_0$ is abelian, then $\mathfrak{F}\mathfrak{B}_i\mathfrak{S}_0\mathfrak{H}_i$ is a Frobenius group by Lemma 26.21, (together with $\mathfrak{F}\langle M_i\rangle$ a Frobenius group), and $\hat{\mathfrak{N}}_i$ is a T.I. set in $\mathfrak{G}$. If $\mathfrak{F}\mathfrak{B}_i\mathfrak{S}_0$ is non abelian, then since $\mathfrak{F}$ is prime on $\mathfrak{M}'$, and $\mathfrak{F}$ is prime on $\mathfrak{H}_i$, $\mathfrak{F}$ is prime on $\mathfrak{B}_i\mathfrak{S}_0\mathfrak{H}_i$. If $|\mathfrak{F}|$ is not a prime, then $[\mathfrak{F}, \mathfrak{B}_i\mathfrak{S}_0]$ centralizes $\mathfrak{H}_i$. Since $\mathfrak{S}_0$ is cyclic and every subgroup of $\mathfrak{S}_0$ is normal in $\mathfrak{M}$, we have $\mathfrak{S}_0 = 1$. But $N(\mathfrak{B}_i) \subseteq \mathfrak{M}$ since $\Omega_1(\mathfrak{B}_i) \subseteq \boldsymbol{Z}(\mathfrak{M}')$. Thus, we can suppose $|\mathfrak{F}|$ is a prime. If $\mathfrak{F}$ centralizes $\mathfrak{B}_i$, Lemma 26.21 implies that $\mathfrak{N}_i$ is of type I. Thus, we can suppose that $\mathfrak{F}\mathfrak{B}_i$ is a Frobenius group. Hence $\mathfrak{F}\mathfrak{B}_i\mathfrak{S}_0$ is a Frobenius group, as is $\mathfrak{F}\langle M_i\rangle\mathfrak{B}_i\mathfrak{S}_0$. Since $\mathfrak{B}_i\mathfrak{H}_i$ is a Frobenius group and $\mathfrak{F}\mathfrak{B}_i$ is also a Frobenius group, $\mathfrak{H}_i$ is a nilpotent T.I. set in $\mathfrak{G}$. Hence $\mathfrak{F}^* = \boldsymbol{C}_{\mathfrak{H}_i}(\mathfrak{F})$ is a non identity cyclic subgroup and $\mathfrak{F}\mathfrak{F}^*$ satisfies the hypotheses of Lemma 26.22 with the obvious factorization $\mathfrak{F}\mathfrak{F}^* = \mathfrak{F} \times \mathfrak{F}^*$. But $\mathfrak{E}\mathfrak{E}_1$ also satisfies the hypotheses of Lemma 26.22, so $\mathfrak{F}\mathfrak{F}^*$ and $\mathfrak{E}\mathfrak{E}_1$ are isomorphic. In particular, p divides $|\mathfrak{F}\mathfrak{F}^*|$, so divides $|\mathfrak{F}^*|$. This is absurd, since p divides $|\mathfrak{B}_i|$ and $\mathfrak{B}_i\mathfrak{H}_i$ is a Frobenius group with Frobenius kernel $\mathfrak{H}_i \supseteq \mathfrak{F}^*$. Hence, this case cannot arise. Hence, $\hat{\mathfrak{N}}_i$ is a T.I. set in $\mathfrak{G}$, and in fact (26.8) holds. Since $\mathfrak{H}_i$ is a S-subgroup of $\mathfrak{N}_i$, we have $\mathfrak{N}_i = N(\hat{\mathfrak{N}}_i)$.

Since $\mathfrak{N}_i$ and $\mathfrak{N}_j$ are not conjugate in $\mathfrak{G}$, $1 \leqq i, j \leqq n, i \neq j$, by construction, we have $(|\mathfrak{H}_i|, |\mathfrak{H}_j|) = 1$ if $i \neq j$. The factorization of $\boldsymbol{C}(M_k)$ is now immediate, $1 \leqq k \leqq m$. We have already shown that $(|\mathfrak{M}|, |\mathfrak{H}_i|) = 1$. Thus, $\mathfrak{M}'$ is tamely imbedded in $\mathfrak{G}$.

Hypothesis 26.1.

(i) *$\mathfrak{S} \in \mathscr{M}$ and $\mathfrak{S}'$ is a S-subgroup of $\mathfrak{S}$.*

(ii) *$|\mathfrak{S} : \mathfrak{S}'| = q$ is a prime and $\mathfrak{Q}^*$ is a complement to $\mathfrak{S}'$ in $\mathfrak{S}$.*

(iii) *$\mathfrak{S}'$ is not nilpotent.*

(iv) *$\mathfrak{H}^* = \boldsymbol{C}_{\mathfrak{S}'}(\mathfrak{Q}^*)$.*

LEMMA 26.25. *Under Hypothesis 26.1, $\mathfrak{H}^*$ is cyclic and $\mathfrak{Q}^*\mathfrak{H}^*$ satisfies the hypotheses of Lemma 26.22 with the factorization $\mathfrak{Q}^*\mathfrak{H}^* = \mathfrak{Q}^* \times \mathfrak{H}^*$; $N(\mathfrak{Q}^*)$ is contained in a unique maximal subgroup $\mathfrak{T}$ of $\mathfrak{G}$; $\mathfrak{S} \cap \mathfrak{T} = \mathfrak{Q}^*\mathfrak{H}^*$; $\mathfrak{Q}^* \subseteq \mathfrak{T}'$; every element of $\mathscr{M}$ is of type I or is conjugate to $\mathfrak{S}$ or $\mathfrak{T}$.*

Proof. Since $\mathfrak{S}'$ is not nilpotent, $\mathfrak{H}^* \neq 1$. Let $\mathfrak{T}$ be any maximal subgroup of $\mathfrak{G}$ containing $N(\mathfrak{Q}^*)$.

Let $\tilde{\pi}$ consist of those p in $\pi(\mathfrak{S}')$ such that either $p \in \pi^*$ or $p \notin \pi(\mathfrak{H}^*)$ or $p \notin \pi(\boldsymbol{H}(\mathfrak{S}))$, and let $\mathfrak{U}$ be a $\mathfrak{Q}^*$-invariant $S_{\tilde{\pi}}$-subgroup of

$\mathfrak{S}$, and let $\mathfrak{H}$ be a $S_{\tilde{\pi}'}$-subgroup of $\mathfrak{S}'$. We will show that $\mathfrak{U}$ is nilpotent and that $\mathfrak{H} \lhd \mathfrak{S}$.

