CHAPTER III

9. Tamely Imbedded Subsets of a Group

The character ring of a group has a metric structure which is derived from the inner product. Let $\mathfrak{L}$ be a subgroup of the group $\mathfrak{X}$. The purpose of this chapter is to state conditions on $\mathfrak{L}$ and $\mathfrak{X}$ which ensure the existence of an isometry τ that maps suitable subsets of the character ring of $\mathfrak{L}$ into the character ring of $\mathfrak{X}$ and has certain additional properties. If α is in the character ring of $\mathfrak{L}$ and α^τ is defined then these additional properties will yield information concerning $\alpha^\tau(L)$ for some elements L of $\mathfrak{L}$. Once the existence of τ is established it will enable us to derive information about certain generalized characters of $\mathfrak{X}$ provided we know something about the character ring of $\mathfrak{L}$. In this way it is possible to get global information about $\mathfrak{X}$ from local information about $\mathfrak{L}$.

There are two stages in establishing the existence of τ. First we will require that $\mathfrak{L}$ is in some sense "nicely" imbedded in $\mathfrak{X}$. When this requirement is fulfilled it is possible to define α^τ for certain generalized characters α of $\mathfrak{L}$ with $\alpha(1) = 0$. In this situation α^τ is explicitly defined in terms of induced characters of various subgroups of $\mathfrak{X}$. Secondly it is necessary that the character ring of $\mathfrak{L}$ have certain special properties. These properties make it possible to extend the definition of τ to a wider domain. In particular it is then possible to define α^τ for some generalized characters α of $\mathfrak{L}$ with $\alpha(1) \neq 0$. The precise conditions that the character ring of $\mathfrak{L}$ needs to satisfy will be stated later. In this section we are concerned with the imbedding of $\mathfrak{L}$ in $\mathfrak{X}$. The following definition is appropriate.

DEFINITION 9.1. Let $\hat{\mathfrak{L}}$ be a subset of the group $\mathfrak{X}$ such that

$$(9.1) \qquad \langle 1\rangle \subseteqq \hat{\mathfrak{L}} \subseteqq N(\hat{\mathfrak{L}}) = \mathfrak{L}\ .$$

Let $\mathfrak{L}_0$ be the set of elements L in $\hat{\mathfrak{L}}$ such that $C(L) \subseteqq \mathfrak{L}$, and let $\mathfrak{D} = \hat{\mathfrak{L}}^\# - \mathfrak{L}_0$.

We say that $\hat{\mathfrak{L}}$ is *tamely imbedded* in $\mathfrak{X}$ if the following conditions are satisfied:

(i) If two elements of $\hat{\mathfrak{L}}$ are conjugate in $\mathfrak{X}$, they are conjugate in $\mathfrak{L}$.

(ii) If $\mathfrak{D}$ is non empty, then there are non identity subgroups $\mathfrak{H}_1, \cdots, \mathfrak{H}_n$ of $\mathfrak{X}$, $n \geqq 1$, with the following properties:

(a) $(|\mathfrak{H}_i|, |\mathfrak{H}_j|) = 1$ for $i \neq j$;
(b) $\mathfrak{H}_i$ is a S-subgroup of $\mathfrak{N}_i = N(\mathfrak{H}_i)$;
(c) $\mathfrak{N}_i = \mathfrak{H}_i(\mathfrak{L} \cap \mathfrak{N}_i)$ and $\mathfrak{H}_i \cap \mathfrak{L} = 1$;
(d) $(|\mathfrak{H}_i|, |C_{\mathfrak{L}}(L)|) = 1$ for $L \in \hat{\mathfrak{L}}^*$;
(e) For $1 \leq i \leq n$, define

$$\hat{\mathfrak{N}}_i = \left\{\bigcup_{H \in \mathfrak{H}_i^*} C_{\mathfrak{N}_i}(H)\right\} - \mathfrak{H}_i^* .$$

Then $\hat{\mathfrak{N}}_i^*$ is a non empty T. I. set in $\mathfrak{X}$ and $\mathfrak{N}_i = N(\hat{\mathfrak{N}}_i)$.

(iii) If $L_0 \in \mathfrak{D}$, then there is a conjugate L of L_0 in $\hat{\mathfrak{L}}$ and an index i such that

$$C(L) = C_{\mathfrak{H}_i}(L) \cdot C_{\mathfrak{L}}(L) \subseteq \mathfrak{N}_i .$$

If $\hat{\mathfrak{L}}$ is a tamely imbedded subset of $\mathfrak{X}$ then for $1 \leq i \leq n$, each of the groups $\mathfrak{H}_i$ is called a *supporting subgroup* of $\hat{\mathfrak{L}}$. The collection $\{\mathfrak{H}_i \mid 1 \leq i \leq n\}$ is called a *system of supporting subgroups* of $\hat{\mathfrak{L}}$.

In one important special case, the definition of tamely imbedded subset of $\mathfrak{X}$ is fairly easy to master. Namely, if $\mathfrak{D}$ is empty, the reader can check that $\hat{\mathfrak{L}}$ is a T. I. set.

If $\hat{\mathfrak{L}}$ is a tamely imbedded subset of $\mathfrak{X}$ with $\mathfrak{L} = N(\hat{\mathfrak{L}})$ then in this section $\mathscr{I}(\hat{\mathfrak{L}})$ denotes the set of generalized characters of $\mathfrak{L}$ which vanish outside $\hat{\mathfrak{L}}$ and $\mathscr{C}(\hat{\mathfrak{L}})$ denotes the complex valued class functions of $\mathfrak{L}$ which vanish outside $\hat{\mathfrak{L}}$. Similarly, $\mathscr{I}_0(\hat{\mathfrak{L}})(\mathscr{C}_0(\hat{\mathfrak{L}}))$ is the subset of $\mathscr{I}(\hat{\mathfrak{L}})(\mathscr{C}(\hat{\mathfrak{L}}))$ vanishing at 1. R. Brauer and M. Suzuki noted that if $\hat{\mathfrak{L}}$ is a T. I. set in $\mathfrak{X}$ then the mapping τ from $\mathscr{C}_0(\hat{\mathfrak{L}})$ into the ring of class functions of $\mathfrak{X}$ defined by

$$\alpha^\tau = \alpha^*$$

is an isometry ([24], p. 662). They were then able to extend this isometry to certain subsets of $\mathscr{C}(\hat{\mathfrak{L}})$. Several authors have since then used this technique and it has played an important role in recent work in group theory.

In this chapter these results will be generalized in two ways. First we will consider tamely imbedded subsets of $\mathfrak{X}$ rather than T. I. sets in $\mathfrak{X}$. Secondly we will show that under a variety of conditions τ can be extended to various large subsets of $\mathscr{C}(\hat{\mathfrak{L}})$. The results proved in this chapter are important for the proof of the main theorem of this paper. However it is unnecessary in general to assume that $\mathfrak{X}$ has odd order or that $\mathfrak{X}$ is a minimal simple group.

The following notation will be used throughout this section.

For a tamely imbedded subset $\hat{\mathfrak{L}}$ of $\mathfrak{X}$ let $\mathfrak{L} = N(\hat{\mathfrak{L}})$ and for

$1 \leqq i \leqq n$ let $\mathfrak{H}_i$ and $\mathfrak{N}_i$ have the same meaning as in Definition 9.1. Define $\mathfrak{H}_0 = 1$ and

$$\mathfrak{L}_i = \{L \mid L \in \mathfrak{D}, C(L) \subseteq \mathfrak{N}_i\} \qquad \text{for } 1 \leqq i \leqq n .$$

For $L \in \mathfrak{L}_i$, $0 \leqq i \leqq n$ let

$$\mathfrak{A}_L = \{LH \mid LH = HL, H \in \mathfrak{H}_i\} = L\{\mathfrak{H}_i \cap C(L)\} . \tag{9.2}$$

Since $\hat{\mathfrak{L}}$ is tamely imbedded in $\mathfrak{X}$ it follows from (9.2) and Definition 9.1 that for $L \in \mathfrak{L}_i$, $0 \leqq i \leqq n$

$$|C(L)| = |C(L) \cap \mathfrak{L}| \; |\mathfrak{A}_L| . \tag{9.3}$$

For $\alpha \in \mathscr{C}_0(\hat{\mathfrak{L}})$ and $1 \leqq i \leqq n$ define

$$\alpha_i = \alpha |_{\mathfrak{L} \cap \mathfrak{N}_i} .$$

Let α_{i1} be the class function of $\mathfrak{N}_i/\mathfrak{H}_i$ which satisfies

$$\alpha_{i1} |_{\mathfrak{L} \cap \mathfrak{N}_i} = \alpha_i .$$

Let α_{i2} be the class function of $\mathfrak{N}_i$ induced by α_i. Define

$$\alpha^\tau = \alpha^* + \sum_{i=1}^{n} (\alpha_{i1} - \alpha_{i2})^* . \tag{9.4}$$

If $\alpha \in \mathscr{I}_0(\hat{\mathfrak{L}})$ then (9.4) implies that α^τ is a generalized character of $\mathfrak{X}$. It is an immediate consequence of the definition of induced characters that for $1 \leqq i \leqq n$

$$\begin{aligned} &\alpha_{i1}(A) = \alpha(L) && \text{for } L \in \mathfrak{L}_i, A \in \mathfrak{A}_L \\ &\alpha_{i2}(A) = 0 && \text{for } L \in \mathfrak{L}_i, A \in \mathfrak{A}_L, A \neq L \\ &\alpha_{i2}(L) = |C(L) \cap \mathfrak{H}_i| \alpha(L) && \text{for } L \in \mathfrak{L}_i . \end{aligned} \tag{9.5}$$

LEMMA 9.1. *Suppose that $\hat{\mathfrak{L}}$ is a tamely imbedded subset of $\mathfrak{X}$. If $\alpha \in \mathscr{C}_0(\hat{\mathfrak{L}})$ let α^τ be defined by* (9.4). *Then $\alpha^\tau(X) = 0$ if X is not conjugate to an element of $\mathfrak{A}_L$ for any $L \in \bigcup_{i=0}^{n} \mathfrak{L}_i$, while*

$$\alpha^\tau(A) = \alpha(L) \quad \text{for} \quad A \in \mathfrak{A}_L, L \in \bigcup_{i=0}^{n} \mathfrak{L}_i .$$

Proof. If $N \in \mathfrak{N}_i$ then a complement of $\mathfrak{H}_i$ in $\mathfrak{H}_i\langle N\rangle$ is solvable. Thus ([28] p. 162) for $1 \leqq i \leqq n$ every element of $\mathfrak{N}_i$ is conjugate to an element of the form $HL = LH$ with $L \in \mathfrak{L} \cap \mathfrak{N}_i$, $H \in \mathfrak{H}_i$. Suppose that L is not conjugate to an element of $\hat{\mathfrak{L}}^{\sharp}$; then since $\alpha \in \mathscr{C}_0(\hat{\mathfrak{L}})$, (9.4) implies that $\alpha^{\tau}(HL) = 0$. This implies that $\alpha^{\tau}(X) = 0$ unless X is conjugate to an element of $\mathfrak{A}_L$ for some $L \in \bigcup_{i=0}^{n} \mathfrak{L}_i$.

Let $A \in \mathfrak{A}_L, L \in \mathfrak{L}_i$ for some i with $0 \leqq i \leqq n$. Suppose that $X^{-1}LX \in \hat{\mathfrak{N}}_j$ for some $X \in \mathfrak{X}$ and some j with $1 \leqq j \leqq n, i \neq j$. Then $(|\mathfrak{H}_j|, |C(L)|) \neq 1$. Thus $i \neq 0$ and $L \in \hat{\mathfrak{N}}_i$. Furthermore $C(L) = C_{\mathfrak{L}}(L)C_{\mathfrak{H}_i}(L)$. By assumption $(|\mathfrak{H}_i|, |\mathfrak{H}_j|) = 1$ and $(|\mathfrak{H}_j|, |C_{\mathfrak{L}}(L)|) = 1$. Thus $(|C(L)|, |\mathfrak{H}_j|) = 1$ contrary to the choice of L. Since $\alpha \in \mathscr{C}_0(\hat{\mathfrak{L}})$, $\alpha_{j1} - \alpha_{j2}$ vanishes on $\mathfrak{N}_j - \hat{\mathfrak{N}}_j^{\sharp}$. Consequently (9.4) implies that

$$\begin{aligned} &\alpha^{\tau}(A) = \alpha^*(A) && \text{for } i = 0 \\ &\alpha^{\tau}(A) = \alpha^*(A) + (\alpha_{i1} - \alpha_{i2})^*(A) && \text{for } 1 \leqq i \leqq n . \end{aligned} \tag{9.6}$$

Since $\hat{\mathfrak{N}}_i$ is a T. I. set in $\mathfrak{X}$ with $N(\hat{\mathfrak{N}}_i) = \mathfrak{N}_i$ we get that

$$(\alpha_{i1} - \alpha_{i2})^*(A) = (\alpha_{i1} - \alpha_{i2})(A) .$$

Thus (9.6) yields that

$$\alpha^{\tau}(A) = \alpha^*(A) + (\alpha_{i1} - \alpha_{i2})(A) \quad \text{for } 1 \leqq i \leqq n . \tag{9.7}$$

Assume first that $A = L$. Then $\alpha^*(L) = |C(L) \cap \mathfrak{H}_i| \alpha(L)$. Hence (9.5), (9.6) and (9.7) yield that $\alpha^{\tau}(A) = \alpha(L)$. If $A \neq L$, then $\alpha^*(A) = 0$ and $1 \leqq i \leqq n$. Thus (9.5) and (9.7) yield that also in this case $\alpha^{\tau}(A) = \alpha(L)$. The proof is complete in all cases.

LEMMA 9.2. *Suppose that $\hat{\mathfrak{L}}$ is a tamely imbedded subset of $\mathfrak{X}$. If $\alpha \in \mathscr{C}_0(\hat{\mathfrak{L}})$ let α^{τ} be defined by* (9.4). *Then for $1 \leqq i \leqq n$*

$$\alpha^{\tau}(N) = \alpha_{i1}(N) \quad \textit{for } N \in \hat{\mathfrak{N}}_i \cup \mathfrak{H}_i .$$

Furthermore $\alpha^{\tau}|_{\mathfrak{N}_i}$ is a linear combination of characters of $\mathfrak{N}_i/\mathfrak{H}_i$.

Proof. If $N \in \mathfrak{H}_i$ then by Lemma 9.1 and the definition of α_{i1}

$$\alpha^{\tau}(N) = 0 = \alpha_{i1}(1) = \alpha_{i1}(N) .$$

If $N \in \hat{\mathfrak{N}}_i$, and $\alpha^{\tau}(N) \neq 0$, then N is conjugate to an element A of $\mathfrak{A}_L$ for some $L \in \mathfrak{L}_i$. Thus by (9.5) and Lemma 9.1 $\alpha^{\tau}(N) = \alpha_{i1}(N)$ as required.

Let θ be an irreducible character of $\mathfrak{N}_i$ which does not have $\mathfrak{H}_i$ in its kernel. Then

$$(9.8) \qquad (\alpha^\tau|_{\mathfrak{N}_i}, \theta) = \frac{1}{|\mathfrak{N}_i|}\Sigma_{\mathfrak{N}_i}\alpha^\tau(N)\overline{\theta(N)} .$$

By Lemma 4.3 θ vanishes on $\mathfrak{N}_i - \hat{\mathfrak{N}}_i - \mathfrak{H}_i$ hence (9.8) and the first part of the lemma yield that

$$\begin{aligned}(\alpha^\tau|_{\mathfrak{N}_i}, \theta) &= \frac{1}{|\mathfrak{N}_i|}\Sigma_{\mathfrak{N}_i}\alpha^\tau(N)\overline{\theta(N)} \\ &= \frac{1}{|\mathfrak{N}_i|}\Sigma_{\mathfrak{N}_i}\alpha_{i1}(N)\overline{\theta(N)} = (\alpha_{i1}, \theta) .\end{aligned}$$

Since α_{i1} is a linear combination of characters of $\mathfrak{N}_i/\mathfrak{H}_i$ this yields that $(\alpha^\tau|_{\mathfrak{N}_i}, \theta) = 0$. The lemma is proved.

LEMMA 9.3. *Suppose that $\hat{\mathfrak{L}}$ is a tamely imbedded subset of $\mathfrak{X}$. If $\alpha \in \mathscr{C}_0(\hat{\mathfrak{L}})$ and α^τ is defined by* (9.4) *then*

$$(\alpha^\tau, 1_{\mathfrak{X}})_{\mathfrak{X}} = (\alpha, 1_{\mathfrak{L}})_{\mathfrak{L}} .$$

Proof. Let $\mathfrak{C}_1, \mathfrak{C}_2, \cdots$ be all the conjugate classes of $\mathfrak{X}$ which contain elements of $\hat{\mathfrak{L}}^\sharp$. Let $L_1, L_2, \cdots$ be elements in $\bigcup_{i=0}^{n}\mathfrak{L}_i$ such that $L_j \in \mathfrak{C}_j \cap \hat{\mathfrak{L}}^\sharp$. The number of elements in $\mathfrak{X}$ which are conjugate to an element of $\mathfrak{A}_{L_j}$ is easily seen to be

$$|\mathfrak{C}_j|\,|\mathfrak{A}_{L_j}| = \frac{|\mathfrak{X}|}{|C(L_j)|}|\mathfrak{A}_{L_j}| .$$

Thus by Lemma 9.1 and (9.3)

$$(9.9) \qquad \begin{aligned}(\alpha^\tau, 1_{\mathfrak{X}})_{\mathfrak{X}} &= \frac{1}{|\mathfrak{X}|}\Sigma_j\frac{|\mathfrak{X}|}{|C(L_j)|}|\mathfrak{A}_{L_j}|\,\alpha(L_j) \\ &= \frac{1}{|\mathfrak{L}|}\Sigma_j\frac{|\mathfrak{L}|}{|C(L_j)\cap\mathfrak{L}|}\alpha(L_j) .\end{aligned}$$

By assumption $\mathfrak{C}_1 \cap \hat{\mathfrak{L}}^\sharp, \mathfrak{C}_2 \cap \hat{\mathfrak{L}}^\sharp, \cdots$ are the conjugate classes of $\mathfrak{L}$ which contain elements of $\hat{\mathfrak{L}}^\sharp$. Since $\alpha \in \mathscr{C}_0(\hat{\mathfrak{L}})$ this yields that

$$(\alpha, 1_{\mathfrak{L}})_{\mathfrak{L}} = \frac{1}{|\mathfrak{L}|}\Sigma_j\frac{|\mathfrak{L}|}{|C(L_j)\cap\mathfrak{L}|}\alpha(L_j) .$$

Therefore (9.9) implies the desired equality.

LEMMA 9.4. *Suppose that $\hat{\mathfrak{L}}$ is a tamely imbedded subset of $\mathfrak{X}$.*

Let θ be a generalized character of $\mathfrak{X}$ such that for $L \in \bigcup_{i=0}^{n} \mathfrak{L}_i$, θ is constant on $\mathfrak{A}_L$. If $\alpha, \beta \in \mathscr{I}_0(\hat{\mathfrak{L}})$ and if α^τ, β^τ are defined by (9.4) *then*

$$(\alpha^\tau, \theta)_{\mathfrak{X}} = (\alpha, \theta_{|\mathfrak{L}})_{\mathfrak{L}}$$
$$(\alpha^\tau, \beta^\tau)_{\mathfrak{X}} = (\alpha, \beta)_{\mathfrak{L}} .$$

Proof. Since θ is constant on $\mathfrak{A}_L$ for $L \in \bigcup_{i=0}^{n} \mathfrak{L}_i$ it follows from Lemma 9.1 that

$$\{\alpha\bar{\theta}_{|\mathfrak{L}}\}^\tau = \alpha^\tau\bar{\theta} .$$

Thus by Lemma 9.3

$$(\alpha^\tau, \theta)_{\mathfrak{X}} = (\alpha^\tau\bar{\theta}, 1_{\mathfrak{X}})_{\mathfrak{X}} = (\alpha\bar{\theta}_{|\mathfrak{L}}, 1_{\mathfrak{L}})_{\mathfrak{L}} = (\alpha, \theta_{|\mathfrak{L}})_{\mathfrak{L}} .$$

By Lemma 9.1 β^τ is a generalized character of $\mathfrak{X}$ which is constant on $\mathfrak{A}_L$ for $L \in \bigcup_{i=0}^{n} \mathfrak{L}_i$. If now θ is replaced by β^τ in the first equation of the lemma the second equation follows.