Choose $p \in \tilde{\pi}$ and let $\mathfrak{P}$ be a $\mathfrak{Q}^*$-invariant S_p-subgroup of $\mathfrak{S}$. If $p \in \pi^*$ or $p \notin \pi(\boldsymbol{H}(\mathfrak{S}))$, then $\mathfrak{S}$ has p-length one, by Lemma 26.17. Hence, $\mathfrak{S}'$ centralizes $\boldsymbol{O}_{p',p}(\mathfrak{S})/\boldsymbol{O}_{p'}(\mathfrak{S})$, so $\mathfrak{S}'$ has a normal p-complement. If $p \notin \pi(\mathfrak{H}^*)$, then by 3.16 (i) or Lemma 13.4, $\mathfrak{S}'$ centralizes $\boldsymbol{O}_{p',p}(\mathfrak{S})/\boldsymbol{O}_{p'}(\mathfrak{S})$, so in this case, too, $\mathfrak{S}'$ has a normal p-complement. Hence, $\mathfrak{U}$ is nilpotent and $\mathfrak{H} \lhd \mathfrak{S}$. Since $\mathfrak{S}'$ is not nilpotent, $\mathfrak{H} \neq 1$. Furthermore, $\mathfrak{H}^* \cap \mathfrak{U} \subseteq \mathfrak{U}'$. By construction, $\pi(\mathfrak{H}) \subseteq \pi_0 - \pi^*$, so $N(\hat{\mathfrak{H}}) \subseteq \mathfrak{S}$ for every non empty subset $\hat{\mathfrak{H}}$ of $\mathfrak{H}^\sharp$. Thus, $\mathfrak{H}$ is a T.I. set in $\mathfrak{G}$. Since $\mathfrak{H}^* \cap \mathfrak{U} \subseteq \mathfrak{U}'$, Lemma 26.14 implies that $N(\hat{\mathfrak{H}}) \subseteq \mathfrak{S}$ for every non empty subset $\hat{\mathfrak{H}}$ of $\mathfrak{H}^{*\sharp}$. Thus $\mathfrak{H}^*\mathfrak{Q}^* = \mathfrak{H}^* \times \mathfrak{Q}^*$ satisfies Hypothesis (ii) in Lemma 26.22.

Let $\mathfrak{H}^{**} = \mathfrak{S}' \cap \mathfrak{T} \supseteq \mathfrak{H}^*$. $\mathfrak{T}$ is not conjugate to $\mathfrak{S}$, either because $\mathfrak{Q}^*$ is not a S_q-subgroup of $\mathfrak{G}$ or because $\mathfrak{Q}^* \subseteq \mathfrak{T}'$. Thus, $\mathfrak{H}^{**} \cap \boldsymbol{H}(\mathfrak{T}) = 1$. If $\mathfrak{H}^* \subset \mathfrak{H}^{**}$, then $\mathfrak{Q}^* \not\subseteq \mathfrak{T}'$ since $[\mathfrak{Q}^*, \mathfrak{H}^{**}] \neq 1$. But in that case, some S_q-subgroup of $\mathfrak{T}$ normalizes $\mathfrak{H}^{**}$, so $\mathfrak{Q}^*$ is a S_q-subgroup of $\mathfrak{G}$. But in that case, $\mathfrak{Q}^* \subseteq N(\mathfrak{Q}^*)' \subseteq \mathfrak{T}'$. Hence, $\mathfrak{H}^* = \mathfrak{S}' \cap \mathfrak{T}$, so $\mathfrak{H}^*\mathfrak{Q}^* = \mathfrak{S} \cap \mathfrak{T}$. Since $N(\hat{\mathfrak{H}}) \subseteq \mathfrak{S}$ for every non empty subset $\hat{\mathfrak{H}}$ of $\mathfrak{H}^{*\sharp}$, it follows that $\mathfrak{H}^*$ has a normal complement in $\mathfrak{T}$, say $\mathfrak{T}_1$, and $\mathfrak{T}_1$ is a S-subgroup of $\mathfrak{T}$. Suppose $\mathfrak{Q}^* \not\subseteq \mathfrak{T}'$. Then $\mathfrak{T}_1 \cap \mathfrak{T}'$ is disjoint from $\mathfrak{Q}^*$, $\mathfrak{H}^*(\mathfrak{T}_1 \cap \mathfrak{T}')$ is a Frobenius group, and $\mathfrak{T}_1 = (\mathfrak{T}_1 \cap \mathfrak{T}')\mathfrak{Q}^*$. Furthermore, a $\mathfrak{H}^*$-invariant S_q-subgroup $\mathfrak{Q}$ of $\mathfrak{T}_1$ has a normal complement in $\mathfrak{T}_1$, and $\mathfrak{Q}$ is abelian, by Lemmas 26.10 and 26.11. Thus $\mathfrak{Q}^*$ is a direct factor of $\mathfrak{Q}$, and $\mathfrak{Q}^* \subset \mathfrak{Q}$, since $\mathfrak{Q}^* \not\subseteq \mathfrak{T}'$ and $N(\mathfrak{Q}^*) \subseteq \mathfrak{T}$. If a S_q-subgroup of $\mathfrak{G}$ is abelian, then $N(\mathfrak{H}^*)$ dominates $\mathfrak{Q}$, so $\mathfrak{Q}^* \subseteq \mathfrak{S}'$, which is not the case. If a S_q-subgroup of $\mathfrak{G}$ is non abelian, then since $\mathfrak{T}_1 \cap \mathfrak{T}'$ is nilpotent, $\mathfrak{Q}^*$ is contained in the center of some S_q-subgroup of $\mathfrak{G}$. This is absurd, since $N(\mathfrak{Q}^*) \subseteq \mathfrak{T}$ and $\mathfrak{Q}$ is an abelian S_q-subgroup of $\mathfrak{T}$. Hence, $\mathfrak{Q}^* \subseteq \mathfrak{T}'$.

Again, let $\mathfrak{Q}$ be a S_q-subgroup of $\mathfrak{T}$ normalized by $\mathfrak{H}^*$, and let $\mathfrak{B}$ be a $S_{q'}$-subgroup of $\mathfrak{T}_1$ normalized by $\mathfrak{H}^*$. Then either $\mathfrak{B} = 1$ or $\mathfrak{H}^*\mathfrak{B}$ is a Frobenius group. In both these case, we conclude that $\mathfrak{Q} \lhd \mathfrak{T}$. If $\mathfrak{B}$ does not centralize $\mathfrak{Q}$, then by Lemma 26.16, $q \in \pi_0 - \pi^*$, so $\mathfrak{T}$ is the unique maximal subgroup of $\mathfrak{G}$ containing $N(\mathfrak{Q}^*)$. If $\mathfrak{B}$ centralizes $\mathfrak{Q}$, then $\mathfrak{Q}^* \subseteq \mathfrak{Q}'$, so if $q \in \pi_0$, $\mathfrak{T}$ is the unique maximal subgroup of $\mathfrak{G}$ containing $\mathfrak{Q}^*$. But if $q \notin \pi_0$, then $\mathfrak{Q}^* \lhd \mathfrak{T}$, so of course $\mathfrak{T}$ is the unique maximal subgroup of $\mathfrak{G}$ containing $N(\mathfrak{Q}^*)$. Thus, in all cases, $\mathfrak{T}$ is the unique maximal subgroup of $\mathfrak{G}$ containing $\mathfrak{Q}^*$.

We next see that if p_1, p_2 are primes then every subgroup of $\mathfrak{H}^*$ of order p_1p_2 is cyclic. We next show that $\mathfrak{H}^* \cap \mathfrak{U} \subseteq \boldsymbol{Z}(\mathfrak{H}^*)$. Suppose false and $\mathfrak{H}_r^* = \mathfrak{H}^* \cap \mathfrak{U}_r \not\subseteq \boldsymbol{Z}(\mathfrak{H}^*)$ where $\mathfrak{U}_r$ is the S_r-subgroup of $\mathfrak{U}$. If $r \in \pi_1 \cup \pi_2$, then since $\mathfrak{U}_r \subseteq \mathfrak{S}'$, it follows that $r \in \pi_2$ and $\mathfrak{U}_r$ is the non abelian group of order r^3 and exponent r, so that $|\mathfrak{H}_r^*| = r$. Since $\mathfrak{H}^* \cap \mathfrak{U}$ has a normal complement in $\mathfrak{H}^*$ and every subgroup of $\mathfrak{H}^*$ of order $p_1 p_2$ is cyclic, $\mathfrak{H}_r^* \subseteq \boldsymbol{Z}(\mathfrak{H}^*)$. Thus, we can suppose that $r \in \pi_0$. By definition of $\tilde{\pi}$, we also have $r \in \pi^*$. Apply Lemma 8.17 and conclude that q divides $r - 1$. Since $\mathfrak{H}^*$ is a Z-group, Lemma 13.4 applied to $\mathfrak{Q}^*\mathfrak{U}_r$ acting on the $S_{r'}$-subgroup of $\mathfrak{S}'$ implies that $\mathfrak{U}_r'$ centralizes the $S_{r'}$-subgroup of $\mathfrak{S}'$; since $\mathfrak{H}_r^* \subseteq \mathfrak{U}_r'$, it follows once again that $\mathfrak{H}_r^* \subseteq \boldsymbol{Z}(\mathfrak{H}^*)$. Hence, $\mathfrak{H}^* = (\mathfrak{H}^* \cap \mathfrak{U}) \times (\mathfrak{H}^* \cap \mathfrak{H})$ with cyclic $\mathfrak{H}^* \cap \mathfrak{U}$.

If $\mathfrak{H}^* \cap \mathfrak{H} \subseteq \boldsymbol{F}(\mathfrak{S})$, then $\mathfrak{H}^*$ is cyclic. Suppose $\mathfrak{H}^*$ is non cyclic. Since $\mathfrak{U}$ is nilpotent and since $\mathfrak{S}'/\boldsymbol{F}(\mathfrak{S})$ is nilpotent by Lemma 26.4, it follows that $\pi(\mathfrak{H}^* \cap \mathfrak{H})$ contains a prime s such that a S_s-subgroup of $\mathfrak{S}'/\boldsymbol{F}(\mathfrak{S}) \cap \mathfrak{H}$ is non abelian. Hence, $\boldsymbol{C}_{\mathfrak{S}}(\mathfrak{U})$ contains a non abelian S_s-subgroup. By construction, $s \in \pi_0 - \pi^*$, so $\boldsymbol{C}_{\mathfrak{S}}(\mathfrak{U}) \in \mathscr{H}_1$. This implies that $\mathfrak{S}'$ is a T.I. set in $\mathfrak{G}$.

Since $\mathfrak{H}^*$ is assumed non cyclic, hence non abelian, and since every subgroup of $\mathfrak{H}^*$ of order $p_1 p_2$ is cyclic, it follows that $|\mathfrak{H}^* : \mathfrak{H}^{*\prime}|$ is not a prime. By Lemma 26.23 (i), $\mathfrak{T}_1$ is a nilpotent T.I. set in $\mathfrak{G}$. Set $g = |\mathfrak{G}|$, $|\mathfrak{S}'| = m_1$, $|\mathfrak{T}_1| = m_2$, $|\mathfrak{H}^*| = h$, $|\mathfrak{Q}^*| = q$. If $G_1, G_2, G_3 \in \mathfrak{G}$, the sets $G_1^{-1}\mathfrak{S}'^{\sharp}G_1$, $G_2^{-1}\mathfrak{T}_1^{\sharp}G_2$, $G_3^{-1}(\mathfrak{H}^*\mathfrak{Q}^* - \mathfrak{H}^* - \mathfrak{Q}^*)G_3$ have pairwise empty intersections. Hence,

$$g \geqq \frac{g}{m_1 q}(m_1 - 1) + \frac{g}{m_2 h}(m_2 - 1) + \frac{g}{hq}(h - 1)(q - 1) ,$$

so that

$$\frac{1}{m_1 q} + \frac{1}{m_2 h} \geqq \frac{1}{hq} .$$

Since $m_1 \geqq 3h$, $m_2 \geqq 3q$, the last inequality is not possible. Hence, $\mathfrak{H}^*$ is cyclic.

Let $\mathfrak{L}$ be a maximal subgroup of $\mathfrak{G}$ which is not conjugate to either $\mathfrak{S}$ or $\mathfrak{T}$. If $\mathfrak{L}'$ is not a S-subgroup of $\mathfrak{L}$, then Lemmas 26.10, 26.11 and 26.21 imply that $\mathfrak{L}$ is of type I. If $\mathfrak{L}'$ is a S-subgroup of $\mathfrak{L}$ but $\mathfrak{L}/\mathfrak{L}'$ is non cyclic, Lemma 26.21 implies that $\mathfrak{L}$ is of type I. If $\mathfrak{L}'$ is a S-subgroup of $\mathfrak{L}$, $\mathfrak{L}/\mathfrak{L}'$ is cyclic, and $|\mathfrak{L} : \mathfrak{L}'|$ is not a prime, then by Lemma 26.23, $\mathfrak{L}$ is of type I or $\mathfrak{L}$ contains a subgroup $\mathfrak{Z} = \mathfrak{Z}_1 \times \mathfrak{Z}_2$ which satisfies the hypotheses of Lemma 26.22. But $\mathfrak{Q}^*\mathfrak{H}^*$ also satisfies the hypotheses of Lemma 26.22, so $\mathfrak{Z}$ is conjugate to $\mathfrak{Q}^*\mathfrak{H}^*$. Since $\mathfrak{Z}_1 \subseteq \boldsymbol{H}(\mathfrak{L})$ can be assumed, either $(|\mathfrak{Z}_1|, |\mathfrak{Q}^*|) \neq 1$, or $(|\mathfrak{Z}_1|, |\mathfrak{H}^*|) \neq 1$. The first case yields $\mathfrak{L} = \mathfrak{T}^G$, $G \in \mathfrak{G}$, the second case yields $\mathfrak{L} = \mathfrak{S}^{G_1}$, $G_1 \in \mathfrak{G}$ and we are done in this case. Lemmas 26.22 and 26.23 complete the proof.

LEMMA 26.26. *Under Hypothesis* 26.1 $\mathfrak{T}$ *is either of type* V, *or*

(i) $|\mathfrak{H}^*| = p$ *is a prime.*

(ii) $\mathfrak{T}$ *satisfies*

(a) $|\mathfrak{T} : \mathfrak{T}'| = p$, *and* $\mathfrak{T}'$ *is a S-subgroup of* $\mathfrak{T}$.

(b) $\mathfrak{T}'$ *is not nilpotent.*

Proof. By Lemma 26.25, $\mathfrak{Q}^* \subseteq \mathfrak{T}'$ and $\mathfrak{H}^*$ is cyclic. As $\mathfrak{H}^* \cap \mathfrak{U} \subseteq \mathfrak{U}'$ and $\pi(\mathfrak{H}) \subseteq \pi_0 - \pi^*$, it follows that $N(\hat{\mathfrak{H}}) \subseteq \mathfrak{S}$ for every non empty subset $\hat{\mathfrak{H}}$ of $\mathfrak{H}^{*\sharp}$. Since $\mathfrak{S} \cap \mathfrak{T} = \mathfrak{Q}^*\mathfrak{H}^*$, this implies that $\mathfrak{H}^*$ has $\mathfrak{T}'$ as a complement. If $|\mathfrak{H}^*|$ is not a prime, $\mathfrak{T}'$ is nilpotent, by Lemma 26.3. This implies directly that $\mathfrak{T}$ is of type V, condition (ii) in the definition of type V following easily, since $\mathfrak{T}'$ is non abelian.