LEMMA 9.5. *Suppose that $\hat{\mathfrak{L}}$ is a tamely imbedded subset of $\mathfrak{X}$. Let θ be a class function of $\mathfrak{X}$ which is constant on $\mathfrak{A}_L$ for $L \in \bigcup_{i=0}^{n} \mathfrak{L}_i$. Let $\mathfrak{X}_0$ be the set of all elements in $\mathfrak{X}$ which are conjugate to some element of $\mathfrak{A}_L$ with $L \in \bigcup_{i=0}^{n} \mathfrak{L}_i$. Then*

$$\frac{1}{|\mathfrak{X}|}\Sigma_{\mathfrak{X}_0}\theta(X) = \frac{1}{|\mathfrak{L}|}\Sigma_{\hat{\mathfrak{L}}^\sharp}\theta(L) .$$

Proof. Define $\alpha \in \mathscr{C}_0(\hat{\mathfrak{L}})$ by

$$\alpha(L) = \theta(L) \quad \text{if } L \in \hat{\mathfrak{L}}^\sharp$$
$$\alpha(L) = 0 \quad \text{if } L \in \hat{\mathfrak{L}} - \hat{\mathfrak{L}}^\sharp .$$

By Lemma 9.1

$$\alpha^\tau(X) = \theta(X) \quad \text{if } X \in \mathfrak{X}_0$$
$$\alpha^\tau(X) = 0 \quad \text{if } X \in \mathfrak{X} - \mathfrak{X}_0 .$$

Consequently Lemma 9.3 implies that

$$\frac{1}{|\mathfrak{X}|}\sum_{\mathfrak{X}_0}\theta(X) = (\alpha^\tau, 1_{\mathfrak{X}})_{\mathfrak{X}} = (\alpha, 1_{\mathfrak{L}})_{\mathfrak{L}} = \frac{1}{|\mathfrak{L}|}\sum_{\hat{\mathfrak{L}}^\sharp}\alpha(L)$$
$$= \frac{1}{|\mathfrak{L}|}\sum_{\hat{\mathfrak{L}}^\sharp}\theta(L) .$$

Lemma 9.5 is of great importance. Even the special case in which $\theta = 1_{\mathfrak{X}}$ is of considerable interest and plays a role in section 26. In this special case, Lemma 9.5 asserts simply that $|\mathfrak{X}_0|/|\mathfrak{X}| = |\hat{\mathfrak{L}}^\sharp|/|\mathfrak{L}|$.

10. Coherent Sets of Characters

Throughout this section let $\hat{\mathfrak{L}}$ be a tamely imbedded subset of the group $\mathfrak{X}$. Let $\mathfrak{L} = N(\hat{\mathfrak{L}})$ and let $\mathscr{I}(\hat{\mathfrak{L}})$ be the set of generalized characters of $\mathfrak{L}$ which vanish outside $\hat{\mathfrak{L}}$. Let τ be defined by (9.4).

DEFINITION 10.1. A set $\mathscr{S}$ of generalized characters of $\mathfrak{L}$ is *coherent* if and only if

(i) $\mathscr{I}_0(\mathscr{S}) \neq 0$.

(ii) It is possible to extend τ from $\mathscr{I}_0(\mathscr{S})$ to a linear isometry mapping $\mathscr{I}(\mathscr{S})$ into the set of generalized characters of $\mathfrak{X}$.

(iii) $\mathscr{I}_0(\mathscr{S}) \subseteq \mathscr{I}_0(\hat{\mathfrak{L}})$.

It is easily seen that if $\mathscr{S}$ is a coherent set and $\mathscr{T} \subseteq \mathscr{S}$ with $\mathscr{I}_0(\mathscr{T}) \neq 0$ then also $\mathscr{T}$ is a coherent set. It is more difficult to decide whether the union of two coherent subsets of $\mathscr{I}(\hat{\mathfrak{L}})$ is coherent. Examples are known in which $\mathscr{S}$ consists of irreducible characters of $\mathfrak{L}$ and is not coherent though $\mathscr{I}_0(\mathscr{S}) \neq 0$ [25]. In these examples $\hat{\mathfrak{L}}$ is even a T. I. set in $\mathfrak{X}$. The main purpose of this section is to give some sufficient conditions which ensure that a subset $\mathscr{S}$ of $\mathscr{I}(\hat{\mathfrak{L}})$ is coherent.

LEMMA 10.1. *Suppose that $\hat{\mathfrak{L}}$ is a tamely imbedded subset of $\mathfrak{X}$. Let $\mathscr{S} = \{\lambda_i \mid 1 \leq i \leq n\}$ with $n \geq 2$. Assume that for $1 \leq i \leq n$, λ_i is an irreducible character of $\mathfrak{L}$. Furthermore $\lambda_i(L) = \lambda_1(L)$ for $L \in \mathfrak{L} - \hat{\mathfrak{L}}^\sharp$. Then $\mathscr{S}$ is coherent. Furthermore, if τ_1 and τ_2 are extensions of τ to $\mathscr{S}$ then either $\tau_1 = \tau_2$ or $|\mathscr{S}| = 2$ and $\lambda_i^{\tau_1} = -\lambda_{3-i}^{\tau_2}$, $i = 1, 2$.*

Proof. For $1 \leq i, j \leq n$ let $\alpha_{ij} = \lambda_i - \lambda_j$, then $\alpha_{ij} \in \mathscr{I}_0(\mathscr{S})$. Thus $\mathscr{I}_0(\mathscr{S}) \neq 0$ since $n \geq 2$. Furthermore α_{ij}^τ is defined. Since τ is an isometry this yields that

$$(10.1) \qquad (\alpha_{ij}^\tau, \alpha_{i'j'}^\tau) = (\alpha_{ij}, \alpha_{i'j'}) = \delta_{ii'} - \delta_{ji'} - \delta_{ij'} + \delta_{jj'} .$$

In particular (10.1) implies that if $i \neq j$ then $\|\alpha_{ij}^{\tau}\|^2 = 2$. By Lemma 9.1 $\alpha_{ij}^{\tau}(1) = 0$, therefore α_{ij}^{τ} is the difference of two irreducible characters of $\mathfrak{X}$.

If $n > 2$, then it follows from equation (10.1) that $(\alpha_{1i}^{\tau}, \alpha_{1j}^{\tau}) = 1$ if $1 < i, j$ and $i \neq j$. It is now a simple consequence of (10.1) that there exists a unique irreducible character of $\mathfrak{X}$ which is not orthogonal to any α_{1i}^{τ} for $2 \leqq i \leqq n$. Furthermore if Δ_1 is chosen to be plus or minus this character then it may be assumed that

$$(\alpha_{1i}^{\tau}, \Delta_1) = 1 \quad \text{for } 2 \leqq i \leqq n .$$

Now define Δ_i by

$$\alpha_{1i}^{\tau} = \Delta_1 - \Delta_i, \quad 2 \leqq i \leqq n .$$

This implies that

$$\alpha_{ij}^{\tau} = \Delta_i - \Delta_j .$$

Hence (10.1) yields that the generalized characters Δ_i, $1 \leqq i \leqq n$ are pairwise orthogonal and that they each have weight one. It is easily shown that a rational integral linear combination of the characters λ_i of degree zero is a rational integral linear combination of the generalized characters α_{1i}. Hence if $\mathscr{S}^{\tau}$ is the set of generalized characters Δ_i, $1 \leqq i \leqq n$, then the linear mapping sending λ_i into Δ_i is an isometry. Thus, $\mathscr{S}$ is coherent and the extension of τ to $\mathscr{S}$ is unique in this case.

If $n = 2$, define Δ_i for $i = 1, 2$ by $\alpha_{12}^{\tau} = \Delta_1 - \Delta_2$, where Δ_i has weight one. Any rational integral linear combination of λ_1 and λ_2 of degree zero is a multiple of α_{12}. Thus, if τ_1 is any extension of τ to $\mathscr{S}$, $\lambda_i^{\tau_1} = \Delta_i$ or $\lambda_i^{\tau_1} = -\Delta_{3-i}$ for $i = 1, 2$. The proof is complete.

Before proving the main result of this section, another definition is needed. The following notation is introduced temporarily.

Let $\mathscr{S}$ be a subset of $\mathscr{I}(\hat{\mathfrak{L}})$ which consists of pairwise orthogonal characters. If $\mathscr{S}_1 \subseteq \mathscr{S}$, let $x(\mathscr{S}_1)$ denote the smallest weight of any character in $\mathscr{S}_1$ of minimum degree. If $\mathscr{S}_1$ and $\mathscr{T}$ are coherent subsets of $\mathscr{S}$ and τ_1 and τ_2 are extensions of τ to $\mathscr{S}_1$ and $\mathscr{T}$ respectively, define

$$\mathscr{A}(\mathscr{S}_1, \tau_1; \mathscr{T}, \tau_2) = \{\alpha \mid$$

(i) $\alpha \in \mathscr{I}_0(\mathscr{S})$.
(ii) $\alpha^{\tau} = \Delta_1 + \Delta_2$, where
  (a) $\Delta_2 \in \mathscr{C}(\mathscr{T}^{\tau_2})$,
  (b) Δ_1 is not orthogonal to $\mathscr{I}_0(\mathscr{S}_1)^{\tau}$,
  (c) $\|\Delta_1\|^2 \leqq x(\mathscr{S}_1)\}$.

DEFINITION 10.2. Let $\mathscr{S}_1$ be a coherent subset of $\mathscr{S}$ and let τ^* be an extension of τ to $\mathscr{S}_1$. The *pair* $(\mathscr{S}_1, \tau^*)$ *is subcoherent in* $\mathscr{S}$ if the following conditions are satisfied: If $\mathscr{T}$ is any coherent subset of $\mathscr{S}$ which is orthogonal to $\mathscr{S}_1$ and if τ_1 and τ_2 are extensions of τ to $\mathscr{S}_1$ and $\mathscr{T}$ respectively, then

(i) $\mathscr{S}_1^{\tau_1}$ is orthogonal to $\mathscr{T}^{\tau_2}$.

(ii) If $\alpha \in \mathscr{A}(\mathscr{S}_1, \tau_1; \mathscr{T}, \tau_2)$, then α^τ is a sum of two generalized characters, one of which is orthogonal to $\mathscr{S}_1^{\tau^*}$ and the other is in $\pm\mathscr{S}_1^{\tau^*}$.

If $(\mathscr{S}_1, \tau^*)$ is subcoherent in $\mathscr{S}$, we also say that $\mathscr{S}_1$ is subcoherent in $\mathscr{S}$, which causes no confusion in case τ^* has been designated.

Hypothesis 10.1.

(i) $\hat{\mathfrak{L}}$ *is a tamely imbedded subset of the group* $\mathfrak{X}$.

(ii) *For* $1 \leqq i \leqq k$, $\mathscr{S}_i = \{\lambda_{is} \mid 1 \leqq s \leqq n_i\}$ *is a subset of* $\mathscr{I}(\hat{\mathfrak{L}})$.

(iii) $\mathscr{S} = \bigcup_{i=1}^{k} \mathscr{S}_i$ *consists of pairwise orthogonal characters.*

(iv) *For any* i *with* $1 \leqq i \leqq k$, $\mathscr{S}_i$ *is coherent with isometry* τ_i. $\mathscr{S}_i$ *is partitioned into sets* $\mathscr{S}_{ij}$ *such that each* $\mathscr{S}_{ij}$ *either consists of irreducible characters of the same degree and* $|\mathscr{S}_{ij}| \geqq 2$ *or* $(\mathscr{S}_{ij}, \tau_{ij})$ *is subcoherent in* $\mathscr{S}$ *where* $\tau_{ij} = \tau_i$ *on* $\mathscr{S}_{ij}$.

(v) *For* $1 \leqq i \leqq k, 1 \leqq s \leqq n_i$, *there exist integers* ℓ_{is} *such that*

$$1 = \ell_{11} \leqq \ell_{21} \leqq \cdots \leqq \ell_{k1},$$
$$\lambda_{is}(1) = \ell_{is}\lambda_{11}(1), \ \ell_{i1} \mid \ell_{is}.$$

(vi) λ_{11} *is an irreducible character of* $\mathfrak{L}$.

(vii) *For any integer* m *with* $1 < m \leqq k$,

$$\sum_{i=1}^{m-1} \sum_{s=1}^{n_i} \frac{\ell_{is}^2}{\|\lambda_{is}\|^2} > 2\ell_{m1}. \tag{10.2}$$

THEOREM 10.1. *Suppose that Hypothesis* 10.1 *is satisfied. Then* $\mathscr{S}$ *is coherent. There is an extension* τ^* *of* τ *to* $\mathscr{I}(\mathscr{S})$ *such that either* τ^* *agrees with* τ_i *on* $\mathscr{S}_i$ *or* $\mathscr{S}_i = \{\lambda_1, \lambda_2\}$ *and* $\lambda_j^{\tau_i} = -\lambda_{3-j}^{\tau^*}$ *for* $j = 1, 2$.

Proof. The proof is by induction on k. If $k = 1$ the theorem follows by assumption.

It is easily seen that $\bigcup_{i=1}^{k-1} \mathscr{S}_i$ satisfies the assumption of the theorem. Hence by induction it may be assumed that $\bigcup_{i=1}^{k-1} \mathscr{S}_i$ is coherent. Let τ^* denote an extension of τ to $\bigcup_{i=1}^{k-1} \mathscr{S}_i$, with the property that for

$1 \leqq i \leqq k-1$, τ^* agrees with τ_i on $\mathscr{S}_i$, or $\mathscr{S}_i = \{\lambda_1, \lambda_2\}$ and $\lambda_j^{\tau^*} = -\lambda_{3-j}^{\tau_i}$, $j = 1, 2$.

Choose the notation so that λ_{k1} has minimum weight among the characters in $\mathscr{S}_k$ of degree $\ell_{k1}\lambda_{11}(1)$. Let $\mathscr{S}_{k1}$ be the subset $\mathscr{S}_{ki}$ which contains λ_{k1}. For $1 \leqq s \leqq n_k$ define

$$\beta_s = \ell_{ks}\lambda_{11} - \lambda_{ks} .$$

Thus $\beta_s \in \mathscr{I}_0(\mathscr{S})$ and β_s^τ is defined. Define the integer y by

$$(\lambda_{11}^{\tau^*}, \beta_1^\tau) = \ell_{k1} - y . \tag{10.3}$$

If $(i, t) \neq (1, 1)$ and $1 \leqq i \leqq k-1, 1 \leqq t \leqq n_i$, then by (10.3)

$$\begin{aligned}(\lambda_{it}^{\tau^*}, \beta_1^\tau) &= (\ell_{it}\lambda_{11}^{\tau^*}, \beta_1^\tau) - (\ell_{it}\lambda_{11}^{\tau^*} - \lambda_{it}^{\tau^*}, \beta_1^\tau) \\ &= \ell_{it}(\ell_{k1} - y) - \ell_{it}\ell_{k1} = -y\ell_{it} .\end{aligned} \tag{10.4}$$

Since λ_{11} is irreducible and τ is an isometry on $\mathscr{I}_0(\mathscr{S})$

$$\| \beta_s^\tau \|^2 = \ell_{ks}^2 + \| \lambda_{ks} \|^2 \quad \text{for } 1 \leqq s \leqq n_k . \tag{10.5}$$

By (10.4)

$$\beta_1^\tau = \ell_{k1}\lambda_{11}^{\tau^*} - y \sum_{i=1}^{k-1} \sum_{s=1}^{n_i} \frac{\ell_{is}}{\| \lambda_{is} \|^2} \lambda_{is}^{\tau^*} + \Delta \tag{10.6}$$

where $(\Delta, \lambda_{is}^{\tau^*}) = 0$ for $1 \leqq i \leqq k-1, 1 \leqq s \leqq n_i$. Equations (10.5) and (10.6) now yield that

$$\ell_{k1}^2 - 2\ell_{k1}y + y^2 \sum_{i=1}^{k-1} \sum_{s=1}^{n_i} \frac{\ell_{is}^2}{\| \lambda_{is} \|^2} + \| \Delta \|^2 = \ell_{k1}^2 + \| \lambda_{k1} \|^2 . \tag{10.7}$$

If $y \neq 0$ then since y is an integer (10.2) and (10.7) imply that

$$0 \leqq 2\ell_{k1}(y^2 - y) < \| \lambda_{k1} \|^2 - \| \Delta \|^2 .$$

Therefore

$$\| \Delta \|^2 < \| \lambda_{k1} \|^2 \quad \text{if } y \neq 0 . \tag{10.8}$$

We will show that $y = 0$. By Hypothesis 10.1 (iv), τ can be extended from $\mathscr{I}_0(\mathscr{S}_k)$ to a linear isometry τ_k on $\mathscr{I}(\mathscr{S}_k)$. For $1 \leqq s \leqq n_k$ let Δ_s be the image of λ_{ks} under this extension. If $(\mathscr{S}_{kj}, \tau_{kj})$ is subcoherent in $\mathscr{S}$, then $\mathscr{S}_{kj}^{\tau_{kj}}$ is orthogonal to $\bigcup_{i=1}^{k-1} \mathscr{S}_i^{\tau^*}$. Suppose that $\mathscr{S}_{kj}$ consists of irreducible characters of the same degree. If $\mathscr{S}_{kj}^{\tau_{kj}}$ is not orthogonal to $\bigcup_{i=1}^{k-1} \mathscr{S}_i^{\tau^*}$, then there exists $\lambda \in \mathscr{S}_{kj}$ and $\lambda_1 \in \mathscr{S}_{im}$ for some i and m with $1 \leqq i \leqq k-1$, such that $(\lambda^{\tau_{kj}}, \lambda_1^{\tau^*}) \neq 0$. Assume first that $\mathscr{S}_{im}$ consists of irreducible characters of the same degree.

Then it may be assumed that $\lambda = \lambda_{kt}, \lambda_{kt'} \in \mathscr{S}_{kj}, \lambda_{kt} \neq \lambda_{kt'}$ and $\lambda_1 = \lambda_{is}, \lambda_{is'} \in \mathscr{S}_{im}, \lambda_{is} \neq \lambda_{is'}$. Thus $\lambda_{kt}^{\tau kj} = \varepsilon\lambda_{is}^{\tau*}$ for suitable $\varepsilon = \pm 1$. Hence

$$0 = (\lambda_{is}^{\tau*} - \lambda_{is'}^{\tau*}, \lambda_{kt}^{\tau k} - \lambda_{kt'}^{\tau k}) = \varepsilon + (\lambda_{is'}^{\tau*}, \lambda_{kt'}^{\tau k}) .$$

Hence $\lambda_{is'}^{\tau*} = -\varepsilon\lambda_{kt'}^{\tau k}$. Therefore

$$0 = (\lambda_{is}^{\tau*} - \lambda_{is'}^{\tau*})(1) = \varepsilon(\lambda_{kt}^{\tau kj} + \lambda_{kt'}^{\tau kj})(1) = 2\varepsilon\lambda_{kt}^{\tau kj}(1) ,$$

which is not the case as $\| \lambda^{\tau kj} \|^2 = 1$. Suppose now that $\mathscr{S}_{im}$ is subcoherent in $\mathscr{S}$. Then $\mathscr{S}_{im}^{\tau*}$ is orthogonal to $\mathscr{S}_k^{\tau k}$ by definition. Therefore, for $2 \leqq s \leqq n_k$,

$$(10.9) \qquad \left(\Delta, \frac{\ell_{ks}}{\ell_{k1}}\Delta_1 - \Delta_s\right) = \left(\beta_1^\tau, \frac{\ell_{ks}}{\ell_{k1}}\Delta_1 - \Delta_s\right) = -\frac{\ell_{ks}}{\ell_{k1}}\| \lambda_{k1} \|^2 .$$

Thus, Δ is not orthogonal to $\mathscr{I}_0(\mathscr{S}_{k1})^\tau$. If $\mathscr{S}_{k1}$ consists of irreducible characters this yields that $\| \Delta \|^2 \geqq 1$. Hence, $y = 0$ by (10.8). Suppose that $(\mathscr{S}_{k1}, \tau_{k1})$ is subcoherent in $\mathscr{S}$. If $y \neq 0$, (10.8) implies that

$$(10.10) \qquad \beta_1^\tau = \Delta_1 + \Delta$$

where $\Delta \in \pm\mathscr{S}_{k1}^{\tau k1}$ and Δ_1 is orthogonal to $\mathscr{S}_{k1}^{\tau k1}$. By changing notation if necessary it may be assumed that

$$(10.11) \qquad \Delta = \pm\Delta_1$$

by (10.9). Now (10.9), (10.10) and (10.11) yield that

$$(10.12) \qquad \| \lambda_{k1} \|^4 = | (\Delta, \Delta) |^2 \leqq \| \Delta \|^2 \cdot \| \lambda_{k1} \|^2 .$$

Hence, (10.8) and (10.12) imply that $y = 0$ in all cases. Thus, (10.3) becomes

$$(10.13) \qquad (\lambda_{11}^{\tau*}, \beta_1^\tau) = \ell_{k1} .$$

For $1 \leqq s \leqq n_k$,

$$\beta_s = \frac{\ell_{ks}}{\ell_{k1}}\beta_1 + \left(\frac{\ell_{ks}}{\ell_{k1}}\lambda_{k1} - \lambda_{ks}\right).$$

Therefore, (10.13) implies that

$$(10.14) \qquad (\lambda_{11}^{\tau*}, \beta_s^\tau) = \ell_{ks} , \qquad 1 \leqq s \leqq n_k .$$

For $1 \leqq s \leqq n_k$, define $\lambda_{ks}^{\tau*}$ by

$$(10.15) \qquad \beta_s^\tau = \ell_{ks}\lambda_{11}^{\tau*} - \lambda_{ks}^{\tau*} ,$$

and extend the definition to $\mathscr{I}(\mathscr{S})$ by linearity. This implies that $\lambda_{ks}^{\tau*} = \lambda_{ks}^{\tau k}$ or $\mathscr{S}_k = \{\lambda_1, \lambda_2\}$ and $\lambda_i^{\tau*} = -\lambda_{3-i}^{\tau k}$ for $i = 1, 2$. Hence, $\mathscr{S}_k^{\tau*}$ is orthogonal to $\bigcup_{i=1}^{k-1} \mathscr{S}_i^{\tau^*}$ and thus τ^* is an isometry on $\mathscr{I}(\mathscr{S})$. The proof is complete.

If $\mathscr{S}$ is a coherent subset of $\mathscr{I}(\hat{\mathfrak{L}})$, then τ will be used to denote an extension of τ to $\mathscr{I}(\mathscr{S})$.

Hypothesis 10.2.

(i) *$\hat{\mathfrak{L}}$ is a tamely imbedded subset of $\mathfrak{X}$ and $\mathfrak{H}_i$ is a supporting subgroup of $\hat{\mathfrak{L}}$. $\mathfrak{N}_i = N(\mathfrak{H}_i)$.*

(ii) *If θ is any non-principal irreducible character of $\mathfrak{H}_i$ and $\tilde{\theta}$ is the character of $\mathfrak{N}_i$ induced by θ, then $\tilde{\theta}$ is a sum of irreducible characters of $\mathfrak{N}_i$, all of which have the same degree and occur with the same multiplicity in $\tilde{\theta}$.*

LEMMA 10.2. *Suppose that Hypothesis* 10.2 *is satisfied. For any character α of $\mathfrak{H}_i$ let $\mathscr{S}_\alpha$ be the set of irreducible characters of $\mathfrak{N}_i$ whose restriction to $\mathfrak{H}_i$ coincides with α. If Θ is a generalized character of $\mathfrak{X}$ which is orthogonal to $\mathscr{I}_0(\mathscr{S}_\alpha)^*$ for all α with $(\alpha, 1_{\mathfrak{H}_i}) = 0$ then Θ is constant on the cosets of $\mathfrak{H}_i$ which lie in $\mathfrak{N}_i - \mathfrak{H}_i$.*

Proof. We first remark that by Lemma 4.3 characters in $\mathscr{S}_\alpha$ vanish on $\mathfrak{N}_i - \hat{\mathfrak{N}}_i - \mathfrak{H}_i$, and so generalized characters in $\mathscr{I}_0(\mathscr{S}_\alpha)$ vanish on $\mathfrak{N}_i - \hat{\mathfrak{N}}_i$. Suppose that θ_1, θ_2 are distinct characters in $\mathscr{S}_\alpha$ with $(\alpha, 1_{\mathfrak{H}_i}) = 0$. By assumption $(\Theta, (\theta_1 - \theta_2)^*) = 0$. Thus by the Frobenius reciprocity theorem $(\Theta_{|\mathfrak{N}_i}, \theta_1 - \theta_2) = 0$. Hence by Hypothesis 10.2 $\Theta_{|\mathfrak{N}_i} = \tilde{\gamma} + \beta$, where $\tilde{\gamma}$ is a class function of $\mathfrak{N}_i$ induced by a class function γ of $\mathfrak{H}_i$ and β is a generalized character of $\mathfrak{N}_i/\mathfrak{H}_i$. Thus $\Theta(N) = \beta(N)$ for $N \in \mathfrak{N}_i - \mathfrak{H}_i$. The proof is complete.

LEMMA 10.3. *Suppose that Hypothesis* 10.2 *is satisfied. Let $\mathscr{S}$ be a coherent subset of $\mathscr{I}(\hat{\mathfrak{L}})$ which consists of pairwise orthogonal characters of $\mathfrak{L}$. Assume further that $\mathscr{S}$ contains at least two irreducible characters. Then if $\lambda \in \mathscr{S}$, λ^τ is constant on the cosets of $\mathfrak{H}_i$ which lie in $\mathfrak{N}_i - \mathfrak{H}_i$.*

Proof. Suppose that θ_1, θ_2 are distinct irreducible characters of $\mathfrak{N}_i$ which do not contain $\mathfrak{H}_i$ in their kernel such that $\theta_{1|\mathfrak{H}_i} = \theta_{2|\mathfrak{H}_i}$. We will show that

$$(10.16) \qquad (\lambda^\tau_{|\mathfrak{N}_i}, \theta_1 - \theta_2) = 0 .$$

By Lemma 4.3 θ_1 and θ_2 vanish on $\mathfrak{N}_i - \hat{\mathfrak{N}}_i - \mathfrak{H}_i$. Since $\hat{\mathfrak{N}}_i$ is a T. I. set in $\mathfrak{X}$ and $\mathfrak{N}_i = N(\hat{\mathfrak{N}}_i)$ the mapping sending $\theta_1 - \theta_2$ into

$(\theta_1 - \theta_2)^*$ defines an isometry on $\mathscr{I}_0(\{\theta_1, \theta_2\})$. By Lemma 10.1 this can be extended to an isometry of $\mathscr{I}(\{\theta_1, \theta_2\})$. Let Θ_1, Θ_2 be the respective images of θ_1, θ_2 under this isometry. By assumption $\mathscr{S}$ contains two irreducible characters λ_1 and λ_2. Since

$$\lambda_j(1)\lambda - \lambda(1)\lambda_j \in \mathscr{I}_0(\mathscr{S})$$

for $j = 1, 2$, Lemma 9.2 implies that if (10.16) is violated then

$$(\lambda^{\tau}_{j\,|\mathfrak{N}_t}, \theta_1 - \theta_2) \neq 0 \quad \text{for } j = 1, 2 .$$

Thus by the Frobenius reciprocity theorem

$$(\lambda^{\tau}_j, \Theta_1 - \Theta_2) = (\lambda^{\tau}_j, (\theta_1 - \theta_2)^*) \neq 0 \quad \text{for } j = 1, 2 .$$

Thus by changing notation if necessary it may be assumed that $\lambda^{\tau}_j = \pm\Theta_j$ for $j = 1, 2$, where the sign is independent of j. Hence

$$(10.17) \qquad (\lambda_1(1)\lambda^{\tau}_2 - \lambda_2(1)\lambda^{\tau}_1, \Theta_1 - \Theta_2) = \pm(\lambda_1(1) + \lambda_2(1)) \neq 0 .$$

Since $\lambda_1(1)\lambda_2 - \lambda_2(1)\lambda_1 \in \mathscr{I}_0(\mathscr{S})$ Lemma 9.2 implies that

$$((\lambda_1(1)\lambda^{\tau}_2 - \lambda_2(1)\lambda^{\tau}_1)_{|\mathfrak{N}_t}, \theta_1 - \theta_2) = 0 .$$

Thus by the Frobenius reciprocity theorem

$$(\lambda_1(1)\lambda^{\tau}_2 - \lambda_2(1)\lambda^{\tau}_1, \Theta_1 - \Theta_2) = (\lambda_1(1)\lambda^{\tau}_2 - \lambda_2(1)\lambda^{\tau}_1, (\theta_1 - \theta_2)^*) = 0$$

contrary to (10.17). Therefore (10.16) must hold. The result now follows from Lemma 10.2.

LEMMA 10.4. *Suppose that the assumptions of Lemma* 10.3 *are satisfied. Let a be the least common multiple of all the orders of elements in* $\hat{\mathfrak{L}}$. *If* λ *is an irreducible character in* $\mathscr{S}$, *then* $\mathscr{Q}_a$ *contains all the values assumed by* λ^{τ}.

Proof. By assumption $\mathscr{S}$ contains another irreducible character λ_1. Let σ be any automorphism of $\mathscr{Q}_{|\mathfrak{X}|}$ whose fixed field contains $\mathscr{Q}_a$. Then since $\lambda_1(1)\lambda - \lambda(1)\lambda_1 \in \mathscr{I}_0(\mathscr{S})$ it follows directly from (9.4) that

$$\begin{aligned}\sigma[\{\lambda_1(1)\lambda - \lambda(1)\lambda_1\}^{\tau}] &= \{\lambda_1(1)\sigma(\lambda) - \lambda(1)\sigma(\lambda_1)\}^{\tau} \\ &= \{\lambda_1(1)\lambda - \lambda(1)\lambda_1\}^{\tau} .\end{aligned}$$

Therefore

$$\lambda_1(1)\sigma(\lambda^{\tau}) - \lambda(1)\sigma(\lambda^{\tau}_1) = \lambda_1(1)\lambda^{\tau} - \lambda(1)\lambda^{\tau}_1 .$$

As $\|\lambda^{\tau}\|^2 = \|\lambda^{\tau}_1\|^2 = 1$, this implies that $\sigma(\lambda^{\tau}) = \lambda^{\tau}$. As σ may be an arbitrary automorphism of $\mathscr{Q}_{|\mathfrak{X}|}$ whose fixed field contains $\mathscr{Q}_a$ the result is proved.

LEMMA 10.5. *Suppose that* $\hat{\mathfrak{L}}$ *is a tamely imbedded subset of* $\mathfrak{X}$. *Let* $\mathfrak{A}_L$ *have the same meaning as in* (9.2) *and let* Θ *be a generalized character of* $\mathfrak{X}$ *which is constant on* $\mathfrak{A}_L$ *for* $L \in \bigcup_{i=0}^{n} \mathfrak{L}_i$. *Let* $\mathscr{S}$ *be a coherent subset of* $\mathscr{I}(\hat{\mathfrak{L}})$ *consisting of irreducible characters. Then there exist rational numbers b, c, and generalized characters* β, γ *of* $\mathfrak{L}$ *which are orthogonal to* $\mathscr{S}$ *such that if* $L \in \hat{\mathfrak{L}}^{\sharp}$ *then* $\Theta(L) = b\beta(L)$ *if* Θ *is orthogonal to* $\mathscr{S}^{\tau}$, *and* $\lambda^{\tau}(L) = \lambda(L) + c\gamma(L)$ *if* $\Theta = \lambda^{\tau} \in \mathscr{S}^{\tau}$.

Proof. It is an immediate consequence of Lemma 9.4 that if Θ is orthogonal to $\mathscr{S}^{\tau}$ and if $\xi = \Sigma_i \lambda_i(1)\lambda_i$, where λ_i ranges over $\mathscr{S}$, then

$$\Theta(L) = b_1\xi(L) + b_2\beta_1(L) \quad \text{for } L \in \hat{\mathfrak{L}}^{\sharp} \tag{10.18}$$

where b_1, b_2 are rational numbers and β_1 is a generalized character of $\mathfrak{L}$ which is orthogonal to $\mathscr{S}$. If $\Theta = \lambda^{\tau}$, then Lemma 9.4 yields that

$$\lambda^{\tau}(L) = \lambda(L) + c_1\xi(L) + c_2\gamma_1(L) \quad \text{for } L \in \hat{\mathfrak{L}}^{\sharp} \tag{19.19}$$

where c_1, c_2 are rational numbers and γ_1 is a generalized character of $\mathfrak{L}$ which is orthogonal to $\mathscr{S}$. There exists a generalized character ξ' of $\mathfrak{L}$ which is orthogonal to $\mathscr{S}$ such that

$$\xi + \xi' = \rho_{\mathfrak{L}} .$$

Since $\rho_{\mathfrak{L}}(L) = 0$ for $L \in \mathfrak{L}^{\sharp}$ (10.18) and (10.19) imply respectively that

$$\Theta(L) = -b_1\xi'(L) + b_2\beta_1(L)$$
$$\lambda^{\tau}(L) = \lambda(L) - c_1\xi'(L) + c_2\gamma_1(L) .$$

The lemma follows by a suitable change in notation.

It is worth noting that if the hypotheses of Lemma 10.3 are satisfied for every subgroup in a system of supporting subgroups of $\hat{\mathfrak{L}}$, then that lemma implies that λ^{τ} satisfies the hypotheses of Lemma 10.5. This fact will be used later in this paper.

11. Some Applications of Theorem 10.1

In this section we are concerned with the problem of finding conditions under which it is possible to apply Theorem 10.1. That theorem will then allow us to conclude that certain sets of characters are coherent. To clarify matters the main Hypothesis is stated separately. This also serves to introduce the notation.

Hypothesis 11.1.

(i) $\hat{\mathfrak{L}}_0$ *is a tamely imbedded subset of the group* $\mathfrak{X}$ *and* $\mathfrak{L}_0 = N(\hat{\mathfrak{L}}_0)$ *has odd order.* $\mathfrak{H}_0 \lhd \mathfrak{L}_0$ *and* $\hat{\mathfrak{L}}_0$ *is a union of cosets of* $\mathfrak{H}_0$. *Let* $\mathfrak{L} = \mathfrak{L}_0/\mathfrak{H}_0$ *and let* $\hat{\mathfrak{L}}$ *be the image of* $\hat{\mathfrak{L}}_0$ *in* $\mathfrak{L}$.

(ii) $\mathfrak{H}$ *and* $\mathfrak{K}$ *are normal subgroups of* $\mathfrak{L}$ *such that* $\mathfrak{H}$ *is nilpotent and*

$$(11.1) \qquad \mathfrak{H} \subseteq \bigcup_{H \in \mathfrak{H}^\#} C(H) \cap \mathfrak{K} \subseteq \hat{\mathfrak{L}} \subseteq \mathfrak{K} \subseteq \mathfrak{L} .$$

(iii) $\mathscr{S}$ *is the set of all characters of* $\mathfrak{L}$ *which are induced by non principal irreducible characters of* $\mathfrak{K}$, *each of which vanishes outside* $\hat{\mathfrak{L}}$. *Then* $\mathscr{S}$ *consists of pairwise orthogonal characters.*

(iv) *There exists an integer d such that* $d \,|\, \mathfrak{L}:\mathfrak{K} \,|\, \mid \lambda(1)$ *for* $\lambda \in \mathscr{S}$. *Furthermore* $\mathscr{S}$ *contains an irreducible character of degree* $d\,|\,\mathfrak{L}:\mathfrak{K}\,|$.

(v) *Define an equivalence relation on* $\mathscr{S}$ *by the condition that two characters in* $\mathscr{S}$ *are equivalent if and only if they have the same degree and the same weight. Then each equivalence class of* $\mathscr{S}$ *is either subcoherent in* $\mathscr{S}$ *or consists of irreducible characters.*

(vi) *For any subgroup* $\mathfrak{A}$ *of* $\mathfrak{H}$ *which is normal in* $\mathfrak{L}$ *let* $\mathscr{S}(\mathfrak{A})$ *be the subset of* $\mathscr{S}$ *consisting of those characters which are equivalent to some character in* $\mathscr{S}$ *that has* $\mathfrak{A}$ *in its kernel.*

In the application to the main theorem of this paper (11.1) will always be augmented by one of the following conditions.

$$(11.2) \qquad \mathfrak{H} = \hat{\mathfrak{L}} = \mathfrak{K} \subset \mathfrak{L} .$$

$$(11.3) \qquad \mathfrak{H} \subset \hat{\mathfrak{L}} = \mathfrak{K} \subset \mathfrak{L} .$$

$$(11.4) \qquad \mathfrak{H} \subseteq \bigcup_{H \in \mathfrak{H}^\#} C(H) \cap \mathfrak{K} = \hat{\mathfrak{L}} \subseteq \mathfrak{K} \subseteq \mathfrak{L} .$$

THEOREM 11.1. *Suppose that Hypothesis* 11.1 *is satisfied. Let* $\mathfrak{H}_1$ *be a normal subgroup of* $\mathfrak{L}$ *which is contained in* $\mathfrak{H}$ *such that*

$$(11.5) \qquad |\,\mathfrak{H}:\mathfrak{H}_1\,| > 4d^2\,|\,\mathfrak{L}:\mathfrak{K}\,|^2 + 1 .$$

If $\mathscr{S}(\mathfrak{H}_1)$ *is coherent and contains an irreducible character of degree* $d\,|\,\mathfrak{L}:\mathfrak{K}\,|$ *then* $\mathscr{S}$ *is coherent.*

Proof. Let $\mathfrak{H}_2$ be a normal subgroup of $\mathfrak{L}$ which is contained in $\mathfrak{H}_1$ and is minimal with the property that $\mathscr{S}(\mathfrak{H}_2)$ is coherent. Suppose that $\mathfrak{H}_2 \neq \langle 1 \rangle$. Choose $\mathfrak{H}_3 \subset \mathfrak{H}_2$ such that $\mathfrak{H}_2/\mathfrak{H}_3$ is a chief factor of $\mathfrak{L}$. Let $\mathscr{S}(\mathfrak{H}_2) = \mathscr{S}_1 = \{\lambda_{1s} \mid 1 \leqq s \leqq n_1\}$, where λ_{11} is irreducible and $\lambda_{11}(1) = d\,|\,\mathfrak{L}:\mathfrak{K}\,|$. Let $\mathscr{S}_2, \cdots \mathscr{S}_k$ be the subsets of $\mathscr{S}(\mathfrak{H}_3) - \mathscr{S}(\mathfrak{H}_2)$ consisting of all characters of a given weight and a given degree. For $2 \leqq i \leqq k$ let $\ell_{i1}\lambda_{11}(1)$ be the common degree of the characters in $\mathscr{S}_i$.

By Hypothesis 11.1 all the assumptions of Theorem 10.1, except possibly inequality (10.2), are satisfied for $\mathscr{S}(\mathfrak{H}_3)$. We will now verify that also inequality (10.2) is satisfied.