We can suppose that $\mathfrak{T}$ is not of type V. Hence, (i) is satisfied. Since $\mathfrak{T}'$ is not nilpotent, (ii) (a) and (ii) (b) also hold.

Lemma 26.26 is important, since if $\mathfrak{T}$ is not of type V, then $\mathfrak{T}$ satisfies Hypothesis 26.1, as does $\mathfrak{S}$.

LEMMA 26.27. *Under Hypothesis* 26.1, *one of the following holds*: (i) $N(\mathfrak{U}) \not\subseteq \mathfrak{S}$; (ii) $\mathfrak{S}'$ *is a tamely imbedded subset of* $\mathfrak{G}$, *and* $\mathfrak{U}$ *is a S-subgroup of* $\mathfrak{G}$.

Proof. Suppose $N(\mathfrak{U}) \subseteq \mathfrak{S}$. If $\mathfrak{S}'$ is a T.I. set in $\mathfrak{G}$ we are done. Hence, we can suppose that $\mathfrak{S}'$ is not a T.I. set in $\mathfrak{G}$.

Since $\mathfrak{S}'$ is not a T.I. set in $\mathfrak{G}$ and since $\mathfrak{H}$ is a T.I. set in $\mathfrak{G}$ ($\pi(\mathfrak{H}) \subseteq \pi_0 - \pi^*$, so Lemma 26.5 (ii) applies), $\mathfrak{U} \neq 1$. We first treat the case in which $\mathfrak{U}$ is non abelian. Let $\mathfrak{U} = \mathfrak{R} \times \mathfrak{R}_0$, where $\mathfrak{R}$ is a non abelian S_r-subgroup of $\mathfrak{R}$, and $\mathfrak{R}_0$ is the $S_{r'}$-subgroup of $\mathfrak{U}$. We show that $\mathfrak{S}$ is the unique maximal subgroup of $\mathfrak{G}$ containing $\mathfrak{R}$.

Suppose $\mathfrak{R} \subseteq \mathfrak{L}, \mathfrak{L} \in \mathscr{M}$. By Lemma 26.1, $N(\Omega_1(Z(\mathfrak{R}))) \subseteq \mathfrak{L} \cap \mathfrak{S}$. In particular, $N(\mathfrak{R}) \subseteq \mathfrak{L} \cap \mathfrak{S}$, so $\mathfrak{R}$ is a S_r-subgroup of $\mathfrak{G}$. If $\mathfrak{L} = \mathfrak{S}^G$, $G \in \mathfrak{G}$, then by Sylow's theorem, $\mathfrak{R}$ is conjugate to $G\mathfrak{R}G^{-1}$ in $\mathfrak{S}$, $\mathfrak{R} = S^{-1}G\mathfrak{R}G^{-1}S$, so that $S^{-1}G \in N(\mathfrak{R}) \subseteq \mathfrak{S}$, and $G \in S$. Hence, we can suppose $\mathfrak{L}$ is not conjugate to $\mathfrak{S}$. Clearly, $\mathfrak{L}$ is not conjugate to $\mathfrak{T}$, since $q \nmid |\mathfrak{T} : \mathfrak{T}'|$. Hence, $\mathfrak{L}$ is of type I. But then $\mathfrak{R} \subseteq \boldsymbol{H}(\mathfrak{L})$, so that $\mathfrak{L} = N(\mathfrak{R}) \subseteq \mathfrak{S}$, contrary to assumption. Hence, $\mathfrak{R}$ is contained in $\mathfrak{S}$ and no other maximal subgroup of $\mathfrak{G}$. This implies that $\mathfrak{U}$ is a S-subgroup of $\mathfrak{G}$.

Choose $S \in \mathfrak{S}'^\sharp \cap \mathfrak{S}'^G$, $G \in \mathfrak{G} - \mathfrak{S}$. There are such elements S and G since $\mathfrak{S}'$ is not a T.I. set in $\mathfrak{G}$. If S is not a $\tilde{\pi}$-element, then $S_1 = S^n \in \mathfrak{H}^\sharp \cap \mathfrak{H}^{\sharp G}$ for some integer n, contrary to the fact that $\mathfrak{H}$ is a T.I. set in $\mathfrak{G}$. Hence S is a $\tilde{\pi}$-element and we can suppose that $S \in \mathfrak{U}$. If $S \notin \mathfrak{R}$, then $S_2 = S^m \in \mathfrak{R}_0^\sharp \cap S'^{\sharp G}$ for some m, and $C(S_2)$ contains a S_r-subgroup of both $\mathfrak{S}$ and $\mathfrak{S}^G$, which is not the case. Hence,

$S \in \mathfrak{R}$. Since $\mathfrak{R}$ was any non abelian Sylow subgroup of $\mathfrak{U}$, it follows that $\mathfrak{R}_0$ is abelian.

Let $\mathfrak{L} \in \mathscr{M}$, $C(S) \subseteq \mathfrak{L}$. A S_r-subgroup of $\mathfrak{L}$ is non cyclic. Let $\tilde{\mathfrak{R}}$ be a S_r-subgroup of $\mathfrak{L}$ containing $C_{\mathfrak{R}}(S)$. If $r \in \pi_0$, then by Lemma 26.7, $N(C_{\mathfrak{R}}(S)) \subseteq \mathfrak{S}$, so $\tilde{\mathfrak{R}} = C_{\mathfrak{R}}(S)$. If $r \in \pi_2$, the same equality holds by Lemma 26.14 and the containment $N(C_{\tilde{\mathfrak{R}}}(S)) \subseteq N(\Omega_1(Z(\mathfrak{R})))$. Thus, $\mathfrak{L}$ is not conjugate to $\mathfrak{S}$. Since $\tilde{\mathfrak{R}}$ is non cyclic, $\mathfrak{L}$ is not conjugate to $\mathfrak{T}$. Hence, $\mathfrak{L}$ is of type I, and this implies directly that $\mathfrak{L} = H(\mathfrak{L})(\mathfrak{L} \cap \mathfrak{S})$, $\mathfrak{S} \cap H(\mathfrak{L}) = 1$. Since a S_r-subgroup of $\mathfrak{G}$ is non abelian, Lemmas 26.12 and 26.18 imply that

$$\left\{\bigcup_{H \in H(\mathfrak{L})^\sharp} C_{\mathfrak{L}}(H)\right\} - H(\mathfrak{L})^\sharp = H(\mathfrak{L})\langle S\rangle^\sharp ,$$