Let $\theta_1, \theta_2, \cdots$ be all the irreducible characters of $\mathfrak{K}$ which do not have $\mathfrak{H}$ in their kernel. Let $\tilde{\theta}_j$ denote the character of $\mathfrak{L}$ induced by θ_j. Then each $\tilde{\theta}_j$ is in $\mathscr{S}$ by Lemma 4.3. Furthermore if θ_j ranges only over characters of $\mathfrak{K}/\mathfrak{H}_2$ then

$$\Sigma\theta_j(1)\theta_j = \rho_{\mathfrak{K}/\mathfrak{H}_2} - \rho_{\mathfrak{K}/\mathfrak{H}} .$$

Therefore

$$\Sigma\theta_j(1)\tilde{\theta}_j = \rho_{\mathfrak{L}/\mathfrak{H}_2} - \rho_{\mathfrak{L}/\mathfrak{H}} . \tag{11.6}$$

If $\tilde{\theta}_i \neq \tilde{\theta}_j$ then $(\tilde{\theta}_i, \tilde{\theta}_j) = 0$. Suppose that for a given j there are a_j values of i such that $\tilde{\theta}_j = \tilde{\theta}_i$. Then (11.6) implies that

$$\Sigma\{\theta_j(1)a_j\}^2 \| \tilde{\theta}_j \|^2 = |\mathfrak{L}:\mathfrak{H}_2| - |\mathfrak{L}:\mathfrak{H}| \tag{11.7}$$

where the summation in (11.7) ranges over the distinct ones among the $\tilde{\theta}_j$. Since

$$\{\theta_j(1)a_j\}^2 \| \tilde{\theta}_j \|^2 = \theta_j(1)^2 |\mathfrak{L}:\mathfrak{K}| a_j = \tilde{\theta}_j(1)\theta_j(1)a_j = \frac{\tilde{\theta}_j(1)^2}{\| \tilde{\theta}_j \|^2}$$

(11.7) yields that

$$\sum_s \frac{\lambda_{1s}(1)^2}{\| \lambda_{1s} \|^2} \geqq |\mathfrak{L}:\mathfrak{H}_2| - |\mathfrak{L}:\mathfrak{H}| ,$$

where $\mathscr{S}_1 = \{\lambda_{1s}\}$ or equivalently

$$\sum_s \frac{\ell_{1s}^2}{\| \lambda_{1s} \|^2} \geqq \frac{|\mathfrak{L}:\mathfrak{H}_2| - |\mathfrak{L}:\mathfrak{H}|}{d^2 |\mathfrak{L}:\mathfrak{K}|^2} . \tag{11.8}$$

Since $\mathfrak{H}$ is nilpotent $\mathfrak{H}_2/\mathfrak{H}_3$ is in the center of $\mathfrak{H}/\mathfrak{H}_3$. Every irreducible character of $\mathfrak{K}$ is a constituent of a character induced by an irreducible character of $\mathfrak{H}$. Thus for $2 \leqq m \leqq k$, Lemma 4.1 implies that

$$\ell_{m1} d |\mathfrak{L}:\mathfrak{K}| \leqq \sqrt{|\mathfrak{H}:\mathfrak{H}_2|}\, |\mathfrak{L}:\mathfrak{H}| ,$$

or equivalently

$$\ell_{m1} \leqq \frac{|\mathfrak{K}:\mathfrak{H}| \sqrt{|\mathfrak{H}:\mathfrak{H}_2|}}{d} . \tag{11.9}$$

Suppose now that inequality (10.2) is violated for some value of m. Then (11.8) and (11.9) yield that

$$\frac{|\mathfrak{L}:\mathfrak{H}_2| - |\mathfrak{L}:\mathfrak{H}|}{d^2 |\mathfrak{L}:\mathfrak{K}|^2} \leqq \frac{2 |\mathfrak{K}:\mathfrak{H}| \sqrt{|\mathfrak{H}:\mathfrak{H}_2|}}{d} .$$

Thus

$$\{|\mathfrak{H}:\mathfrak{H}_2| - 1\} \leqq 2d\,|\mathfrak{L}:\mathfrak{K}|\sqrt{|\mathfrak{H}:\mathfrak{H}_2|}\,,$$

or

$$|\mathfrak{H}:\mathfrak{H}_2|^2 - 2\,|\mathfrak{H}:\mathfrak{H}_2| + 1 \leqq 4d^2\,|\mathfrak{L}:\mathfrak{K}|^2\,|\mathfrak{H}:\mathfrak{H}_2|\,.$$

Since every term is an integer this implies that

$$(11.10) \qquad |\mathfrak{H}:\mathfrak{H}_2| - 1 \leqq 4d^2\,|\mathfrak{L}:\mathfrak{K}|^2\,.$$

However $\mathfrak{H}_2 \subseteq \mathfrak{H}_1$, thus $|\mathfrak{H}:\mathfrak{H}_2| \geqq |\mathfrak{H}:\mathfrak{H}_1|$. Now (11.5) and (11.10) are incompatible. Therefore inequality (10.2), and thus all the assumptions of Theorem 10.1, are satisfied. Hence by that theorem $\mathscr{S}(\mathfrak{H}_3)$ is coherent contrary to the minimal nature of $\mathfrak{H}_2$. This finally implies that $\mathfrak{H}_2 = \langle 1 \rangle$. Therefore $\mathscr{S} = \mathscr{S}(\mathfrak{H}_2)$ is coherent. The proof is complete.

The remainder of this section consists of applications of Theorem 11.1. Lemmas 11.1 and 11.2 are closely related to Theorem 2 of [8]. By using the argument of that theorem the assumption that $|\mathfrak{L}|$ is odd in the following lemmas can be replaced by suitable weaker assumptions. However the stronger results are not relevant to this paper and will not be proved here.

Hypothesis 11.2.

(i) *Hypothesis* 11.1 *and equation* (11.2) *are satisfied. Thus* $d = 1$.

(ii) $|\mathfrak{L}|$ *is odd and* $\mathfrak{L}/\mathfrak{H}'$ *is a Frobenius group with Frobenius kernel* $\mathfrak{H}/\mathfrak{H}'$.

LEMMA 11.1. *Suppose that Hypothesis* 11.2 *is satisfied. If*

$$|\mathfrak{H}:\mathfrak{H}'| > 4\,|\mathfrak{L}:\mathfrak{H}|^2 + 1$$

then $\mathscr{S}$ *is coherent.*

Proof. By Lemma 10.1 and 3.16 (iii) $\mathscr{S}(\mathfrak{H}')$ is coherent. The result now follows from Theorem 11.1.

LEMMA 11.2. *Suppose that Hypothesis* 11.2 *is satisfied. Then* $\mathscr{S}$ *is coherent except possibly if* $\mathfrak{H}$ *is a non abelian p-group for some prime p and*

$$|\mathfrak{H}:\mathfrak{H}'| \leqq 4\,|\mathfrak{L}:\mathfrak{H}|^2 + 1\,.$$

Proof. If $\mathfrak{H} = \mathfrak{H}_1 \times \mathfrak{H}_2$, where $\mathfrak{H}_1$ and $\mathfrak{H}_2$ are proper normal subgroups of $\mathfrak{L}$, then

$$|\mathfrak{H}_i:\mathfrak{H}_i'| \equiv 1 \pmod{|\mathfrak{L}:\mathfrak{H}|} \quad \text{for } i = 1, 2 .$$

Since $|\mathfrak{L}|$ is odd, this implies that

$$|\mathfrak{H}_i:\mathfrak{H}_i'| \geqq 2|\mathfrak{L}:\mathfrak{H}| + 1 \quad \text{for } i = 1, 2 .$$

Hence $|\mathfrak{H}:\mathfrak{H}'| > 4|\mathfrak{L}:\mathfrak{H}|^2 + 1$ and $\mathscr{S}$ is coherent by Lemma 11.1. As $\mathfrak{H}$ is nilpotent this implies that $\mathscr{S}$ is coherent if $\mathfrak{H}$ is not a p-group for any prime p. Since $|\mathfrak{L}|$ is odd

$$|\mathscr{S}| \geqq \frac{|\mathfrak{H}:\mathfrak{H}'| - 1}{|\mathfrak{L}:\mathfrak{H}|} \geqq 2 .$$

Thus by Lemma 10.1 $\mathscr{S}$ is coherent if $\mathfrak{H}$ is abelian. The result now follows directly from Lemma 11.1.

LEMMA 11.3. *Suppose that Hypothesis* 11.2 *is satisfied and* $\mathfrak{L}$ *is a Frobenius group with Frobenius kernel* $\mathfrak{H}$. *Assume that* $\mathfrak{H}$ *is a p-group for some prime p and* $|\mathfrak{H}:D(\mathfrak{H})| = p^2$. *Then* $\mathscr{S}$ *is coherent.*

Proof. If $\mathfrak{H}$ is abelian Lemma 11.2 implies that $\mathscr{S}$ is coherent. If $\mathfrak{H}$ is not abelian then the second term of the descending central series modulo the third is cyclic. Thus

$$p \equiv 1 \pmod{|\mathfrak{L}:\mathfrak{H}|} .$$

Therefore $(p - 1) \geqq 2|\mathfrak{L}:\mathfrak{H}|$ as $|\mathfrak{L}|$ is odd. Hence

$$|\mathfrak{H}:\mathfrak{H}'| \geqq p^2 > 4|\mathfrak{L}:\mathfrak{H}|^2 + 1$$

and the result follows from Lemma 11.1.

LEMMA 11.4. *Suppose that Hypothesis* 11.2 *is satisfied and* $\mathfrak{L}$ *is a Frobenius group with Frobenius kernel* $\mathfrak{H}$. *Assume that* $\mathfrak{H}$ *is a p-group for some prime p and* $|\mathfrak{H}:D(\mathfrak{H})| = p^3$. *If*

$$p^3 - 1 > 2p|\mathfrak{L}:\mathfrak{H}| \tag{11.11}$$

then $\mathscr{S}$ *is coherent.*

Proof. If $\mathfrak{H}$ is abelian Lemma 11.2 implies that $\mathscr{S}$ is coherent. If $\mathfrak{H}$ is non-abelian let $\mathfrak{H}_1$ be a subgroup of $D(\mathfrak{H})$ such that $D(\mathfrak{H})/\mathfrak{H}_1$ is a chief factor of $\mathfrak{L}$. As $\mathfrak{H}$ is nilpotent $D(\mathfrak{H})/\mathfrak{H}_1$ is in the center of $\mathfrak{H}/\mathfrak{H}_1$. Thus by Lemma 4.1 the degree of any irreducible character of $\mathfrak{H}/\mathfrak{H}_1$ is either 1 or p. Hence the degree of any character in $\mathscr{S}(\mathfrak{H}_1)$

is either $|\mathfrak{L}:\mathfrak{H}|$ or $|\mathfrak{L}:\mathfrak{H}|p$. Let $\mathscr{S}_1, \mathscr{S}_2$ be the subsets of $\mathscr{S}(\mathfrak{H}_1)$ consisting of all the characters of degree $|\mathfrak{L}:\mathfrak{H}|, |\mathfrak{L}:\mathfrak{H}|p$ respectively. Let $\ell_1 = 1, \ell_2 = p$. By (11.11)

$$|\mathscr{S}_1| \geqq \frac{p^3 - 1}{|\mathfrak{L}:\mathfrak{H}|} > 2p = 2\ell_2 .$$

Thus by Theorem 10.1 $\mathscr{S}(\mathfrak{H}_1)$ is coherent.

If $|D(\mathfrak{H}):\mathfrak{H}_1| = p$ or p^2, then $p \equiv 1 \pmod{|\mathfrak{L}:\mathfrak{H}|}$ or

$$p^2 - 1 \equiv 0 \pmod{|\mathfrak{L}:\mathfrak{H}|} .$$

As $(p^3 - 1, p^2 - 1) = p - 1$ this yields that in either case

$$p \equiv 1 \pmod{|\mathfrak{L}:\mathfrak{H}|} .$$

Therefore $p - 1 \geqq 2|\mathfrak{L}:\mathfrak{H}|$. Hence

$$|\mathfrak{H}:\mathfrak{H}'| \geqq |\mathfrak{H}:D(\mathfrak{H})| = p^3 > 4|\mathfrak{L}:\mathfrak{H}|^2 + 1$$

and $\mathscr{S}$ is coherent by Lemma 11.1. Suppose that $|D(\mathfrak{H}):\mathfrak{H}_1| \geqq p^3$. Then by (11.11)

$$|\mathfrak{H}:\mathfrak{H}_1| \geqq p^6 > 4|\mathfrak{L}:\mathfrak{H}|^2 + 1 .$$

Since $\mathscr{S}(\mathfrak{H}_1)$ is coherent the result now follows from Theorem 11.1.

The next two lemmas involve the following situation:

Hypothesis 11.3.

(i) *Hypothesis* 11.2 *is satisfied.*

(ii) *There exist primes p, q and positive integers a, b such that* $|\mathfrak{L}:\mathfrak{H}| = p^b, |\mathfrak{H}:\mathfrak{H}'| = |\mathfrak{H}:D(\mathfrak{H})| = q^a$. *Thus $|\mathfrak{H}|$ is a power of q.*

LEMMA 11.5. *Suppose that Hypothesis* 11.3 *is satisfied and $a = 2c$ is even. Then $\mathscr{S}$ is coherent except possibly if $q^c + 1 = 2p^b$, q^c is the smallest degree of any non linear irreducible character of $\mathfrak{H}$ whose kernel contains $[\mathfrak{H}, \mathfrak{H}']$ and for no subgroup $\mathfrak{H}_1$ of $\mathfrak{H}'$ with $\mathfrak{H}_1 \neq \mathfrak{H}'$, $\mathfrak{H}_1 \lhd \mathfrak{L}$ is $\mathfrak{L}/\mathfrak{H}_1$ a Frobenius group.*

Proof. Suppose that $\mathscr{S}$ is not coherent. Then by Lemma 11.1 $4p^{2b} + 1 \geqq q^a$. As $(q^c + 1, q^c - 1) = 2$ it follows that $2p^b \mid q^c + 1$ or $2p^b \mid q^c - 1$. If $2p^b \neq q^c + 1$ this implies that $4p^{2b} + 1 < q^a$ contrary to what has been proved above. Therefore $q^c + 1 = 2p^b$.

Let $\mathscr{T}_i = \{\theta_{ij}\}$ be the set of non principal irreducible characters of $\mathfrak{H}/[\mathfrak{H}, \mathfrak{H}']$ of degree q^i. Lemma 4.1 implies that $\mathscr{T}_i$ is empty for $i > c$. Let $\mathscr{S}_i = \{\lambda_{ij}\}$ be the set of characters in $\mathscr{S}$ of degree $q^i p^b$.

Since $|\mathscr{S}_0| = 2(q^c - 1) > 2q^{c-1}$, it follows from Hypothesis 11.1 and Theorem 10.1 that $\bigcup_{i=0}^{c-1} \mathscr{S}_i$ is coherent. Suppose that $\bigcup_{i=1}^{c-1} \mathscr{T}_i$ is non empty. Then 3.15 implies that

$$\sum_{i=1}^{c-1} \sum_j \theta_{ij}(1)^2 \geqq q^{2c} .$$

Therefore

$$\sum_{i=0}^{c-1} \sum_j \frac{q^{2i}}{||\lambda_{ij}||^2} = \frac{1}{p^{2b}} \sum_{i=0}^{c-1} \sum_j \frac{\lambda_{ij}(1)^2}{||\lambda_{ij}||^2} = |\mathscr{S}_0| + \frac{1}{p^b} \sum_{i=1}^{c-1} \sum_j \theta_{ij}(1)^2 > 2q^c .$$

Thus by Theorem 10.1

$$\bigcup_{i=0}^{c} \mathscr{S}_i \supseteqq \mathscr{S}([\mathfrak{H}, \mathfrak{H}'])$$

is coherent. Since

$$4\,|\mathfrak{L}:\mathfrak{H}|^2 + 1 = 4p^{2b} + 1 < q^{a+1} \leqq |\mathfrak{H}:[\mathfrak{H}, \mathfrak{H}']| ,$$

Theorem 11.1 implies that $\mathscr{S}$ is coherent. Thus it may be assumed that q^c is the smallest degree of any non linear irreducible character of $\mathfrak{H}/[\mathfrak{H}, \mathfrak{H}']$.

Suppose now that $\mathfrak{H}'$ contains a subgroup $\mathfrak{H}_1 \neq \mathfrak{H}'$ such that $\mathfrak{L}/\mathfrak{H}_1$ is a Frobenius group. Then $\mathfrak{H}_1$ may be chosen so that $\mathfrak{H}'/\mathfrak{H}_1$ is a chief factor of $\mathfrak{L}$. Thus $[\mathfrak{H}, \mathfrak{H}'] \subseteqq \mathfrak{H}_1$ and by the earlier part of the lemma every irreducible character of $\mathfrak{H}/\mathfrak{H}_1$ has degree either 1 or q^c. As $q^c + 1 = 2p^b$, q^{2c} is the smallest power of q which satisfies $q^{2c} \equiv 1 \pmod{p^b}$. Since $\mathfrak{H}'/\mathfrak{H}_1$ is a chief factor of $\mathfrak{L}$ this implies that $\mathfrak{H}'/\mathfrak{H}_1$ is in the center of $\mathfrak{H}/\mathfrak{H}_1$ and $|\mathfrak{H}':\mathfrak{H}_1| = q^{2c}$. If θ is an irreducible character of $\mathfrak{H}/\mathfrak{H}_1$ of degree q^c, then the orthogonality relations yield that $\theta(H) = 0$ for $H \in \mathfrak{H}/\mathfrak{H}_1 - \mathfrak{H}'/\mathfrak{H}_1$. As every non linear character of $\mathfrak{H}/\mathfrak{H}_1$ has degree q^c the orthogonality relations may once again be used. They imply that

$$(11.13) \qquad |C(H)| = q^{2c} \quad \text{for } H \in \mathfrak{H}/\mathfrak{H}_1 - \mathfrak{H}'/\mathfrak{H}_1 .$$

However

$$\langle H, \mathfrak{H}'/\mathfrak{H}_1 \rangle \subseteqq C(H)$$

which contradicts (11.13). Thus $\mathfrak{H}'$ contains no subgroup $\mathfrak{H}_1 \neq \mathfrak{H}'$ such that $\mathfrak{L}/\mathfrak{H}_1$ is a Frobenius group. All statements in the lemma are proved.

LEMMA 11.6. *Suppose that Hypothesis 11.3 is satisfied. Assume further that a is odd and $p = 3$. Then $\mathscr{S}$ is coherent.*

Proof. As a is odd and $q^a \equiv 1 \pmod 3$, it follows that $q \equiv 1 \pmod 3$. Define the integer $c \geqq 1$ by

$$q \equiv 1 \pmod{3^c}, \qquad q \not\equiv 1 \pmod{3^{c+1}}.$$

If $b \leqq c$, then $q \geqq 2.3^b + 1$. Thus if $a \neq 1, 4\cdot 3^{2b} + 1 < q^a$ and $\mathscr{S}$ is coherent by Lemma 11.1. If $a = 1$, then $\mathfrak{H}$ is cyclic. Therefore $\mathscr{S}$ is coherent by Lemma 10.1.

Suppose now that $b > c$. Then since $q^a \equiv 1 \pmod{3^b}$ we must have $a = 3^{b-c}x$ for some integer x. Therefore

$$q^a \geqq (q^{3^{b-c-1}})^3 .$$

Since $q^{3^{b-c-1}} \equiv 1 \pmod{3^{b-1}}$, this yields that

$$q^a \geqq (1 + 2\cdot 3^{b-1})^3 . \tag{11.14}$$

If $4.3^{2b} + 1 < q^a$ then $\mathscr{S}$ is coherent by Lemma 11.1. Thus if $\mathscr{S}$ is not coherent (11.14) implies that

$$4.3^{2b} + 1 \geqq q^a \geqq (1 + 2.3^{b-1})^3 > 8.3^{3(b-1)} + 1 .$$

Therefore $3^3 > 2.3^b$. Hence $b = 1$ or $b = 2$. In either case this implies that $q^a \leqq 4.3^4 + 1 < 7^3$. As $a \equiv 0 \pmod 3$ we get that $q < 7$. However $q \equiv 1 \pmod 3$. This contradiction arose from assuming that $\mathscr{S}$ is not coherent. The proof is complete.

12. Further Results about Tamely Imbedded Subsets

In this section a fairly special situation is studied. Our purpose here is to get some information about certain sets of characters which may not be coherent.

Hypothesis 12.1.