and it is obvious that $H(\mathfrak{L})\langle S\rangle^\sharp$ is a T.I. set in $\mathfrak{G}$ with $\mathfrak{L}$ as its normalizer. We have verified all the properties in the definition of a tamely imbedded subset except the conjugacy condition for $\mathfrak{S}'$ and the coprime conditions. By definition of $H(\mathfrak{L})$, together with the fact that $\mathfrak{S}'$ is a S-subgroup of $\mathfrak{G}$, it follows that $(|H(\mathfrak{L})|, |\mathfrak{S}'|) = 1$. If $(|H(\mathfrak{L})|, |\mathfrak{Q}^*|) \neq 1$, then $\mathfrak{L}$ is conjugate to $\mathfrak{T}$. This is not the case, as $\tilde{\mathfrak{R}}$ is non cyclic. Thus, if $\mathfrak{L}_1, \cdots, \mathfrak{L}_m$ is a set of representatives for the conjugate classes of maximal subgroups of $\mathfrak{G}$ which contain $C(S)$ for some S in $\mathfrak{S}'$ and are different from $\mathfrak{S}$, it follows that $(|H(\mathfrak{L}_i)|, |H(\mathfrak{L}_j)|) = 1$ for $i \neq j$. It remains only to verify the conjugacy condition for elements of $\mathfrak{S}'^\sharp$. Let S, S_1 be elements of $\mathfrak{S}'^\sharp$ which are conjugate in $\mathfrak{G}$. We can suppose that S and S_1 have order r and are in $\mathfrak{R}$; otherwise it is immediate that S and S_1 are conjugate in $\mathfrak{S}$. Let $S = G^{-1}S_1G$, then $C(S) \supseteq \langle \Omega_1(Z(\mathfrak{R})), \Omega_1(Z(\mathfrak{R}^G))\rangle$. Since $N(\Omega_1(Z(\mathfrak{R}))) \subseteq \mathfrak{S}$, it follows that S and S_1 are conjugate in $\mathfrak{S}$. (It is at this point that we once again have made use of the fact that the subgroups in $\mathscr{T}(\mathfrak{R})$ have two conjugate classes of subgroups of order r.) Thus, $\mathfrak{S}'$ is a tamely imbedded subset of $\mathfrak{G}$ in this case.

We now assume that $\mathfrak{U}$ is abelian. We first show that $\mathfrak{U}$ is a S-subgroup of $\mathfrak{G}$. Otherwise, $\mathfrak{U}$ is not a S-subgroup of $N(\mathfrak{U}_r)$ for some non identity S_r-subgroup $\mathfrak{U}_r$ of $\mathfrak{U}$. Let $N(\mathfrak{U}_r) \subseteq \mathfrak{L} \in \mathscr{M}$. Then $\mathfrak{L}$ is not conjugate to $\mathfrak{S}$, since $|\mathfrak{L}|_{\pi}^{\cdot} \neq |\mathfrak{S}|_{\pi}^{\cdot}$. Suppose $\mathfrak{L}$ is conjugate to $\mathfrak{T}$. Since $\mathfrak{U}\mathfrak{Q}^*$ is a Frobenius group, we have $\mathfrak{U} \subseteq \mathfrak{L}'$. Thus $\mathfrak{L}'$ is not nilpotent, since by hypothesis $N(\mathfrak{U}) \subseteq \mathfrak{S}$. Hence, $\mathfrak{T}$ is not of type V. By Lemma 26.26, $|\mathfrak{H}^*| = p$ is a prime. Since $|\mathfrak{Q}^*| = q$ is also a prime, it follows that if $\mathfrak{V}$ is a $S_{q'}$-subgroup of $\mathfrak{T}'$ normalized by $\mathfrak{H}^*$, then $\mathfrak{H}^*\mathfrak{V}$ is a Frobenius group, ($\mathfrak{V} \neq 1$, since $\mathfrak{T}'$ is not nilpotent). If $\pi(\mathfrak{U}) \subseteq \pi(\mathfrak{V})$, then since $N(\mathfrak{U}) \subseteq \mathfrak{S}$, it follows that $\mathfrak{U}$ is conjugate to $\mathfrak{V}$. But p divides $|N(\mathfrak{V}) : C(\mathfrak{V})|$, and so $p = q$, which is not the case. Hence $\pi(\mathfrak{U}) \not\subseteq \pi(\mathfrak{V})$. But $\pi(\mathfrak{U}) \subseteq \pi(\mathfrak{S}) \cap \pi(\mathfrak{T}') \subseteq$

$\pi(\mathfrak{V}) \cup \{q\}$, so $q \in \pi(\mathfrak{U})$, which is absurd since $\mathfrak{S}'$ is a q'-group. Hence, $\mathfrak{L}$ is not conjugate to either $\mathfrak{S}$ or $\mathfrak{T}$, so $\mathfrak{L}$ is of type I. Since $\mathfrak{Q}^*$ is of prime order and $\mathfrak{Q}^*\mathfrak{U}$ is a Frobenius group, $\mathfrak{U} \subseteq \boldsymbol{H}(\mathfrak{L})$. Since $\boldsymbol{N}(\mathfrak{U}) \subseteq \mathfrak{S}$, we have $\mathfrak{U} = \boldsymbol{H}(\mathfrak{L})$. Hence $\mathfrak{L} \subseteq \boldsymbol{N}(\mathfrak{U}) \subseteq \mathfrak{S}$, which is absurd. Hence, $\mathfrak{U}$ is a S-subgroup of $\mathfrak{G}$. This implies directly that $\boldsymbol{N}(\mathfrak{U}_r) \subseteq \mathfrak{S}$ for all non identity Sylow subgroups $\mathfrak{U}_r$ of $\mathfrak{S}$.

Since $\mathfrak{U}$ is an abelian S-subgroup of $\mathfrak{G}$, and $\mathfrak{H}$ is a T.I. set in $\mathfrak{G}$, the condition $\boldsymbol{N}(\mathfrak{U}) \subseteq \mathfrak{S}$ implies that two elements of $\mathfrak{S}'$ are conjugate in $\mathfrak{G}$ if and only if they are conjugate in $\mathfrak{S}$.

Suppose $S \in \mathfrak{S}'^{\#}$, and $\boldsymbol{C}(S) \not\subseteq \mathfrak{S}$. Then S is a $\tilde{\pi}$-element, and we can suppose $S \in \mathfrak{U}$. Let $\mathfrak{L} \in \mathscr{M}$, $\boldsymbol{C}(S) \subseteq \mathfrak{L}$. Since $\mathfrak{U}$ is an abelian S-subgroup of $\mathfrak{G}$ and since $\mathfrak{U} \subseteq \boldsymbol{C}(S) \subseteq \mathfrak{L}$, it follows that $\mathfrak{L}$ is not conjugate to $\mathfrak{S}$ or $\mathfrak{T}$. It is now straightforward to verify that $\mathfrak{S}'$ is tamely imbedded in $\mathfrak{G}$.

LEMMA 26.28. *Under Hypothesis* 26.1, *either* $\mathfrak{S}$ *or* $\mathfrak{T}$ *is of type* II. *If* $\mathfrak{S}$ *is of type* II, *then*

$$\bigcup_{H \in \mathfrak{H}^{\#}} C'_{\mathfrak{S}}(H)$$

is a T.I. *set in* $\mathfrak{G}$. *Both* $\mathfrak{S}$ *and* $\mathfrak{T}$ *are of type* II, III, IV *or* V.