(i) *Let q be a prime and let $\mathfrak{Q}$ be a S_q-subgroup of the group $\mathfrak{X}$. Assume that $\mathfrak{Q} = \hat{\mathfrak{L}}$ is tamely imbedded in $\mathfrak{X}$ and $\mathfrak{L} = N(\mathfrak{Q}) \neq \mathfrak{Q}$ has odd order. Let $\mathfrak{Q}_1 \lhd \mathfrak{L}, \mathfrak{Q}_1 \subset \mathfrak{Q}$ and let $\bar{\mathfrak{Q}} = \mathfrak{Q}/\mathfrak{Q}_1, \bar{\mathfrak{L}} = \mathfrak{L}/\mathfrak{Q}_1$.*

(ii) *$\mathscr{L}$ is the set of all characters of $\mathfrak{L}$ which are induced by non-principal irreducible characters of $\bar{\mathfrak{Q}}$. Define an equivalence relation on $\mathscr{L}$ by the condition that two characters are equivalent if and only if they have the same degree and the same weight. Then each equivalence class of $\mathscr{L}$ is either subcoherent in $\mathscr{L}$ or consists of irreducible characters.*

(iii) *Let $1 = q^{f_0} < q^{f_1} \cdots$ be all the integers which are degrees of irreducible characters of $\bar{\mathfrak{Q}}$. Let $n > 0$ be a fixed integer. For $0 \leqq i \leqq n-1$ let $\mathscr{S}_i$ be the set of all characters in $\mathscr{L}$ of degree*

$q^{f_i} |\mathfrak{L}:\mathfrak{Q}|$. *Assume that each* $\mathscr{S}_i$ *consists of irreducible characters. Let* $\mathscr{S}_n$ *be an equivalence class in* $\mathscr{L}$ *consisting of characters of degree* $q^{f_n} |\mathfrak{L}:\mathfrak{Q}|$. *Let* $\mathscr{S} = \bigcup_{i=0}^{n} \mathscr{S}_i$.

In case Hypothesis 12.1 is satisfied the following notation will be used.

$$(12.1) \qquad |\bar{\mathfrak{Q}}:\bar{\mathfrak{Q}}'| = q^a, \qquad |\bar{\mathfrak{L}}:\bar{\mathfrak{Q}}| = e > 1 .$$

Since $|\mathfrak{L}|$ is odd, $|\mathscr{S}_i| \geqq 2$ and $\mathscr{I}_0(\mathscr{S}_i) \neq 0$ for $0 \leqq i \leqq n$. Thus by Lemma 10.1 $\mathscr{S}_i$ is coherent for $0 \leqq i \leqq n-1$.

For $0 \leqq i < n$ let a_i be the number of non principal irreducible characters of $\bar{\mathfrak{Q}}$ of degree q^{f_i}. By Hypothesis 12.1 $\bar{\mathfrak{L}}/\bar{\mathfrak{Q}}$ acts regularly as a permutation group on the non principal irreducible characters of degree q^{f_i} for $0 \leqq i < n$. Since $|\mathfrak{L}|$ is odd, no non principal irreducible character of $\mathfrak{Q}$ is real. Thus a_i is even. Therefore

$$(12.2) \qquad a_i \equiv 0 \pmod{2e}, |\mathscr{S}_i| = \frac{a_i}{e} \quad \text{for } 0 \leqq i \leqq n-1 .$$

Let $j_0 = 0$. Define j_s inductively to be the largest integer not exceeding $n+1$ such that $\bigcup_{i=j_{s-1}}^{j_s-1} \mathscr{S}_i$ is coherent. Suppose that

$$0 = j_0 < \cdots < j_t < j_{t+1} = n+1 .$$

For $0 \leqq s \leqq t$, define

$$(12.3) \qquad \mathscr{T}_s = \bigcup_{i=j_s}^{j_{s+1}-1} \mathscr{S}_i$$

and let $m_s = f_{j_s}$. Define

$$(12.4) \qquad c_s = \sum_i a_i q^{2(f_i - m_s)} \quad \text{for } 0 \leqq s \leqq t ,$$

where i ranges from j_s to $j_{s+1}-1$. Define

$$(12.5) \qquad d_s = q^{m_{s+1}-m_s} \quad \text{for } 0 \leqq s < t .$$

Then by Theorem 10.1 applied to $\mathscr{T}_s \cup \mathscr{S}_{j_{s+1}}$

$$(12.6) \qquad c_s \leqq 2ed_s \quad \text{for } 0 \leqq s < t .$$

By (12.2)

$$(12.7) \qquad c_s \equiv 0 \pmod{2e} \quad \text{for } 0 \leqq s < t .$$

By 3.15

$$(12.8) \qquad 1 + \sum_{j=0}^{s} c_j q^{2m_j} \equiv 0 \pmod{q^{2m_{s+1}}} \quad \text{for } 0 \leqq s < t .$$

LEMMA 12.1. *Suppose that Hypothesis* 12.1 *is satisfied. Assume that*

$$|\bar{\mathfrak{Q}}:\bar{\mathfrak{Q}}'| = q^a \leqq 4e^2 + 1 .$$

Then

$$\frac{d_s^2 e}{c_s} < e + 1 \quad \textit{for } 0 \leqq s < t .$$

Furthermore if a is odd, $c_s < e^2$ and $c_s \not\equiv 0 \pmod q$, then

$$\frac{d_s^2 e}{c_s} < e - 1 .$$

Proof. We will first prove that

$$(12.9) \qquad 1 + \sum_{j=0}^{s-1} c_j q^{2m_j} < eq^{2m_s} \quad \text{for } 0 \leqq s < t .$$

This is true if $s = 0$ since $1 < e$. Suppose that $s > 0$. Then by (12.5) and (12.6)

$$\begin{aligned} 1 + \sum_{j=0}^{s-1} c_j q^{2m_j} &\leqq 1 + 2e \sum_{j=0}^{s-1} q^{m_j + m_{j+1}} \\ &\leqq 1 + 2e(1 + q + \cdots + q^{2m_s - 1}) \\ &\leqq 1 + 2e\frac{(q^{2m_s} - 1)}{(q-1)} \leqq 1 + e(q^{2m_s} - 1) < eq^{2m_s} . \end{aligned}$$

Assume now that the lemma is false and choose s minimum tc violate the result. Let $c = c_s$, $d = d_s$.

By (12.8) and (12.9)

$$q^{2m_{s+1}} < eq^{2m_s} + cq^{2m_s} .$$

Hence by (12.5)

$$(12.10) \qquad d^2 < e + c .$$

Inequalities (12.6) and (12.10) yield that $d^2 < e + 2ed$ or $d^2 - 2ed - e < 0$. This implies that

$$e - \sqrt{e^2 + e} \leqq d \leqq e + \sqrt{e^2 + e} .$$

Consequently

$$(12.11) \qquad d \leqq e + \sqrt{e^2 + e} < 3e .$$

Suppose that

$$1 + \sum_{j=0}^{s} c_j q^{2m_j} \geqq 3q^{2m_{s+1}} .$$

Then by (12.9)

$$3q^{2m_{s+1}} < (e + c)q^{2m_s} .$$

Hence by (12.7) $3d_s^2 < e + c \leqq 3c/2$. Thus

$$\frac{d^2e}{c} \leqq \frac{e}{2} < e - 1$$

since $e > 2$. This contradicts the choice of s. Hence

$$1 + \sum_{j=0}^{s} c_j q^{2m_j} < 3q^{2m_{s+1}} .$$

As c_j is even for $0 \leqq j \leqq s$, (12.8) implies that

$$1 + \sum_{j=0}^{s} c_j q^{2m_j} = q^{2m_{s+1}} . \tag{12.12}$$

The group $\bar{\mathfrak{Q}}$ contains a normal subgroup $\bar{\mathfrak{Q}}_0$ of index $q^{2m_{s+1}}$. Every irreducible character of $\bar{\mathfrak{Q}}/\bar{\mathfrak{Q}}_0$ has degree strictly less than $q^{m_{s+1}}$ and the sum of the squares of the degrees of these characters is equal to $q^{2m_{s+1}}$. Hence (12.12) implies that every character of $\bar{\mathfrak{Q}}$ whose degree is strictly less than $q^{m_{s+1}}$ has $\bar{\mathfrak{Q}}_0$ in its kernel. Thus $\bar{\mathfrak{Q}}_0$ is a normal subgroup of $\bar{\mathfrak{L}}$ and $\bar{\mathfrak{L}}/\bar{\mathfrak{Q}}_0$ is a Frobenius group with Frobenius kernel $\bar{\mathfrak{Q}}/\bar{\mathfrak{Q}}_0$. Therefore

$$q^{2m_{s+1}} \equiv d^2q^{2m_s} \equiv 1 \pmod{e} , \tag{12.13}$$

and the center of $\bar{\mathfrak{Q}}/\bar{\mathfrak{Q}}_0$ has order at least q^a. Thus by Lemma 4.1 $q^{2m_s} \leqq q^{2m_{s+1}-a}$. This yields that

$$q^a \leqq d^2 . \tag{12.14}$$

Define the integer k by

$$c + k = d^2 . \tag{12.15}$$

By (12.10) $k < e$ and by (12.12) $0 < k$. Thus

$$0 < k < e . \tag{12.16}$$

Define the integer b by

$$q^{2m_s} \equiv q^{a-b} \pmod{e} , \qquad 0 \leqq b \leqq a - 1 . \tag{12.17}$$

Equations (12.7), (12.13), (12.15) and (12.17) imply that

$$k \equiv d^2 \equiv q^{b-a} \equiv q^b \pmod{e} . \tag{12.18}$$

If $b = 0$, then by (12.16) and (12.18) $k = 1$. Thus by (12.15) $c = d^2 - 1$, hence by (12.7)

$$\frac{d^2e}{c} = \frac{(c+1)e}{c} = e + \frac{e}{c} < e + 1 .$$

If $c < e^2$ and a is odd, then

$$d^2 = c + 1 < e^2 + 1 < q^{2a} .$$

Thus by (12.18) $d^2 = q^a$. However this is impossible as a is odd.

Assume now that $b \neq 0$. As d^2 is a power of q, (12.14) and (12.18) imply that either $d^2 = q^{a+b}$ or $d^2 \geqq q^{2a+b}$. Since $b \neq 0$, the latter case leads to

$$d^2 \geqq q^{2a+b} = q^{2a}q^b > 4e^2q > 9e^2 .$$

Hence $d > 3e$ contrary to (12.11). Thus

$$d^2 = q^{a+b}, \quad 2 \leqq a - b . \tag{12.19}$$

The inequality follows from (12.17) and the fact that $a + b$ is even. Now (12.11) and (12.19) yield that

$$q^{2b} = \frac{q^{a+b}}{q^{a-b}} = \frac{d^2}{q^{a-b}} < \frac{9e^2}{q^2} \leqq e^2 .$$

Thus $1 \leqq q^b < e$. (12.16) and (12.18) imply that

$$k = q^b , \qquad b > 0 . \tag{12.20}$$

Equation (12.15) now becomes $d^2 = c + q^b$. Hence

$$c \equiv 0 \pmod q .$$

Furthermore by (12.19)

$$c = d^2 - q^b = q^b(q^a - 1) .$$

Consequently

$$\frac{d^2e}{c} = \frac{q^{a+b}e}{q^b(q^a-1)} = \frac{q^ae}{q^a-1} = e + \frac{e}{q^a-1} < e + 1 .$$

THEOREM 12.1. *Suppose that Hypothesis* 12.1 *is satisfied. Assume that for some* j *with* $0 \leqq j \leqq n-1$, $\lambda_1 \in \mathscr{S}_j$ *and* $\lambda_2 \in \mathscr{S}_{j+1}$. *Define*

$$\alpha = q^{f_{j+1}-f_j}\lambda_1 - \lambda_2 .$$

Suppose that $\mathscr{S}_j \subseteqq \mathscr{T}$, *and*

$$\alpha^\tau = \Delta + \Delta_1$$

where $\Delta_1 \in \mathscr{I}(\mathscr{T}_s^\tau)$ *and* Δ *is orthogonal to* $\mathscr{I}(\mathscr{T}_s^\tau)$. *Then*

$$\|\Delta\|^2 \leqq e + \|\lambda_s\|^2 .$$

Furthermore if a *is odd,* $c = c_s < e^2$ *and* $c \not\equiv 0 \pmod q$ *then*

$$\|\Delta\|^2 \leqq e + \|\lambda_s\|^2 - 2 .$$

Proof. Let $\mathscr{T} = \mathscr{T}_s$. If $\mathscr{S}_{j+1} \subseteq \mathscr{T}$ then $\alpha^\tau \in \mathscr{I}(\mathscr{T})^\tau$ and $\Delta = 0$. Thus the result is trivial in this case. Hence it may be assumed that $\mathscr{S}_{j+1} \not\subseteq \mathscr{T}$. In particular, $\mathscr{S}$ is not coherent, hence $\mathscr{L}$ is not coherent, so by Lemma 11.1 $|\bar{\mathfrak{Q}}:\bar{\mathfrak{Q}}'| \leqq 4e^2 + 1$. Consequently Lemma 12.1 may be applied. Furthermore $f_{j+1} = m_{s+1}$ and $s < t$. Thus $\mathscr{T}$ consists of irreducible characters. Let $\mathscr{T} = \{\lambda_{si} \mid 1 \leqq i \leqq n_s\}$, where the notation is chosen so that $\lambda_1 \neq \lambda_{s1}$ and $\lambda_{si}(1) \mid \lambda_{s\,i+1}(1)$ for $1 \leqq i < n_s$. Suppose that $\lambda_1 = \lambda_{sk}$. Define the integer x by $(\alpha^\tau, \lambda_{s1}^\tau) = -x$. Then since $\alpha \in \mathscr{I}_0(\mathscr{S})$ Lemma 9.4 implies that

$$(\alpha^\tau, \lambda_{si}^\tau) = -x\frac{\lambda_{si}(1)}{\lambda_{s1}(1)} + \delta_{ik}q^{m_{s+1}-f_j} \quad \text{for } 1 \leqq i \leqq n_s .$$

Then

$$\Delta_1 = q^{m_{s+1}-f_j}\lambda_{sk}^\tau - x\sum_{i=1}^{n}\frac{\lambda_{si}(1)}{\lambda_{s1}(1)}\lambda_{si}^\tau .$$

Therefore

$$\begin{aligned}\|\Delta\|^2 = \|\alpha^\tau\|^2 - \|\Delta_1\|^2 &= q^{2(m_{s+1}-f_j)} \\ &+ \|\lambda_s\|^2 - x^2\frac{c}{e} - q^{2(m_{s+1}-f_j)} + 2x\frac{\lambda_{sk}(1)}{\lambda_{s1}(1)}q^{m_{s+1}-f_j} ,\end{aligned} \tag{12.21}$$

where $c = c_s$ is defined by (12.4). Let $d = d_s$ be defined by (12.5). Since $\lambda_{s1}(1) = q^{m_s}$ and $\lambda_{sk}(1) = q^{f_j}$ (12.21) yields that

$$\|\Delta\|^2 = \|\lambda_s\|^2 + 2xd - \frac{x^2c}{e} . \tag{12.22}$$

As a function of x, $2xd - (x^2c/e)$ assumes its maximum at $x = ed/c$. Thus (12.22) implies that

$$\|\Delta\|^2 \leqq \|\lambda_s\|^2 + 2\frac{ed^2}{c} - \frac{ed^2}{c} = \|\lambda_s\|^2 + \frac{ed^2}{c} . \tag{12.23}$$

As $\|\Delta\|^2$ is an integer Lemma 12.1 and (12.23) imply that $\|\Delta\|^2 \leqq \|\lambda_s\|^2 + e$. Furthermore if a is odd, $c < e^2$ and $c \not\equiv 0 \pmod q$, then

$$\|\Delta\|^2 \leqq \|\lambda_s\|^2 + e - 2 .$$

The proof is complete.

13. Self Normalizing Cyclic Subgroups

Hypothesis 13.1.

(i) $\mathfrak{W}$ *is a cyclic subgroup of the group* $\mathfrak{X}$ *with* $|\mathfrak{W}| = w$ *odd. Suppose that* $\mathfrak{W} = \mathfrak{W}_1 \times \mathfrak{W}_2$, *where* $w_i = |\mathfrak{W}_i|$ *and* $w_i \neq 1$ *for* $i = 1, 2$. *Let*

$$\hat{\mathfrak{W}} = \mathfrak{W} - \mathfrak{W}_1 - \mathfrak{W}_2 .$$

For any non empty subset $\mathfrak{A}$ *of* $\hat{\mathfrak{W}}$

$$C(\mathfrak{A}) = N(\mathfrak{A}) = \mathfrak{W} . \tag{13.1}$$

(ii) *Let* ω_{10}, ω_{01} *be faithful irreducible characters of* $\mathfrak{W}/\mathfrak{W}_2$, $\mathfrak{W}/\mathfrak{W}_1$ *respectively. Define*

$$\omega_{ij} \doteq \omega_{10}^i \omega_{01}^j$$

for $0 \leqq i \leqq w_1 - 1, 0 \leqq j \leqq w_2 - 1$.

If w_1, w_2 in Hypothesis 13.1 are both primes then (13.1) follows from the assumption that $N(\mathfrak{W}) = \mathfrak{W}$. Thus the situation described above is a generalization of this case.

LEMMA 13.1. *Suppose that Hypothesis* 13.1 *is satisfied. Then* $\hat{\mathfrak{W}}$ *is a* T. I. *set in* $\mathfrak{X}$. *There exists an orthonormal set* $\{\eta_{ij} \mid 0 \leqq i \leqq w_1 - 1, 0 \leqq j \leqq w_2 - 1\}$ *of generalized characters of* $\mathfrak{X}$ *such that for* $0 \leqq i \leqq w_1 - 1, 0 \leqq j \leqq w_2 - 1$, *the values assumed by* $\eta_{ij}, \eta_{i0}, \eta_{0j}$ *lie in* $\mathcal{Q}_w, \mathcal{Q}_{w_1}, \mathcal{Q}_{w_2}$ *respectively.* $\eta_{00} = 1_{\mathfrak{X}}$ *and*

$$\eta_{ij}(W) = \omega_{ij}(W) \quad \textit{for} \quad W \in \hat{\mathfrak{W}} ,$$
$$(1 - \omega_{i0} - \omega_{0j} + \omega_{ij})^* = 1_{\mathfrak{X}} - \eta_{i0} - \eta_{0j} + \eta_{ij} .$$

Furthermore every irreducible character of $\mathfrak{X}$ *distinct from all* $\pm\eta_{ij}$ *vanishes on* $\hat{\mathfrak{W}}$.

Proof. It follows directly from Hypothesis 13.1 that $\hat{\mathfrak{W}}$ is a T. I. set in $\mathfrak{X}$. Define the generalized character α_{ij} of $\mathfrak{W}$ by

$$\alpha_{ij} = (\omega_{00} - \omega_{i0})(\omega_{00} - \omega_{0j}) .$$

Clearly α_{ij} vanishes on $\mathfrak{W} - \hat{\mathfrak{W}}$. Thus

$$\begin{aligned} &\alpha_{ij}^*(W) = \alpha_{ij}(W) \quad \text{for} \quad W \in \hat{\mathfrak{W}} , \\ &(\alpha_{ij}^*, \alpha_{st}^*) = 1 + \delta_{is} + \delta_{jt} + \delta_{is}\delta_{jt} \end{aligned} \tag{13.2}$$

for $1 \leqq i, s \leqq w_1 - 1, 1 \leqq j, t \leqq w_2 - 1$. Therefore $\|\alpha_{ij}^*\|^2 = 4$ and $(\alpha_{ij}^*, \alpha_{it}^*) = 2$ for $i, j, t \neq 0, j \neq t$. It follows directly from the definition of α_{ij} that the values of α_{ij}^* lie in $\mathscr{Q}_w$.

For any algebraic number field $\mathscr{F}$ and any generalized character α of a group let $\mathscr{F}(\alpha)$ denote the field generated by $\mathscr{F}$ and all the values assumed by α. Since $\mathscr{Q}(\alpha_{ij}) = \mathscr{Q}(\alpha_{ij}^*)$ we see that $\mathscr{Q}(\alpha_{ij}^*) = \mathscr{Q}_v$ for some v with $v \mid w$. If $i, j \neq 0$ then $v = v_1 v_2$, where $v_s \mid w_s$ and $v_s > 1$ for $s = 1, 2$. By (13.2)

$$\alpha_{ij}^* = 1_{\mathfrak{X}} \pm \theta_1 \pm \theta_2 \pm \theta_3 ,$$

where $\theta_1, \theta_2, \theta_3$ are distinct irreducible characters of $\mathfrak{X}$.

Suppose that $\mathscr{Q}(\theta_k) \nsubseteq \mathscr{Q}_{v_1}$ for $k = 1, 2, 3$. Let

$$\mathscr{F} = \mathscr{Q}_v(\theta_1, \theta_2, \theta_3) = \mathscr{Q}(\theta_1, \theta_2, \theta_3) .$$

Let $\mathfrak{G}$ be the Galois group of $\mathscr{F}$ over $\mathscr{Q}_{v_1}$. For $k = 1, 2, 3$ let $\mathfrak{G}_k$ be the subgroup of $\mathfrak{G}$ whose fixed field is $\mathscr{Q}_{v_1}(\theta_k)$.

Assume first that $\mathfrak{G} = \mathfrak{G}_1 \cup \mathfrak{G}_2 \cup \mathfrak{G}_3$. By (13.2) $\mathfrak{G}_s \cap \mathfrak{G}_t = 1$ for $1 \leqq s < t \leqq 3$. If $\mathfrak{G} = \mathfrak{G}_k$ for some k then $\mathscr{Q}(\theta_k) \subseteq \mathscr{Q}_{v_1}$ contrary to assumption. Let $|\mathfrak{G}| = g$ and $|\mathfrak{G}_k| = g_k$ for $k = 1, 2, 3$. Then it may be assumed that $g > g_1 \geqq g_2 \geqq g_3$. Since $g = g_1 + g_2 + g_3 - 1 - 1 - 1 + 1$ we must have $g_1 = g/2$. Therefore

$$1 = |\mathfrak{G}_1 \cap \mathfrak{G}_2| \geqq g_2/2, \quad 1 = |\mathfrak{G}_1 \cap \mathfrak{G}_3| \geqq g_3/2 .$$

Hence

$$g/2 = g - g_1 = g_2 + g_3 - 2, \quad g_2, g_3 \leqq 2 .$$

Therefore $g \leqq 4$. $\mathfrak{G}$ is not cyclic as it is the union of proper subgroups. Hence $\mathfrak{G}$ is the non cyclic group of order 4 and $|\mathfrak{G}_k| = 2$ for $k = 1, 2, 3$. As v_2 is odd this implies that $v_2 = 3$. For $k = 1, 2, 3$ let $\mathfrak{G}_k = \langle \sigma_k \rangle$, where the notation is chosen so that $\mathscr{Q}_v = \mathscr{Q}_{v_1}(\theta_1)$. Therefore $\sigma_1(\alpha_{ij}^*) = \alpha_{ij}^*$. Hence $\sigma_1(\theta_2) = \theta_3$. Consequently $\mathscr{Q}_{v_1}(\theta_2) = \mathscr{Q}_{v_1}(\theta_3)$ as $\mathfrak{G}$ is abelian. This implies that $\sigma_2 = \sigma_3$ which is not the case. Thus $\mathfrak{G} \neq \mathfrak{G}_1 \cup \mathfrak{G}_2 \cup \mathfrak{G}_3$.

If $\sigma \in \mathfrak{G} - \mathfrak{G}_1 \cup \mathfrak{G}_2 \cup \mathfrak{G}_3$, then by (13.2) $(\alpha_{ij}^*, \sigma(\alpha_{ij}^*)) \geqq 2$. Hence by choosing the notation suitably it may be assumed that $\sigma(\theta_2) = \theta_3$. If $(\alpha_{ij}^*, \theta_2) \neq (\alpha_{ij}^*, \sigma(\theta_2))$ then replacing σ by σ^{-1} and θ_2 by θ_3 if necessary we get that

$$\alpha_{ij}^* = 1_{\mathfrak{X}} \pm \{\theta_1 + \theta_2 - \sigma(\theta_2)\} .$$

By (13.2) $\sigma(\theta_2) \neq \theta_1, \theta_2$. Hence also $\sigma(\theta_2) \neq \sigma^2(\theta_2)$. Therefore

$$\begin{aligned} 2 \leqq (\sigma(\alpha_{ij}^*), \alpha_{ij}^*) &= 1 - 1 + (\theta_1 + \theta_2, \sigma(\theta_1) - \sigma^2(\theta_2)) \\ &= (\theta_1 + \theta_2, \sigma(\theta_1)) - (\theta_1 + \theta_2, \sigma^2(\theta_2)) \\ &\leqq (\theta_1 + \theta_2, \sigma(\theta_1)) \leqq 1 \end{aligned}$$

since $\theta_1, \theta_2, \sigma(\theta_1)$ and $\sigma^2(\theta_2)$ are all characters. This contradiction establishes that $(\alpha_{ij}^*, \theta_2) = (\alpha_{ij}^*, \sigma(\theta_2))$. Since $\alpha_{ij}^*(1) = 0$ we see that

$$\alpha_{ij}^* = 1_{\mathfrak{X}} \pm \{\theta_2 + \sigma(\theta_2) - \theta_1\} . \tag{13.3}$$

Furthermore $\mathfrak{G}_2 = \mathfrak{G}_3$ and if $\gamma \in \mathfrak{G} - \mathfrak{G}_1 \cup \mathfrak{G}_2$ then $\theta_1 \neq \gamma(\theta_2)$. By definition $\theta_1 \neq \gamma(\theta_2)$ for $\gamma \in \mathfrak{G}_1 \cup \mathfrak{G}_2$. Therefore

$$\theta_1 \neq \gamma(\theta_2) \quad \text{for } \gamma \in \mathfrak{G} .$$

Suppose that $\gamma(\theta_2) = \theta_2$ for some automorphism γ of $\mathscr{F}$. Then $\gamma\sigma(\theta_2) = \sigma(\theta_2)$ and (13.3) implies that $(\alpha_{ij}^*, \gamma(\alpha_{ij}^*)) \geqq 3$. Thus by (13.2) $\gamma(\alpha_{ij}^*) = \alpha_{ij}^*$. Consequently $\gamma(\theta_1) = \theta_1$ and so

$$\mathscr{Q}_v \subseteqq \mathscr{F} = \mathscr{Q}(\theta_2) . \tag{13.4}$$

If now $\gamma \in \mathfrak{G}^\#, \gamma \neq \sigma, \gamma \neq \sigma^{-1}$, then (13.3) yields that

$$2 \leqq (\alpha_{ij}^*, \gamma(\alpha_{ij}^*)) = 1 + (\theta_1, \gamma(\theta_1)) .$$

Therefore $\gamma(\theta_1) = \theta_1$ and $\gamma \in \mathfrak{G}_1$. Thus $|\mathfrak{G}_1| \geqq |\mathfrak{G}| - 2$. Since $\mathfrak{G}_1 \neq \mathfrak{G}$ and $|\mathfrak{G}_1| \,|\, |\mathfrak{G}|$ we get that $|\mathfrak{G}| \leqq 4$. If $|\mathfrak{G}| = 2$ then $\mathscr{F} \subseteqq \mathscr{Q}_v$. Thus (13.2) and (13.3) yield that $2 = (\alpha_{ij}^*, \sigma(\alpha_{ij}^*)) \geqq 3$. Since $|\mathscr{Q}_v : \mathscr{Q}_{v_1}|$ is even we get that $|\mathfrak{G}| = 4$. Thus either $v_2 = 5$ and $\mathscr{F} \subseteqq \mathscr{Q}_v$ or $v_2 = 3$. In the latter case (13.2), (13.3) and (13.4) imply that $\sigma(\theta_1) = \theta_1$. Thus $\mathfrak{G} = \mathfrak{G}_1$ or equivalently $\mathscr{Q}(\theta_1) \subseteqq \mathscr{Q}_{v_1}$ contrary to assumption.

Suppose now that $v_2 = 5$. Thus $v_1 \neq 5$ and the previous argument with v_1 and v_2 interchanged yields that $\mathscr{Q}(\theta_k) \subseteqq \mathscr{Q}_{v_2}$ for $k = 1$ or $k = 2$. Thus by (13.4) $\mathscr{Q}(\theta_1) \subseteqq \mathscr{Q}_{v_2}$. By (13.2) and (13.3) $\mathfrak{G} = \langle\sigma\rangle$. Thus $\sigma^2(\theta_1) = \theta_1$ since $(\sigma^2(\alpha_{ij}^*), \alpha_{ij}^*) = 2$. Let γ be in the Galois group of $\mathscr{Q}_v$ over $\mathscr{Q}_{v_2}$. Then $\gamma\sigma^2(\theta_1) = \theta_1$ and γ can be chosen so that

$$(\alpha_{ij}^*, \gamma\sigma^2(\alpha_{ij}^*)) = 1 .$$

Hence (13.3) yields that

$$(\theta_2 + \sigma(\theta_2) - \theta_1, \gamma\sigma^2(\theta_2) + \gamma\sigma^3(\theta_2) - \theta_1) = 0 .$$

Since θ_1 is not conjugate to θ_2 this implies that

$$(\theta_2 + \sigma(\theta_2), \gamma\sigma^2(\theta_2) + \gamma\sigma^3(\theta_2)) = -1$$

contrary to the fact that $\theta_2, \sigma(\theta_2), \gamma\sigma^2(\theta_2)$ and $\gamma\sigma^3(\theta_2)$ are all characters.

Thus in any case there exists a non principal irreducible character

θ_1 of $\mathfrak{X}$ such that $(\theta_1, \alpha^*_{ij}) \neq 0$ and $\mathscr{Q}(\theta_1) \subseteq \mathscr{Q}_{v_1}$. Suppose that $\mathscr{Q}(\theta_1) = \mathscr{Q}$. Since w is odd

$$(\alpha^*_{ij}, \overline{\alpha^*_{ij}}) = (\alpha_{ij}, \overline{\alpha_{ij}}) = 1 \, .$$

Therefore

$$1 = (1_{\mathfrak{X}} \pm \theta_1 \pm \theta_2 \pm \theta_3, 1_{\mathfrak{X}} \pm \theta_1 \pm \bar{\theta}_2 \pm \bar{\theta}_3) = 2 + (\theta_2 \pm \theta_3, \bar{\theta}_2 \pm \bar{\theta}_3) \, .$$

Hence

$$(\theta_2 \pm \theta_3, \bar{\theta}_2 \pm \bar{\theta}_3) = -1 \, .$$

Since θ_2 and θ_3 are characters this yields that $\theta_k \neq \bar{\theta}_k$ for $k = 2, 3$. Hence $\bar{\theta}_2 = \theta_3$ and so $\bar{\theta}_3 = \theta_2$. Consequently $(\theta_2 \pm \theta_3, \bar{\theta}_2 \pm \bar{\theta}_3) = \pm 2$, which is not the case. Therefore

$$\mathscr{Q} \neq \mathscr{Q}(\theta_1) \subseteq \mathscr{Q}_{v_1} \, . \tag{13.5}$$

Similarly there exists an irreducible character θ_2 of $\mathfrak{X}$ with $(\theta_2, \alpha^*_{ij}) \neq 0$ and $\mathscr{Q} \neq \mathscr{Q}(\theta_2) \subseteq \mathscr{Q}_{v_2}$. Thus by (13.5) $\theta_1 \neq \theta_2$. Now (13.2) yields that

$$\alpha^*_{ij} = 1_{\mathfrak{X}} - \eta_{i0} - \eta_{0j} + \eta_{ij} \tag{13.6}$$

for $1 \leqq i \leqq w_1 - 1, 1 \leqq j \leqq w_2 - 1$. The $\pm\eta_{ij}$ are distinct irreducible characters of $\mathfrak{X}$ whose values lie in the required field. Suppose now that

$$\eta_{s0 \mid \mathfrak{W}} = \sum_{i,j} a_{ij}\omega_{ij} + a\rho_{\mathfrak{W}}$$

with $a_{00} = 0$. Then by the Frobenius reciprocity theorem it follows from (13.6) that

$$-a_{i0} - a_{0j} + a_{ij} = -\delta_{is} \, ,$$

$$\begin{aligned} \eta_{s0 \mid \mathfrak{W}} &= \sum_{i=1}^{w_1-1} a_{i0}\omega_{i0} + \sum_{j=1}^{w_2-1} a_{0j}\omega_{0j} + \sum_{i=1}^{w_1-1} a_{i0} \sum_{j=1}^{w_2-1} \omega_{ij} \\ &\quad + \sum_{j=1}^{w_2-1} a_{0j} \sum_{i=1}^{w_1-1} \omega_{ij} - \sum_{j=1}^{w_2-1} \omega_{sj} + a\rho_{\mathfrak{W}} \\ &= \sum_{i=1}^{w_1-1} a_{i0} \sum_{j=0}^{w_2-1} \omega_{ij} + \sum_{j=1}^{w_2-1} a_{0j} \sum_{i=0}^{w_1-1} \omega_{ij} - \sum_{j=1}^{w_2-1} \omega_{sj} + a\rho \quad . \end{aligned}$$

Consequently for $W \in \hat{\mathfrak{W}}$

$$\eta_{s0}(W) = -\sum_{j=1}^{w_2-1} \omega_{sj}(W) = \omega_{s0}(W) \, .$$

In a similar way it can be shown that $\eta_{0t}(W) = \omega_{0t}(W)$. Then it follows from (13.6) that $\eta_{st}(W) = \omega_{st}(W)$ for $W \in \hat{\mathfrak{W}}$.

This implies that if $W \in \hat{\mathfrak{W}}$ then

$$\sum_{i=0}^{w_1-1}\sum_{j=0}^{w_2-1} |\eta_{ij}(W)|^2 = \sum_{i=0}^{w_1-1}\sum_{j=0}^{w_2-1} |\omega_{ij}(W)|^2 = w = |C(W)| .$$

The orthogonality relations for the irreducible characters of $\mathfrak{X}$ now yield that every irreducible character of $\mathfrak{X}$ distinct from all $\pm\eta_{ij}$ vanishes on $\hat{\mathfrak{W}}$. This completes the proof of the lemma.

LEMMA 13.2. *Suppose that Hypothesis* 13.1 *is satisfied. If Δ is a generalized character of $\mathfrak{X}$ which vanishes on $\hat{\mathfrak{W}}$ then*

$$\Delta = a_{00}1_{\mathfrak{X}} + \sum_{i=1}^{w_1-1} a_{i0} \sum_{j=0}^{w_2-1} \eta_{ij} + \sum_{j=1}^{w_2-1} a_{0j} \sum_{i=0}^{w_1-1} \eta_{ij} - a_{00} \sum_{i=1}^{w_1-1}\sum_{j=1}^{w_2-1} \eta_{ij} + \Delta_0$$

where $(\Delta_0, \eta_{ij}) = 0$ *for* $0 \leqq i \leqq w_1 - 1, 0 \leqq j \leqq w_2 - 1$.

Proof. Let

$$\Delta = \Delta_0 + \sum_{i=0}^{w_1-1}\sum_{j=0}^{w_2-1} a_{ij}\eta_{ij} ,$$

where $(\Delta_0, \eta_{ij}) = 0$ for $0 \leqq i \leqq w_1 - 1, 0 \leqq j \leqq w_2 - 1$. By Lemma 13.1

$$(\Delta, 1_{\mathfrak{X}} - \eta_{i0} - \eta_{0j} + \eta_{ij}) = 0 \qquad \text{for } 0 \leqq i \leqq w_1 - 1, 0 \leqq j \leqq w_2 - 1 .$$

Hence

$$a_{00} - a_{i0} - a_{0j} + a_{ij} = 0 \qquad \text{for } 0 \leqq i \leqq w_1 - 1, 0 \leqq j \leqq w_2 - 1 .$$

This implies the desired result.

Hypothesis 13.2.
(i) *The group* $\mathfrak{L} = \mathfrak{X}$ *satisfies Hypothesis* 13.1.
(ii) $\mathfrak{L}$ *contains a normal subgroup* $\mathfrak{K}$ *such that*

$$\mathfrak{L} = \mathfrak{K}\mathfrak{W}_1, \mathfrak{K} \cap \mathfrak{W}_1 = \langle 1 \rangle$$

and if $\mathfrak{A}$ *is a non empty subset of* $\mathfrak{W} - \mathfrak{W}_2$ *then*

$$C(\mathfrak{A}) = N(\mathfrak{A}) = \mathfrak{W} .$$

Since $\mathfrak{W}_1$ is a S-subgroup of $\mathfrak{W}$, Hypothesis 13.2 (ii) implies that $\mathfrak{W}_1$ is a S-subgroup of $\mathfrak{L}$. Also, if $W \in \mathfrak{W}_1^\sharp$, then $C(W) \cap \mathfrak{K} = \mathfrak{W}_2$.

LEMMA 13.3. *Suppose that* $\mathfrak{L}$ *satisfies Hypothesis* 13.2. *Then* $\mathfrak{W} - \mathfrak{W}_2$ *is a* T. I. *set in* $\mathfrak{L}$. *For* $0 \leqq i \leqq w_1 - 1, 0 \leqq j \leqq w_2 - 1$ *there exist irreducible characters* μ_{ij} *of* $\mathfrak{L}$ *such that*

$$\mu_{ij|\mathfrak{W}} = \pm\,\omega_{ij} + \sum_{t=0}^{w_2-1} a_t \sum_{s=0}^{w_1-1} \omega_{st}\ ,$$

where $\{a_t\}$ *is a set of integers depending on* j *and the sign depends only on* j.

Proof. Hypothesis 13.2 implies that $\mathfrak{W} - \mathfrak{W}_2$ is a T. I. set in $\mathfrak{L}$. For $0 \leqq i, k \leqq w_1 - 1, 0 \leqq j \leqq w_2 - 1, \omega_{ij} - \omega_{kj}$ vanishes on $\mathfrak{W}_2$. Define

$$\mathscr{S}_j = \{\omega_{ij} \mid 0 \leqq i \leqq w_1 - 1\} \quad \text{for } 0 \leqq j \leqq w_2 - 1\ .$$

Then by Lemma 10.1 $\mathscr{S}_j$ is coherent for $0 \leqq j \leqq w_2 - 1$. Let $\mu_{ij} = \pm\omega_{ij}^{\tau}$, where the sign is chosen so that $\mu_{ij}(1) > 0$. Then

$$(\omega_{ij} - \omega_{kj})^{\tau} = (\omega_{ij} - \omega_{kj})^{*} = \pm(\mu_{ij} - \mu_{kj})$$
$$\text{for } 0 \leqq i, k \leqq w_1 - 1, 0 \leqq j \leqq w_2 - 1\ .$$

The Frobenius reciprocity theorem now implies the required result since $(\omega_{ij} - \omega_{kj})^{*}$ vanishes on $\mathfrak{W}_2$.

LEMMA 13.4. *Suppose that* $\mathfrak{L}$ *satisfies Hypothesis* 13.2. *Let* λ *be an irreducible character of* $\mathfrak{L}$. *Then there exists an integer* a *such that*

$$\lambda_{|\mathfrak{W}_1} = a\rho_{\mathfrak{W}_1}\ ,$$

or

$$\lambda_{|\mathfrak{W}_1} = \pm\omega_{ij_{|\mathfrak{W}_1}} + a\rho_{\mathfrak{W}_1}$$

for some i, j *with* $0 \leqq i \leqq w_1 - 1, 0 \leqq j \leqq w_2 - 1$.

Proof. Let μ_{ij} be the characters defined in Lemma 13.3. If $\lambda = \mu_{ij}$ for some i, j with $0 \leqq i \leqq w_1 - 1$, $0 \leqq j \leqq w_2 - 1$ then the result follows from Lemma 13.3. Furthermore Lemma 13.3 implies that

$$\sum_{i=0}^{w_1-1} \sum_{j=0}^{w_2-1} |\mu_{ij}(W)|^2 = w = |C(W)| \quad \text{for } W \in \mathfrak{W}_1^{\sharp}\ .$$

Hence if $\lambda \neq \mu_{ij}$ for all i, j we have that $\lambda(W) = 0$ for $W \in \mathfrak{W}_1^{\sharp}$. This completes the proof of the lemma.

We will use the fact that Lemma 13.4 is valid over fields of characteristic prime to $|\mathfrak{L}|$, provided that λ is absolutely irreducible.