Proof. First, suppose $\mathfrak{T}$ is of type V, but that $\mathfrak{S}$ is not of type II. Suppose $\boldsymbol{N}(\mathfrak{U}) \subseteq \mathfrak{S}$. By Lemma 26.27, $\mathfrak{S}'$ is a tamely imbedded subset of $\mathfrak{G}$. As $\mathfrak{U}$ is a S-subgroup of $\mathfrak{G}$ in this case, we have $(|\mathfrak{S}'|, |\mathfrak{T}'|) = 1$. By Lemma 26.24, $\mathfrak{T}'$ is a tamely imbedded subset of $\mathfrak{G}$. We now use the notation of section 9. Suppose $S \in \mathfrak{S}'^{\#}$, $T \in \mathfrak{T}'^{\#}$ and some element of $\mathfrak{A}_S$ is conjugate to some element of $\mathfrak{A}_T$. This implies the existence of $\mathfrak{L} \in \mathscr{M}$ such that $|\mathfrak{L} : \boldsymbol{H}(\mathfrak{L})|$ divides $(|\mathfrak{S}'|, |\mathfrak{T}'|) = 1$, which is not the case. Setting $\hat{\mathfrak{W}} = \mathfrak{H}^*\mathfrak{Q}^* - \mathfrak{H}^* - \mathfrak{Q}^*$, it follows that no element of $\hat{\mathfrak{W}}$ is conjugate to an element of $\mathfrak{A}_S$ or $\mathfrak{A}_T$. We find, with $h = |\mathfrak{H}^*|$, $s = |\mathfrak{S}'|$, $t = |\mathfrak{T}'|$, that by Lemma 9.5,

$$g \geq \frac{(h-1)(q-1)}{hq} g + \frac{s-1}{sq} g + \frac{t-1}{th} g \,, \tag{26.9}$$

which is not the case. Hence $\boldsymbol{N}(\mathfrak{U}) \not\subseteq \mathfrak{S}$. If $\mathfrak{U}_r$ were a non abelian S_r-subgroup of $\mathfrak{G}$, then $\boldsymbol{N}(\Omega_1(\boldsymbol{Z}(\mathfrak{U}_r))) \subseteq \mathfrak{S}$, by Lemma 26.14. Since $\boldsymbol{N}(\mathfrak{U}) \subseteq \boldsymbol{N}(\Omega_1(\boldsymbol{Z}(\mathfrak{U}_r)))$, this is impossible. Hence $\mathfrak{U}$ is abelian, and $m(\mathfrak{U}) \leq 2$. Thus, $\mathfrak{S}$ is of type II in this case, since the above information implies directly that $\mathfrak{H}$ is nilpotent.

Suppose now that $\mathfrak{T}$ is not of type V. Then from Lemma 26.26 we have $\mathfrak{T} = \mathfrak{H}^*\mathfrak{V}\mathfrak{Q}$, where $\mathfrak{Q}$ is a normal S_q-subgroup of $\mathfrak{T}$, $\mathfrak{H}^*\mathfrak{V}$ is a Frobenius group with Frobenius kernel $\mathfrak{V}$, and $\mathfrak{V}$ is a non identity q'-group. Since $\mathfrak{Q}^*$ is of prime order q, it follows from 3.16 that $\mathfrak{Q}$ contains a subgroup $\mathfrak{Q}_0$ such that $\mathfrak{Q}_0 \triangleleft \mathfrak{T}$, $\mathfrak{Q}/\mathfrak{Q}_0$ is elementary of order q^p $(p = |\mathfrak{H}^*|)$, and $\mathfrak{V}$ centralizes $\mathfrak{Q}_0$.

We next show that $\mathfrak{V}'$ centralizes $\mathfrak{Q}$. This is an immediate application of 3.16. If $N(\mathfrak{V}) \subseteq \mathfrak{T}$, then $\mathfrak{T}$ is of type III or IV according as $\mathfrak{V}$ is abelian or non abelian. If neither $\mathfrak{S}$ nor $\mathfrak{T}$ is of type II, then both $\mathfrak{S}'$ and $\mathfrak{T}'$ are tamely imbedded subsets of $\mathfrak{G}$, by Lemma 26.27, since both $\mathfrak{S}$ and $\mathfrak{T}$ satisfy Hypothesis 26.1. Once again, (26.9) yields a contradiction.

If $\mathfrak{S}$ is of type II, then $\mathfrak{H}$ is a T.I. set in $\mathfrak{G}$. Suppose

$$X,\, Y \in \bigcup_{H \in \mathfrak{H}^\#} C_{\mathfrak{S}'}(H)$$

and $X = G^{-1}YG$. Choose $H_1 \in C_{\mathfrak{H}}(X)^\#$, $H_2 \in C_{\mathfrak{H}}(Y)^\#$. Then $C(X) \supseteq \langle H_1, G^{-1}H_2G\rangle$. If $C(X) \subseteq \mathfrak{S}$, then $G \in \mathfrak{S}$, since $\mathfrak{H}$ is a T.I. set in $\mathfrak{G}$. We can suppose $C(X) \not\subseteq \mathfrak{S}$, and without loss of generality, we assume that X has prime order r, $X \in \mathfrak{U}$. If a S_r-subgroup of $\mathfrak{U}$ is non cyclic, then by Lemmas 26.12 and 26.13, $C(X) \subseteq \mathfrak{S}$. We can suppose that the S_r-subgroup $\mathfrak{U}_r$ of $\mathfrak{U}$ is cyclic, so that $\langle X\rangle = \Omega_1(\mathfrak{U}_r)$. Since $N(\mathfrak{U}) \not\subseteq \mathfrak{S}$, it follows that $N(\langle X\rangle) \not\subseteq \mathfrak{S}$. Choose $\mathfrak{L} \in \mathscr{M}$ with $N(\langle X\rangle) \subseteq \mathfrak{L}$. If $C(X) \cap \mathfrak{H}^* \neq 1$, it follows readily that $C(X) \subseteq \mathfrak{S}$, so we can suppose $C(X) \cap \mathfrak{H}^* = 1$. In this case, $C_{\mathfrak{H}}(X)\mathfrak{Q}^*$ is a Frobenius group, and this implies that $C_{\mathfrak{H}}(X) \subseteq \boldsymbol{H}(\mathfrak{L})$, which is not the case. The proof is complete.

LEMMA 26.29. *If $\mathfrak{L} \in \mathscr{M}$ and $\mathfrak{L}$ is of type* I, *then*

$$\bigcup_{H \in \boldsymbol{H}(\mathfrak{L})^\#} C_{\mathfrak{L}}(H) = \hat{\mathfrak{L}}$$

is a tamely imbedded subset of $\mathfrak{G}$.

Proof. We first show that $\boldsymbol{H}(\mathfrak{L})$ is tamely imbedded in $\mathfrak{G}$.

If $\boldsymbol{H}(\mathfrak{L})$ is a T.I. set in $\mathfrak{G}$ we are done. If $\boldsymbol{H}(\mathfrak{L})$ is abelian, the conjugacy property for elements of $\boldsymbol{H}(\mathfrak{L})$ holds. Suppose $\boldsymbol{H}(\mathfrak{L})$ is abelian, $L \in \boldsymbol{H}(\mathfrak{L})$, and $C(L) \not\subseteq \mathfrak{L}$. Let $\mathfrak{N} \in \mathscr{M}$ with $C(L) \subseteq \mathfrak{N}$.