LEMMA 13.5. *Suppose that* $\mathfrak{L}$ *satisfies Hypothesis* 13.2. *For*

$0 \leqq i \leqq w_1 - 1, 0 \leqq j \leqq w_2 - 1$ let μ_{ij} be the characters defined by Lemma 13.3. *Define*

$$\xi_j = \sum_{i=0}^{w_1-1} \mu_{ij} \quad \text{for } 0 \leqq j \leqq w_2 - 1 .$$

Then ξ_j is induced by an irreducible character μ_j of $\mathfrak{R}$. Furthermore

$$\mu_{ij|\mathfrak{R}} = \mu_{0j|\mathfrak{R}} = \mu_j \quad \text{for } 0 \leqq i \leqq w_1 - 1, 0 \leqq j \leqq w_2 - 1 .$$

Proof. By Lemma 13.4 the characters $\mu_{ij}, 0 \leqq i \leqq w_1 - 1, 0 \leqq j \leqq w_2 - 1$ are the only irreducible characters of $\mathfrak{L}$ which do not vanish on $\mathfrak{W}_1^\#$. Since each μ_{i0} agrees on $\mathfrak{W}_1$ with a suitable linear character of $\mathfrak{L}/\mathfrak{R}$ it follows from Lemma 13.1 that $\{\mu_{i0} \mid 0 \leqq i \leqq w_1 - 1\}$ is the set of irreducible characters of $\mathfrak{L}/\mathfrak{R}$. Therefore $\mu_{i0}\mu_{0j}$ agrees with μ_{ij} on $\hat{\mathfrak{W}}$. Hence Lemma 13.1 implies that $\mu_{i0}\mu_{0j} = \mu_{ij}$. Consequently if $\mu_j = \mu_{0j|\mathfrak{R}}$ then

$$\mu_{ij|\mathfrak{R}} = \mu_{0j|\mathfrak{R}} = \mu_j \quad \text{for } 0 \leqq i \leqq w_1 - 1, 0 \leqq j \leqq w_2 - 1 .$$

Thus the Frobenius reciprocity theorem implies that μ_{ij} is a constituent of μ_j^* for all values of i, j. Since

$$\mu_j^*(1) = w_1\mu_j(1) = \sum_{i=0}^{w_1-1} \mu_{ij}(1) = \xi_j(1)$$

the lemma is proved.

LEMMA 13.6. *Suppose that $\mathfrak{L}$ satisfies Hypothesis* 13.2, *p is a prime, and $\mathfrak{R}$ is an extra special p-group with $\mathfrak{R}' = \mathfrak{W}_2$. Let $|\mathfrak{R}:\mathfrak{R}'| = p^{2n}$. Then w_1 divides either $p^n + 1$ or $p^n - 1$.*

Proof. It is easily seen that a faithful irreducible character of $\mathfrak{R}$ has degree p^n. Thus by Lemmas 13.4 and 13.5

$$p^n = \mu_{11}(1) = aw_1 \pm 1 .$$

This proves the result.

LEMMA 13.7. *Suppose that $\mathfrak{L}$ satisfies Hypothesis* 13.2. *Let μ_j, ξ_j be defined by Lemma* 13.5. *Then an irreducible character of $\mathfrak{R}$ either induces an irreducible character of $\mathfrak{L}$ or it induces ξ_j for some j with $0 \leqq j \leqq w_2 - 1$.*

Proof. The group $\mathfrak{W}_1$ acts as a permutation group on the conjugate classes of $\mathfrak{R}$. If $W \in \mathfrak{W}_1$ and W leaves some conjugate class of $\mathfrak{R}$ fixed, then since $\mathfrak{W}_1$ is a Hall subgroup of $\mathfrak{L}$, W must centralize some element of this conjugate class. Hence by assumption the only conjugate classes of $\mathfrak{K}$ which are fixed by any $W \in \mathfrak{W}_1^\sharp$ are those containing an element of $\mathfrak{W}_2$. There are at most w_2 of these. The group $\mathfrak{W}_1$ also acts as a permutation group on the irreducible characters of $\mathfrak{K}$. Therefore by 3.14 there are at most w_2 irreducible characters of $\mathfrak{K}$ which are fixed by any element $W \in \mathfrak{W}_1^\sharp$. By Lemma 13.5 the w_2 distinct characters μ_j, $0 \leqq j < w_2$ are fixed by every $W \in \mathfrak{W}_1$ and these induce ξ_j, $0 \leqq j < w_2$. Thus every other irreducible character of $\mathfrak{K}$ induces an irreducible character of $\mathfrak{L}$. The proof is complete.

Hypothesis 13.3.

(i) *$\hat{\mathfrak{L}}$ is a tamely imbedded subset of the group $\mathfrak{X}$ and $\mathfrak{L} = N(\hat{\mathfrak{L}})$ has odd order.*

(ii) *$\mathfrak{L}$ satisfies Hypothesis 13.2, and $\mathfrak{X}$ satisfies Hypothesis 13.1 with the same group $\mathfrak{W}$.*

(iii) *$\mathfrak{L}$ contains a normal nilpotent subgroup $\mathfrak{H}$ such that*

$$\mathfrak{W}_2 \subseteq \mathfrak{H} \subseteq \bigcup_{H \in \mathfrak{H}^\sharp} C(H) \cap \mathfrak{K} \subseteq \hat{\mathfrak{L}} \subseteq \mathfrak{K} \subset \mathfrak{L} .$$

$$\hat{\mathfrak{L}}_1 = \hat{\mathfrak{L}} \cup \bigcup_{L \in \mathfrak{L}} L^{-1} \hat{\mathfrak{W}} L .$$

(iv) *There exist subgroups $\mathfrak{H}_1, \cdots, \mathfrak{H}_n$ such that $\{\mathfrak{H}_s \mid 1 \leqq s \leqq n\}$ is a system of supporting subgroups of $\hat{\mathfrak{L}}$ and $\hat{\mathfrak{L}}_1$. Let $\mathfrak{N}_s = N(\mathfrak{H}_s)$ for $1 \leqq s \leqq n$.*

(v) *For $0 \leqq i \leqq w_1 - 1, 0 \leqq j \leqq w_2 - 1$ let $\eta_{ij}, \mu_{ij}, \xi_j$ be defined respectively by Lemmas 13.1, 13.3 and 13.5.*

(vi) *Let $\mathscr{S}$ be the set of characters of $\mathfrak{L}$ which are induced by non principal irreducible characters of $\mathfrak{K}$, each of which vanishes outside $\hat{\mathfrak{L}}$.*

LEMMA 13.8. *Suppose that Hypothesis 13.3 is satisfied. Assume that for some i, j, k with $0 \leqq i \leqq w_1 - 1, 1 \leqq j, k \leqq w_2 - 1$, $\mu_{ij}(1) = \mu_{ik}(1)$. Then $\mu_{ij} - \mu_{ik}$ vanishes in $\mathfrak{L} - \hat{\mathfrak{L}}_1^\sharp$ and*

$$(\mu_{ij} - \mu_{ik})^\tau = \pm(\eta_{ij} - \eta_{ik}) .$$

Proof. By Lemma 13.3 μ_{ij}, μ_{ik} do not contain $\mathfrak{W}_2$ in their kernel, thus they do not contain $\mathfrak{H}$ in their kernel. Hence by Lemma 4.3 μ_{ij}, μ_{ik} vanish on $\mathfrak{K} - \hat{\mathfrak{L}}$. By Lemma 13.3 $\mu_{ij|\mathfrak{W}_1} = \mu_{ik|\mathfrak{W}_1}$. Thus $\mu_{ij} - \mu_{ik}$ vanishes on $\mathfrak{L} - \hat{\mathfrak{L}}_1^\sharp$. Hence $\| (\mu_{ij} - \mu_{ik})^\tau \|^2 = 2$. By Lemmas 9.1 and 13.3

$$\{(\mu_{ij} - \mu_{ik})^\tau \pm (\eta_{ij} - \eta_{ik})\}(W) = 0 \quad \text{for } W \in \hat{\mathfrak{W}} .$$

Thus the result follows from Lemma 13.1.

LEMMA 13.9. *Suppose that Hypothesis* 13.3 *is satisfied. Choose k with $1 \leqq k \leqq w_2 - 1$. Let*

$$\mathscr{S}_1 = \{\xi_j \mid 1 \leqq j \leqq w_2 - 1, \xi_j(1) = \xi_k(1)\} .$$

Then $\mathscr{S}_1$ is coherent and

$$\xi_j^\tau = \varepsilon \sum_{i=0}^{w_1-1} \eta_{ij}$$

is an extension of τ to $\mathscr{S}_1$ where either $\varepsilon = 1$ or $\varepsilon = -1$.

Proof. Since $|\mathfrak{L}|$ is odd $\xi_j \neq \bar{\xi}_j$. Hence $\mathscr{I}_0(\mathscr{S}_1) \neq 0$. By Lemma 13.5

$$\xi_j - \xi_k = \sum_{i=0}^{w_1-1} (\mu_{ij} - \mu_{ik}) .$$

Hence Lemma 13.8 yields that

$$(\xi_j - \xi_k)^\tau = \sum_{i=0}^{w_1-1} \pm (\eta_{ij} - \eta_{ik}) .$$

By Lemma 9.1 $(\xi_j - \xi_k)^\tau$ vanishes on $\hat{\mathfrak{W}}_1$. Thus Lemma 13.2 implies that

$$(13.7) \qquad (\xi_j - \xi_k)^\tau = \pm \sum_{i=0}^{w_1-1} (\eta_{ij} - \eta_{ik}) .$$

Now define

$$\xi_j^\tau = \pm \sum_{i=0}^{w_1-1} \eta_{ij}$$

where the sign is the same as in (13.7). It is easily seen that τ is a linear isometry on $\mathscr{S}_1$. Thus $\mathscr{S}_1$ is coherent.

LEMMA 13.10. *Suppose that Hypothesis* 13.3 *is satisfied. Let $\mathscr{S}_1$ have the same meaning as in Lemma* 13.9. *Then $(\mathscr{S}_1, \tau)$ is subcoherent in $\mathscr{S}$ where τ is defined on $\mathscr{S}_1$ as in Lemma* 13.9.

Proof. By Lemma 13.9 $\mathscr{S}_1$ is coherent. Let $\mathscr{T}$ be a coherent subset of $\mathscr{S}$ which is orthogonal to $\mathscr{S}_1$. Let τ_2 be an extension of τ to $\mathscr{T}$.

Every generalized character in $\mathscr{S}$ vanishes on $\hat{\mathfrak{W}}$. Thus by Lemma 9.1 every generalized character in $\mathscr{I}_0(\mathscr{S})^\tau$ vanishes on $\hat{\mathfrak{W}}$. If λ is an irreducible character in $\mathscr{T}$, then $\lambda \neq \bar{\lambda}$ as $|\mathfrak{L}|$ is odd. Furthermore $(\lambda - \bar{\lambda})^\tau \in \mathscr{I}_0(\mathscr{S})^\tau$ and thus vanishes on $\hat{\mathfrak{W}}$. Hence $\lambda^{\tau_2} \neq \pm\eta_{ij}$ for $0 \leqq i \leqq w_1 - 1, 0 \leqq j \leqq w_2 - 1$. Therefore λ^{τ_2} is orthogonal to $\mathscr{S}_1^\tau$. If $\xi_s \in \mathscr{T}$, then since $(\xi_s^{\tau_2}, (\xi_s - \bar{\xi}_s)^\tau) = w_1$, $\xi_s^{\tau_2}$ is a linear combination of η_{is} and $\bar{\eta}_{is}$ with $0 \leqq i \leqq w_1 - 1$. Hence $\xi_s^{\tau_2}$ is orthogonal to $\mathscr{S}_1^\tau$. Consequently $\mathscr{T}^{\tau_2}$ is orthogonal to $\mathscr{S}_1^\tau$.

Suppose now that $\alpha \in \mathscr{I}_0(\mathscr{S})$ with $\alpha^\tau = \Delta_1 + \Delta_2$, where $\Delta_2 \in \mathscr{C}(\mathscr{T}^{\tau_2})$, Δ_1 is not orthogonal to $\mathscr{I}_0(\mathscr{S}_1^\tau)$ and $||\Delta_1||^2 \leqq w_1$. Let $\alpha^\tau = \Gamma + \Delta$, where Δ is a linear combination of the generalized characters η_{ij} and $(\Gamma, \eta_{ij}) = 0$ for $0 \leqq i \leqq w_1 - 1, 0 \leqq j \leqq w_2 - 1$. Let σ be the set of integers s such that $\xi_s \in \mathscr{T}$. Lemma 13.8 implies that every generalized character in $\mathscr{T}^{\tau_2}$ is orthogonal to η_{ij} for $0 \leqq i \leqq w_1 - 1, j \notin \sigma$. Let $\Delta = \Delta_0 + \Delta_1'$, where Δ_0 is a linear combination of η_{is} with $s \in \sigma$ and $(\Delta_1', \eta_{is}) = 0$ for $0 \leqq i \leqq w_1 - 1, s \in \sigma$. Then

$$||\Delta_1'||^2 \leqq w_1 \,. \tag{13.8}$$

By changing notation it may be assumed that $\xi_1, \xi_2 \in \mathscr{S}_1$ and $(\Delta_1', \xi_1^\tau - \xi_2^\tau) > 0$. By Lemma 9.4

$$(\Delta_1', \xi_1^\tau - \xi_2^\tau)_{\mathfrak{X}} = (\alpha^\tau, \xi_1^\tau - \xi_2^\tau)_{\mathfrak{X}} = (\alpha, \xi_1 - \xi_2)_{\mathfrak{L}} \,.$$

Hence $(\Delta_1', \xi_1^\tau - \xi_2^\tau)$ is a non zero integral multiple of w_1. By (13.8)

$$(\Delta_1', \xi_1^\tau - \xi_2^\tau)^2 \leqq ||\Delta_1'||^2 \, ||\xi_1^\tau - \xi_2^\tau||^2 \leqq 2w_1^2 \,.$$

Therefore

$$(\Delta_1', \xi_1^\tau - \xi_2^\tau) = w_1 \,. \tag{13.9}$$

By Lemma 13.2

$$\Delta_1' = \varepsilon \sum_{i=0}^{w_1-1} a_{i0}\eta_{i0} + \varepsilon \sum_{i=0}^{w_1-1} \{(a_{i0} + a_{01})\eta_{i1} + (a_{i0} + a_{02})\eta_{i2}\} + \Delta_1'' \,, \tag{13.10}$$

where ε is as in Lemma 13.9 and where $(\Delta_1'', \eta_{it}) = 0$ for $0 \leqq i \leqq w_1 - 1$, $t = 0, 1, 2$. Now (13.9) yields that $a_{01} - a_{02} = 1$. Thus (13.8) and (13.10) imply

$$\sum_{i=0}^{w_1-1} a_{i0}^2 + \sum_{i=0}^{w_1-1} \{(a_{i0} + a_{01})^2 + (a_{i0} + a_{01} - 1)^2\} \leqq w_1 \,.$$

Every term in the second summation is non zero. Thus $a_{i0} = 0$ for $0 \leqq i \leqq w_1 - 1$. Hence $a_{01} = 1$ or $a_{01} = 0$. Hence (13.8) and (13.10) yield that $\Delta_1' = \xi_1^\tau$ or $\Delta_1' = -\xi_2^\tau$. This shows that $(\mathscr{S}_1, \tau)$ is subcoherent in $\mathscr{S}$ and completes the proof of the lemma.

In the proof of the main theorem of this paper we will reserve the letter τ to denote the extension of τ to $\mathscr{S}_1$ defined by Lemma 13.9. Thus $(\mathscr{S}_1, \tau)$ will always be subcoherent in $\mathscr{S}$.

DEFINITION. A *Z-group* is a group all of whose Sylow subgroups are cyclic.

Hypothesis 13.4.

(i) $\mathfrak{L} = \mathfrak{B}\mathfrak{K}$ *with* $\mathfrak{B} \cap \mathfrak{K} = 1$, $\mathfrak{K} \lhd \mathfrak{L}$ *and* $\mathfrak{K}$ *solvable. Furthermore* $\mathfrak{B}$ *is a cyclic S-subgroup of* $\mathfrak{L}$ *and* $|\mathfrak{L}|$ *is odd.*

(ii) *For* $B \in \mathfrak{B}^{\#}$, $C_{\mathfrak{K}}(B) = C_{\mathfrak{K}}(\mathfrak{B})$. *Furthermore* $C_{\mathfrak{K}}(\mathfrak{B})$ *is a Z-group and* $\mathfrak{K} \neq C_{\mathfrak{K}}(\mathfrak{B})$.

(iii) $\mathfrak{L}$ *is faithfully and irreducibly represented on a vector space* $\mathscr{V}$ *over a field of characteristic not dividing* $|\mathfrak{L}|$. $\mathscr{V}$ *contains a vector space* $\mathscr{V}_0$ *of dimension at most* 1 *such that if* $B \in \mathfrak{B}^{\#}$, $v \in \mathscr{V}$ *then* $vB = v$ *if and only if* $v \in \mathscr{V}_0$.

LEMMA 13.11. *Suppose that Hypothesis* 13.4 *is satisfied. Then* $\mathfrak{K}$ *is nilpotent. Furthermore* $|\mathfrak{B}|$ *is a prime and the representation of* $\mathfrak{L}$ *on* $\mathscr{V}$ *is absolutely irreducible.*

Proof. Let λ be the character of the representation of $\mathfrak{L}$ on $\mathscr{V}$. Let $\mathfrak{P}$ be a S_p-subgroup of $\mathfrak{K}$ which is normalized but not centralized by $\mathfrak{B}$. Then either $C_{\mathfrak{P}}(\mathfrak{B}) = 1$ or $\mathfrak{P}\mathfrak{B}$ satisfies Hypothesis 13.2. Thus by Lemma 13.4 only one absolutely irreducible constituent of $\lambda_{|\mathfrak{P}\mathfrak{B}}$ is not linear. Hence λ is absolutely irreducible. Furthermore Lemma 13.4 and 3.16 (iii) imply that $\lambda_{|\mathfrak{B}}$ has $\rho_{\mathfrak{B}}$ as a constituent. Thus $|\mathfrak{B}|$ is a prime.

The nilpotence of $\mathfrak{K}$ is proved by induction on $|\mathfrak{K}|$. We assume without loss of generality that the underlying field is algebraically closed. If $\mathfrak{B} \subseteq \boldsymbol{F}(\mathfrak{L})$ then $\mathfrak{K} \subseteq C(\mathfrak{B})$ contrary to assumption. Thus by 3.3 $\mathfrak{B} \nsubseteq C(\boldsymbol{F}(\mathfrak{L}))$. Let $\mathfrak{F}$ be a minimal nilpotent normal subgroup of $\mathfrak{L}$ which is not centralized by $\mathfrak{B}$. Then $\mathfrak{F}$ is a p-group for some prime p. Furthermore $\mathfrak{F}' = \boldsymbol{D}(\mathfrak{F})$ and $\mathfrak{B} \subseteq C(\boldsymbol{D}(\mathfrak{F}))$. By Lemma 13.4 there is exactly one non linear irreducible constituent of $\lambda_{|\mathfrak{F}\mathfrak{B}}$. Let

$$\lambda_{|\mathfrak{F}\mathfrak{B}} = \sum_{i=1}^{n} \mu_i + \theta ,$$

where each μ_i is a linear character of $\mathfrak{F}\mathfrak{B}$. Assume first that $n \neq 0$. If ν is an irreducible constituent of $\theta_{|\mathfrak{F}}$, then $(\nu, \theta_{|\mathfrak{F}}) = 1$. Since $\nu \neq \mu_{i_{|\mathfrak{F}}}$ for $1 \leqq i \leqq n$, we have $(\lambda_{|\mathfrak{F}}, \mu_{i_{|\mathfrak{F}}}) = 1$. Since $\lambda_{|\mathfrak{F}}$ is a sum of conjugate characters this implies that $\mathfrak{F}$ is abelian and the μ_i are distinct. Thus $\mathfrak{F}\mathfrak{B} = \mathfrak{F}_0 \times \mathfrak{F}_1\mathfrak{B}$, where $|\mathfrak{F}_0| = p$ and $\mathfrak{F}_1\mathfrak{B}$ is a Frobenius group. For $L \in \mathfrak{L}$ let $\mu_i^L(X) = \mu_i(L^{-1}XL)$. If $L \in \mathfrak{L}$ such that $\mu_i^L = \mu_j$ for some i, j then $L \in \boldsymbol{N}(\mathfrak{F}_1)$ since $\mathfrak{F}_1$ is the kernel of each $\mu_{i_{|\mathfrak{F}}}$. Since $\mathfrak{L}$ permutes the constituents of $\lambda_{|\mathfrak{F}}$ transitively this implies that $\boldsymbol{N}(\mathfrak{F}_1)$ acts transitively on $\{\mu_1, \cdots, \mu_n\}$. Hence n is odd. Thus $\lambda(1) = n + |\mathfrak{B}|$

is even contradicting the absolute irreducibility of λ. Therefore $n = 0$ and $\lambda_{|\mathfrak{F}\mathfrak{B}}$ is irreducible.

By Lemma 13.4 this implies that $\lambda(1) = |\mathfrak{B}|$ or $\lambda(1) = 2|\mathfrak{B}| - 1$. If $\lambda(1) = |\mathfrak{B}|$ then $\lambda_{|\mathfrak{K}}$ is reducible since $(|\mathfrak{B}|, |\mathfrak{K}|) = 1$. As $|\mathfrak{B}|$ is a prime this implies that $\lambda_{|\mathfrak{K}}$ is a sum of linear characters and $\mathfrak{K}$ is abelian. Thus we can suppose that $\lambda(1) = 2|\mathfrak{B}| - 1$. By Lemma 13.4 $\lambda_{|\mathfrak{F}}$ is irreducible. Thus if $\mathfrak{H}$ is any proper $\mathfrak{B}$-invariant subgroup of $\mathfrak{K}$ with $\mathfrak{F} \subseteq \mathfrak{H}$ then $\mathfrak{B}\mathfrak{H}$ satisfies the induction assumption and $\mathfrak{H}$ is nilpotent. If $\mathfrak{H} = \mathfrak{P} \times \mathfrak{H}_1$ with $\mathfrak{F} \subseteq \mathfrak{P}$ then since $\lambda_{|\mathfrak{P}}$ is irreducible, $\mathfrak{H}_1 \subseteq Z(\mathfrak{L})$. If $\mathfrak{F}$ is not a S_p-subgroup of $\mathfrak{L}$ then $\mathfrak{F}\mathfrak{K}_1$ is a proper subgroup of $\mathfrak{K}$ where $\mathfrak{K}_1$ is a $\mathfrak{B}$-invariant p-complement in $\mathfrak{K}$. Thus $\mathfrak{K}_1 \subseteq Z(\mathfrak{K})$ and $\mathfrak{K}$ is nilpotent. Suppose now that $\mathfrak{F}$ is a S_p-subgroup of $\mathfrak{L}$.

Since $D(\mathfrak{F}) \subseteq C(\mathfrak{B})$, $D(\mathfrak{F})$ is cyclic. Let $\mathfrak{F}_1$ be the subgroup of index p in $D(\mathfrak{F})$. Then $\mathfrak{F}/\mathfrak{F}_1$ is a p-group of class 2 and hence is a regular p-group. If $\mathfrak{F}/\mathfrak{F}_1$ does not have exponent p then there exists a characteristic subgroup of $\mathfrak{F}$ of index p which is normal in $\mathfrak{L}$ but is not centralized by $\mathfrak{B}$ contrary to the minimality of $\mathfrak{F}$. Thus $\mathfrak{F}/\mathfrak{F}_1$ has exponent p. Therefore $\mathfrak{B}$ acts without fixed points on $\mathfrak{F}/D(\mathfrak{F})$ as $C_{\mathfrak{F}}(\mathfrak{B})$ is cyclic and $D(\mathfrak{F}) \subseteq C(\mathfrak{B})$.

Let $\mathfrak{K}/\mathfrak{H}$ be a chief factor of $\mathfrak{L}$ with $\mathfrak{F} \subseteq \mathfrak{H}$. Suppose first that $\mathfrak{B}$ does not centralize $\mathfrak{K}/\mathfrak{H}$. Then $\mathfrak{B}\mathfrak{K}/\mathfrak{H}$ is a Frobenius group which is represented on $\mathfrak{F}/D(\mathfrak{F})$. As $\mathfrak{B}$ has no fixed points on $\mathfrak{F}/D(\mathfrak{F})$ Lemma 4.6 implies that $\mathfrak{K}/\mathfrak{H}$ acts trivially on $\mathfrak{F}/D(\mathfrak{F})$. Thus $\mathfrak{K} = \mathfrak{F}C_{\mathfrak{K}}(\mathfrak{F})$ is nilpotent. Assume now that $\mathfrak{L}/\mathfrak{H}$ is abelian. Then $|\mathfrak{K}:\mathfrak{H}| = q$ for some prime $q \neq p$. If $\mathfrak{B}\mathfrak{K}/\mathfrak{H}$ is represented faithfully on $\mathfrak{F}/D(\mathfrak{F})$, the minimal nature of $\mathfrak{F}$ implies that $\mathfrak{B}\mathfrak{K}/\mathfrak{H}$ is represented irreducibly on $\mathfrak{F}/D(\mathfrak{F})$. Let $\mathfrak{K}/\mathfrak{H} = \langle Q\mathfrak{H}\rangle$. Then Q acts without fixed points on $\mathfrak{F}/D(\mathfrak{F})$. Since $\lambda_{|\mathfrak{F}}$ is irreducible, $Z(\mathfrak{F}) \subseteq Z(\mathfrak{L})$. Thus $Q \in C(\Omega_1(D(\mathfrak{F})))$. Hence $Q \in C(D(\mathfrak{F}))$. We will now reach a contradiction from the fact that $Q \notin C(\mathfrak{F})$. Let $\mathfrak{H} = \mathfrak{F} \times \mathfrak{H}_1$. Then $\mathfrak{H}_1 \subseteq Z(\mathfrak{L})$. Thus $\mathfrak{L}/\mathfrak{F}$ is abelian. Let μ be the linear character of $\mathfrak{L}/\mathfrak{F}$ such that $\lambda(H) = \lambda(1)\mu(H)$ for $H \in \mathfrak{H}_1$. Let $\lambda_0 = \lambda\mu^{-1}$. Then $\lambda_0(1) = \lambda(1) = 2|\mathfrak{B}| - 1$ and λ_0 is an irreducible character of $\mathfrak{L}/\mathfrak{H}_1$. The group $\mathfrak{L}/\mathfrak{H}_1$ satisfies Hypothesis 13.2 where $\mathfrak{F}\mathfrak{H}_1/\mathfrak{H}_1$ is the normal subgroup. Thus by Lemma 13.4 no irreducible character of $\mathfrak{L}/\mathfrak{H}_1$ has degree $2|\mathfrak{B}| - 1$. This completes the proof of the lemma in all cases.

DEFINITION. Let $\mathfrak{A}$ and $\mathfrak{B}$ be subgroups of a group $\mathfrak{L}$ with $\mathfrak{B} \subseteq N(\mathfrak{A})$. We say that $\mathfrak{B}$ *is prime on* $\mathfrak{A}$ if

$$C_{\mathfrak{A}}(B) = C_{\mathfrak{A}}(\mathfrak{B}) \quad \text{for } B \in \mathfrak{B}^{\#} .$$

If $|\mathfrak{B}|$ is a prime then $\mathfrak{B}$ is necessarily prime on $\mathfrak{A}$.

Lemma 13.12. *Let* $\mathfrak{L} = \mathfrak{A}\mathfrak{B}$ *with* $\mathfrak{A} \triangleleft \mathfrak{L}$, $\mathfrak{A}$ *solvable,* $\mathfrak{B}$ *cyclic,* $(|\mathfrak{A}|, |\mathfrak{B}|) = 1$ *and* $|\mathfrak{A}\mathfrak{B}|$ *odd. Suppose that* $\mathfrak{B}$ *is prime on* $\mathfrak{A}$ *and* $C_{\mathfrak{A}}(\mathfrak{B})$ *is a Z-group. If* $C_{\mathfrak{A}}(\mathfrak{B}) \subseteq \mathfrak{A}'$ *then* $\mathfrak{A}/F(\mathfrak{A})$ *is nilpotent. If furthermore* $|\mathfrak{B}|$ *is not a prime then* $\mathfrak{A}$ *is nilpotent.*

Proof. Let $\mathfrak{L}$ be a counter example to the result for which $|\mathfrak{A}|$ has minimum order. Since $(|\mathfrak{A}|, |\mathfrak{B}|) = 1$ the hypotheses are satisfied by all $\mathfrak{B}$-invariant factor groups of $\mathfrak{A}$.

Suppose that $|\mathfrak{B}|$ is not a prime. Let $\mathfrak{M}$ be a minimal normal subgroup of $\mathfrak{L}$. Then $\mathfrak{M}$ is a p-group for some prime p and $\mathfrak{M} \subseteq \mathfrak{A}$. By induction $\mathfrak{A}/\mathfrak{M}$ is nilpotent. If $\mathfrak{Q}$ is a $\mathfrak{B}$-invariant S_q-group of $\mathfrak{A}$ for $q \in \pi(\mathfrak{A}), q \neq p$, then $\mathfrak{M}\mathfrak{Q} \triangleleft \mathfrak{A}\mathfrak{B}$ and $\mathfrak{B}$ has no fixed points on $\mathfrak{Q} - \mathfrak{Q}'$. If $\mathfrak{A}$ is not nilpotent then it is possible to choose q so that $\mathfrak{M}\mathfrak{Q}$ is not nilpotent. Let $\mathfrak{Q}_1 = C_{\mathfrak{Q}}(\mathfrak{M})$. Then $\mathfrak{B}\mathfrak{Q}/\mathfrak{Q}_1$ is faithfully represented on $\mathfrak{M}$. Hypothesis 13.4 is satisfied with $\mathfrak{M}$ in the role of $\mathscr{V}$. Thus by Lemma 13.11 $|\mathfrak{B}|$ is a prime contrary to assumption.

Assume now that $|\mathfrak{B}|$ is a prime. Suppose that $\mathfrak{L}$ contains two distinct minimal normal subgroups $\mathfrak{M}_1$ and $\mathfrak{M}_2$. For $i = 1, 2$ let $\mathfrak{F}_i$ be the inverse image of $F(\mathfrak{A}/\mathfrak{M}_i)$ in $\mathfrak{A}$. By induction $\mathfrak{A}/\mathfrak{F}_i$ is nilpotent for $i = 1, 2$. The result now follows from the fact that $F(\mathfrak{A}) = \mathfrak{F}_1 \cap \mathfrak{F}_2$. Thus it may be assumed that $\mathfrak{L}$ contains a unique minimal normal subgroup $\mathfrak{M}$. Then $\mathfrak{M} \subseteq O_p(\mathfrak{A}) = F(\mathfrak{A})$ for some prime p. Let $\mathfrak{D} = D(O_p(\mathfrak{A}))$. Then $F(\mathfrak{A}/\mathfrak{D})$ is a p-group. Thus the result follows by induction if $\mathfrak{D} \neq 1$. Assume now that $\mathfrak{D} = 1$. Then $C_{\mathfrak{A}}(\mathfrak{M}) = O_p(\mathfrak{A})$.

Let $\mathfrak{A}_1$ be a $\mathfrak{B}$-invariant $S_{p'}$-subgroup of $\mathfrak{A}$. Then $\mathfrak{A}_1\mathfrak{B}$ is faithfully represented of $\mathfrak{M}$. Hypothesis 13.4 is satisfied with $\mathfrak{M}$ in place of $\mathscr{V}$ unless $\mathfrak{A}_1 \subseteq C_{\mathfrak{A}}(\mathfrak{B})$. Thus by Lemma 13.11 $\mathfrak{A}_1$ is nilpotent or $\mathfrak{A}_1 \subseteq C_{\mathfrak{A}}(\mathfrak{B})$.

Let $\mathfrak{A}_0 = \mathfrak{A}/O_p(\mathfrak{A})$ and let $\mathfrak{P}_0$ be a $\mathfrak{B}$-invariant S_p-group of $\mathfrak{A}_0$. If $\mathfrak{P}_0 \subseteq F(\mathfrak{A}_0)$ then $\mathfrak{A}_0/\mathfrak{P}_0$ is nilpotent since it is a p'-group and the result is proved. Assume that $\mathfrak{P}_0 \not\subseteq F(\mathfrak{A}_0)$. By induction $\mathfrak{A}_0/F(\mathfrak{A}_0)$ is nilpotent. Hence $\mathfrak{B}$ does not centralize $\mathfrak{P}_0$ by assumption.

Let $\mathfrak{P}$ be a p-group in $\mathfrak{A}_0$ which is minimal with the property that $\mathfrak{B}$ normalizes $\mathfrak{P}$ but does not centralize $\mathfrak{P}$. Since $F(\mathfrak{A}_0)$ is a p'-group there is a prime $q \neq p$ such that $\mathfrak{P}\mathfrak{Q}$ contains no normal p-subgroup, where $\mathfrak{Q}$ is a S_q-group of $F(\mathfrak{A}_0)$. Thus $\mathfrak{B}\mathfrak{P}$ acts faithfully on $\mathfrak{Q}$. Let $\mathfrak{M}_1 = C_{\mathfrak{M}}(\mathfrak{B})$. As $\mathfrak{Q}\mathfrak{B}$ is faithfully represented on $\mathfrak{M}$ Lemmas 4.6 and 13.4 imply that $\mathfrak{M}_1 \neq 1$. Let $\mathfrak{Q}_1 = C_{\mathfrak{Q}}(\mathfrak{B})$. As $\mathfrak{P}\mathfrak{B}$ is represented faithfully on $\mathfrak{Q}/D(\mathfrak{Q})$, Lemmas 4.6 and 13.4 imply that $\mathfrak{Q}_1 \neq 1$. Thus $C_{\mathfrak{A}}(\mathfrak{B})$ is a Z-group, $\mathfrak{M}_1 \triangleleft C_{\mathfrak{A}}(\mathfrak{B})$ and $pq \mid |C_{\mathfrak{A}}(\mathfrak{B})|$. Therefore

$$p \equiv 1 \pmod{q} . \tag{13.11}$$

By 3.11 $\mathfrak{P}$ is a special p-group and $D(\mathfrak{P}) \subseteq C_{\mathfrak{P}}(\mathfrak{B})$. Thus $D(\mathfrak{P})$ is cyclic. By Lemma 13.11 the representation of $\mathfrak{P}\mathfrak{B}$ on $\mathfrak{Q}/D(\mathfrak{Q})$ has a unique faithful irreducible constituent and this constituent is absolutely irreducible. Let μ be the character of this constituent. If $D(\mathfrak{P}) \neq 1$ then by Lemma 13.4 $\mu_{|\mathfrak{P}}$ remains absolutely irreducible. Hence $q \equiv 1 \pmod p$ contrary to (13.11). Therefore $\mathfrak{P}$ is an elementary abelian group and $\mathfrak{B}\mathfrak{P}$ is a Frobenius group. Thus $\mu(1) = |\mathfrak{B}|$ is a prime. If $\mathfrak{P}$ is not cyclic then $\mu_{|\mathfrak{P}}$ is reducible in the field of q elements as $\mu_{|\mathfrak{P}}$ is faithful. Thus $q \equiv 1 \pmod p$ contrary to (13.11). Therefore $\mathfrak{P}$ is a cyclic group of order p and $\mathfrak{B}\mathfrak{P}$ is a Frobenius group. Hence

$$p \equiv 1 \pmod{|\mathfrak{B}|} . \tag{13.12}$$

Let $\mathfrak{Q}_0$ be a $\mathfrak{B}\mathfrak{P}$ invariant subgroup of $\mathfrak{Q}$ which is minimal subject to $\mathfrak{P} \not\subseteq C_{\mathfrak{A}_0}(\mathfrak{Q}_0)$. Thus the representation of $\mathfrak{B}\mathfrak{P}$ on $\mathfrak{Q}_0/D(\mathfrak{Q}_0)$ is irreducible. Therefore $\mathfrak{Q}_0 \subseteq (\mathfrak{Q}_0\mathfrak{P})'$. Since $O_p(\mathfrak{A})$ is elementary and $C_{\mathfrak{M}}(\mathfrak{B}) \neq 1$ we get that the hypotheses of the lemma are satisfied. Thus the minimal nature of $\mathfrak{A}$ implies that $\mathfrak{A}_0 = \mathfrak{Q}\mathfrak{P}$ and $\mathfrak{Q} = \mathfrak{Q}_0$. Therefore the representation of $\mathfrak{B}\mathfrak{Q}\mathfrak{P}$ on $\mathfrak{M}$ is irreducible. Let $\mathfrak{Q}_1$ be a minimal normal subgroup of $\mathfrak{B}\mathfrak{Q}\mathfrak{P}$ which is not centralized by $\mathfrak{B}$. Thus $\mathfrak{Q}_1 \subseteq \mathfrak{Q}$. Then $\mathfrak{Q}_1' = D(\mathfrak{Q}_1)$ and $\mathfrak{B} \subseteq C(D(\mathfrak{Q}_1))$. Hence $D(\mathfrak{Q}_1)$ is cyclic. Let λ be the character of the representation of $\mathfrak{B}\mathfrak{Q}_1$ on $\mathfrak{M}$. By Lemma 13.4 λ has exactly one irreducible constituent which does not have $(\mathfrak{B}\mathfrak{Q}_1)'$ in its kernel. Let θ be this constituent and let

$$\lambda = \sum_{i=1}^{n} \lambda_i + \theta .$$

Since each λ_i is a character of a group of exponent $q|\mathfrak{B}|$ it follows from (13.11) and (13.12) that each λ_i is absolutely irreducible. Thus $\lambda_i(1) = 1$ for $1 \leq i \leq n$. By Lemma 13.11 θ is absolutely irreducible in the field of p elements. By Lemma 13.4 $\theta(1) \leq 2|\mathfrak{B}| - 1$. Since $|\mathfrak{B}|p$ is odd (B) and (13.12) yield that

$$|\mathfrak{M}| \geq p^p \geq p^{2|\mathfrak{B}|} . \tag{13.13}$$

Thus $n \neq 0$. Let $\theta_{|\mathfrak{Q}_1} = \sum_{j=1}^{m} \nu_j$, where each ν_j is an irreducible character of $\mathfrak{Q}_1$. Thus

$$\lambda = \sum_{i=1}^{n} \lambda_{i|\mathfrak{Q}_1} + \sum_{j=1}^{m} \nu_j . \tag{13.14}$$

Since $\mathfrak{Q}_1 \triangleleft \mathfrak{Q}_1\mathfrak{B}\mathfrak{P}$, $\{\lambda_{i|\mathfrak{Q}_1}, \nu_j\}$ is a set of conjugate characters. Since $n \neq 0$ they are all linear. Thus $\mathfrak{Q}_1' = 1$. Hence $\mathfrak{Q}_1\mathfrak{B} = \mathfrak{Q}_2 \times \mathfrak{Q}_3\mathfrak{B}$, where $\mathfrak{Q}_3\mathfrak{B}$ is a Frobenius group and $|\mathfrak{Q}_2| = q$. Furthermore

$$m = \theta(1) = |\mathfrak{B}| . \tag{13.15}$$

Since $\mathfrak{Q}_3 \subseteq \ker \lambda_i \neq \mathfrak{Q}_1$ for $1 \leqq i \leqq n$ we see that $\lambda_{i_{|\mathfrak{Q}_1}} \neq \nu_j$ for all i, j. Since $\nu_i \neq \nu_j$ for $i \neq j$ we get that no constituent of $\lambda_{|\mathfrak{Q}_1}$ occurs with multiplicity greater than one. Since $\{\lambda_{i_{|\mathfrak{Q}_2}}\}$ is a set of distinct linear characters of $\mathfrak{Q}_2$ we get that $n \leqq q$. Now (13.13), (13.14) and (13.15) yield that

$$p \leqq \lambda(1) = m + n \leqq |\mathfrak{B}| + q .$$

This contradicts (13.11) and (13.12) since $|\mathfrak{B}|\, pq$ is odd. The proof is complete.