Suppose $\mathfrak{N}$ is of type I. Then $\mathfrak{N} \cap \mathfrak{L}$ is disjoint from $\boldsymbol{H}(\mathfrak{N})$, since $\boldsymbol{H}(\mathfrak{L}) \subseteq \mathfrak{N} \cap \mathfrak{L}$. Let $\mathfrak{E}$ be a complement for $\boldsymbol{H}(\mathfrak{N})$ in $\mathfrak{N}$ which contains $\mathfrak{N} \cap \mathfrak{L}$. Lemmas 26.12 and 26.13 imply that $\mathfrak{E} = \mathfrak{N} \cap \mathfrak{L}$.

If $\mathfrak{L}_1, \cdots, \mathfrak{L}_n$ is a set of representatives for the conjugate classes of maximal subgroups of $\mathfrak{G}$ constructed in this fashion, then $(|\boldsymbol{H}(\mathfrak{L}_i)|, |\boldsymbol{H}(\mathfrak{L}_j)|) = 1$ for $i \neq j$. Also, $(|\boldsymbol{H}(\mathfrak{L}_i)|, |\boldsymbol{H}(\mathfrak{L})|) = 1$. Suppose $(|\boldsymbol{H}(\mathfrak{L}_i)|, |C_{\mathfrak{L}}(L)|) \neq 1$ for some $L \in \boldsymbol{H}(\mathfrak{L})^\#$, and some i. We can suppose that

L has prime order r. Let s be a prime divisor of $(|\boldsymbol{H}(\mathfrak{L}_i)|, |C_{\mathfrak{L}}(L)|)$, so that $s \in \pi(\mathfrak{L}) - \pi(\boldsymbol{H}(\mathfrak{L}))$, Since $\mathfrak{L}$ is of type I, this implies that a S_r-subgroup $\mathfrak{S}$ of $\mathfrak{L}$ is non cyclic so that $s \in \pi^*$. Since $\mathfrak{S}$ does not centralize a S_r-subgroup of $\mathfrak{L}$, $s < r$. But now Lemma 8.16 implies that the S_r-subgroup of $\mathfrak{L}$ centralizes a S_s-subgroup of $\boldsymbol{H}(\mathfrak{L}_i)$, which is not the case. Hence, $(|\boldsymbol{H}(\mathfrak{L}_i)|, |C_{\mathfrak{L}}(L)|) = 1$ for every $L \in \boldsymbol{H}(\mathfrak{L})^{\sharp}$.

By construction

$$\hat{\mathfrak{L}}_i = \bigcup_{H \in \boldsymbol{H}(\mathfrak{L}_i)^{\sharp}} C_{\mathfrak{L}_i}(H) - \boldsymbol{H}(\mathfrak{L}_i)^{\sharp}$$

contains a non identity element. From Lemma 26.13 we have $N(\hat{\mathfrak{L}}_i) = \mathfrak{L}_i$ and $\hat{\mathfrak{L}}_i$ is a T.I. set in $\mathfrak{G}$. Thus, if $\boldsymbol{H}(\mathfrak{L})$ is abelian and every $\mathfrak{N}$ with the property that $\mathfrak{N} \in \mathscr{M}$ and $C(L) \subseteq \mathfrak{N}$ for some $L \in \boldsymbol{H}(\mathfrak{L})^{\sharp}$ is of type I, then $\boldsymbol{H}(\mathfrak{L})$ is tamely imbedded in $\mathfrak{G}$.

Suppose $\mathfrak{N}$ is not of type I. Since $\boldsymbol{H}(\mathfrak{L}) \subseteq \mathfrak{N}$, it is obvious that $\mathfrak{N}$ is not of type V. It is equally obvious that $\mathfrak{N}$ is not of type III or IV. Hence, $\mathfrak{N}$ is of type II. Since $\boldsymbol{H}(\mathfrak{L})$ is a S-subgroup of $\mathfrak{G}$, it is a S-subgroup of $\mathfrak{N}$, and it follows that $\mathfrak{N} \cap \mathfrak{L}$ is a complement to $\boldsymbol{H}(\mathfrak{N})$. Since $|\boldsymbol{H}(\mathfrak{N})|$ is relatively prime to $|\boldsymbol{H}(\mathfrak{L})|$ and to each $|\boldsymbol{H}(\mathfrak{L}_i)|$, we only need to show that $|\boldsymbol{H}(\mathfrak{N})|$ is relatively prime to $|C_{\mathfrak{L}}(L)|$, $L \in \boldsymbol{H}(\mathfrak{L})$. Let $q = |\mathfrak{N} : \mathfrak{N}'|$, so that q is a prime and $\mathfrak{N} \cap \mathfrak{L}$ contains a S_q-subgroup $\mathfrak{Q}^*$ of $\mathfrak{N}$. Since $\pi(\boldsymbol{H}(\mathfrak{N})) \subseteq \pi_0 - \pi^*$, it follows that if $\mathfrak{R}$ is a S_{ϖ}-subgroup of $\mathfrak{L}$, $\varpi = \pi(\boldsymbol{H}(\mathfrak{N})) \cap \pi(\mathfrak{L})$, either $\mathfrak{R} = 1$, or $\mathfrak{R}\boldsymbol{H}(\mathfrak{L})$ is a Frobenius group. Thus $(|\boldsymbol{H}(\mathfrak{N})|, |C_{\mathfrak{L}}(\mathfrak{L})|) = 1$ for $L \in \boldsymbol{H}(\mathfrak{L})^{\sharp}$, and $\boldsymbol{H}(\mathfrak{L})$ is a tamely imbedded subset of $\mathfrak{G}$. Since $C(L) \subseteq \mathfrak{L}$ for every element of

$$\{\bigcup_{H \in \boldsymbol{H}(\mathfrak{L})^{\sharp}} C_{\mathfrak{L}}(H)\} - \boldsymbol{H}(\mathfrak{L}) ,$$

by Lemmas 26.12 and 26.13, the lemma is proved if $\boldsymbol{H}(\mathfrak{L})$ is abelian.

We can now suppose that $\boldsymbol{H}(\mathfrak{L})$ is non abelian, and is not a T.I. set in $\mathfrak{G}$. Let $\mathfrak{N}$ be a non abelian $\mathfrak{S}$-subgroup of $\boldsymbol{H}(\mathfrak{L})$, and let $\boldsymbol{H}(\mathfrak{L}) = \mathfrak{N} \times \mathfrak{N}_0$. Since $\boldsymbol{H}(\mathfrak{L})$ is not a T.I. set in $\mathfrak{G}$ Lemmas 26.14 and 26.13 imply that $\mathfrak{N}_0$ is a cyclic T.I. set in $\mathfrak{G}$. It follows directly from Lemma 26.12 that $\boldsymbol{H}(\mathfrak{L})$ is a tamely imbedded subset of $\mathfrak{G}$.

It remains to show that $\hat{\mathfrak{L}}$ is a tamely imbedded subset of $\mathfrak{G}$. This is an immediate consequence of Lemmas 26.12 and 26.13.

LEMMA 26.30. *If $\mathfrak{H}$ is a nilpotent S-subgroup of $\mathfrak{G}$, then two elements of $\mathfrak{H}$ are conjugate in $\mathfrak{G}$ if and only if they are conjugate in $N(\mathfrak{H})$.*

Proof. Let $\mathfrak{L} \in \mathscr{M}$, $N(\mathfrak{H}) \subseteq \mathfrak{L}$. If $\mathfrak{H} \subseteq \boldsymbol{H}(\mathfrak{L})$ and $\mathfrak{L}$ is of type I, we are done. If $\mathfrak{H} \subseteq \boldsymbol{H}(\mathfrak{L})$ and $\mathfrak{L}$ is not of type I, we are done. If $\mathfrak{H} \not\subseteq \boldsymbol{H}(\mathfrak{L})$, then $\mathfrak{H} \cap \boldsymbol{H}(\mathfrak{L}) = 1$. If $\mathfrak{L}$ is of type I, $\mathfrak{H}$ is abelian, and we are done. If $\mathfrak{L}$ is not of type I, then $\mathfrak{L}$ is of type III or IV, and we are done.

We now summarize to show that the proofs of Theorems 14.1 and 14.2 are complete. By Lemma 26.30, the conjugacy property for nilpotent S-subgroups holds. If every element of $\mathscr{M}$ is of type I, we are done by Lemma 26.29. We can therefore suppose that $\mathscr{M}$ contains an element not of type I. Choose $\mathfrak{L} \in \mathscr{M}$, $\mathfrak{L}$ not of type I. By Lemma 26.21, if $p \in \pi(\mathfrak{L}/\mathfrak{L}')$, a S_p-subgroup of $\mathfrak{L}$ is cyclic. This implies that $\mathfrak{L}'$ is a S-subgroup of $\mathfrak{L}$. First, suppose $|\mathfrak{L} : \mathfrak{L}'|$ is not a prime. Then by Lemma 26.23, $\mathfrak{L}$ is of type V or satisfies the conditions listed in Lemma 26.23. Suppose that $\mathfrak{L}$ is not of type V, and $\mathfrak{E}$ is a complement to $\boldsymbol{H}(\mathfrak{L})$ in $\mathfrak{L}$. Let p be the smallest prime such that a S_p-subgroup $\mathfrak{E}_p$ of $\mathfrak{E}$ is not contained in $\boldsymbol{Z}(\mathfrak{E})$ and choose $\mathfrak{L}_1 \in \mathscr{M}$, $N(\Omega_1(\mathfrak{E}_p)) \subseteq \mathfrak{L}_1$. By Lemmas 26.12 and 26.13, $\mathfrak{L}_1$ is not of type I. Lemma 26.21 implies that $\mathfrak{L}_1'$ is a S-subgroup of $\mathfrak{L}_1$ and $\mathfrak{L}_1/\mathfrak{L}_1'$ is cyclic. By construction, $\mathfrak{L}_1'$ is not nilpotent, and also by construction $\mathfrak{L}_1$ is not conjugate to $\mathfrak{L}$. We will now show that $|\mathfrak{L}_1 : \mathfrak{L}_1'|$ is a prime. Otherwise, since $\mathfrak{L}_1$ is not of type I or V, $\mathfrak{L}_1$ satisfies the conditions of Lemma 26.23. In this case, both $\boldsymbol{H}(\mathfrak{L})$ and $\boldsymbol{H}(\mathfrak{L}_1)$ are nilpotent T.I. sets in $\mathfrak{G}$ and $\mathfrak{L} \cap \mathfrak{L}_1$ satisfies the hypotheses of Lemma 26.22. Let $\ell = |\mathfrak{L}|$, $\ell_1 = |\mathfrak{L}_1|$, $|\mathfrak{L} : \boldsymbol{H}(\mathfrak{L})| = e$, $|\mathfrak{L}_1 : \boldsymbol{H}(\mathfrak{L}_1)| = e_1$, $g = |\mathfrak{G}|$, so that

$$g \geqq \frac{(e-1)(e_1-1)}{ee_1} g + \frac{\ell - 1}{\ell e} g + \frac{\ell_1 - 1}{\ell_1 e_1} g , \tag{26.10}$$

which is not the case. Hence $|\mathfrak{L}_1 : \mathfrak{L}_1'|$ is a prime, so that $\mathfrak{L}_1$ satisfies Hypothesis 26.1. But then Lemma 26.25 implies that $\mathfrak{L}$ is of type V. Thus, whenever $\mathfrak{L} \in \mathscr{M}$ satisfies the hypotheses of Lemma 26.23, $\mathfrak{L}$ is of type I or V.

Suppose every element of $\mathscr{M}$ is of type I or V, and there is an element $\mathfrak{L}$ of type V. Let $p \in \pi(\mathfrak{L}/\mathfrak{L}')$, and let $\mathfrak{E}_p$ be a S_p-subgroup of $\mathfrak{L}$. Choose $\mathfrak{L}_1$ so that $N(\mathfrak{E}_p) \subseteq \mathfrak{L}_1 \in \mathscr{M}$. Then $\mathfrak{L}_1$ is not of type I. Suppose $\mathfrak{L}_1$ is of type V. By Lemma 26.20, $\mathfrak{L}'$ and $\mathfrak{L}_1'$ are tamely imbedded subsets of $\mathfrak{G}$. Since $(|\mathfrak{L}'|, |\mathfrak{L}_1'|) = 1$, it follows that $\mathfrak{A}_L$ and $\mathfrak{A}_{L_1}$ do not contain elements in the same conjugate class of $\mathfrak{G}$, $L \in \mathfrak{L}'$, $L_1 \in \mathfrak{L}_1'$. Setting $g = |\mathfrak{G}|$, $|\mathfrak{L}'| = \ell$, $|\mathfrak{L}_1'| = \ell_1$, $|\mathfrak{L} : \mathfrak{L}'| = e$, $|\mathfrak{L}_1 : \mathfrak{L}_1'| = e_1$, then (26.10) holds, by Lemma 9.5, which is not the case.

We can now suppose that $\mathscr{M}$ contains an element $\mathfrak{L}$ not of type I or V. Lemmas 26.21, 26.23 and the previous reduction imply that $\mathfrak{L}'$ is a S-subgroup of $\mathfrak{L}$, $\mathfrak{L}'$ is not nilpotent, and $|\mathfrak{L} : \mathfrak{L}'|$ is a prime. Lemmas 26.25 and 26.28 complete the proof of Theorem 14.1.

As for Theorem 14.2, Lemmas 26.28 and 26.29, together with Theorem 14.1, imply all parts of the theorem, since if $\mathfrak{L}$ is of type II, III, IV, or V, $\hat{\mathfrak{L}}$ is any tamely imbedded subset of $\mathfrak{G}$ which satisfies $N(\hat{\mathfrak{L}}) = \mathfrak{L}$, and $\mathfrak{W} = \mathfrak{W}_1\mathfrak{W}_2$ is a cyclic subgroup of $\mathfrak{L}$ which satisfies the hypothesis of Lemma 26.22, then adjoining all $L^{-1}(\mathfrak{W} - \mathfrak{W}_1 - \mathfrak{W}_2) L$, $L \in \mathfrak{L}$, to $\hat{\mathfrak{L}}$ does not alter the set of supporting subgroups for $\hat{\mathfrak{L}}$, as $C(W) \subseteq \mathfrak{L}$ for all $W \in \mathfrak{W} - \mathfrak{W}_1 - \mathfrak{W}_2$. The proofs are complete.