AN APPROACH TO SINGULAR HOMOLOGY THEORY

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INTRODUCTION

0.1. Given a topological space X, we associate with X a complex $R = R(X)$ as follows. Let E_∞ denote Hilbert space (that is, the space of all sequences $r_1, \cdots, r_n, \cdots$ of real numbers such that the series $r_1^2 + \cdots + r_n^2 + \cdots$ converges, with the usual definition of distance). For $p \geq 0$, let $v_0, \cdots, v_p$ be a sequence of $p + 1$ points in E_∞, which need not be linearly independent or distinct, and let $|v_0, \cdots, v_p|$ denote the convex hull of these points. Finally, let T be a continuous mapping from $|v_0, \cdots, v_p|$ into X. Then the sequence $v_0, \cdots, v_p$ jointly with T is a p-cell of the complex R, and will be denoted by $(v_0, \cdots, v_p, T)^R$. The group C_p^R of (integral) p-chains in R is defined as the free Abelian group with these p-cells as free generators. For $p < 0$, C_p^R is defined by $C_p^R = 0$ (that is, C_p^R consists then of a zero-element alone). The boundary operator ∂_p^R: $C_p^R \longrightarrow C_{p-1}^R$ is defined by the conventional formula

$$\partial_p^R(v_0, \cdots, v_p, T)^R = \sum_{i=0}^{p} (-1)^i (v_0, \cdots, \hat{v}_i, \cdots, v_p, T)^R$$

for $p \geq 1$. For $p \leq 0$, ∂_p^R is defined as the trivial zero-homomorphism. Clearly $\partial\partial = 0$, and thus $R = R(X)$ is a complex which is obviously closure-finite in the sense of [4]. Accordingly, one can define cycles z_p^R, boundaries b_p^R, and so forth, for R in the usual manner. The homology groups of R are defined by $H_p^R = Z_p^R/B_p^R$, where Z_p^R, B_p^R denote the group of p-cycles and p-boundaries respectively in R.

0.2. The complex R, which was introduced and studied recently by the writer [6], differs from the various singular complexes used in previous literature first in the use of Hilbert space. The general practice is to consider continuous mappings T from rectilinear simplexes located in any Euclidean space. Instead, we

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use Hilbert space in its capacity of infinite-dimensional Euclidean space, a procedure which may of course be adopted in all the various versions of singular homology theory. The main departure from previous practice lies however in the fact that no identifications are made in the chain groups C_p^R of R: two p-cells $(v_0', \cdots, v_p', T')^R$, $(v_0'', \cdots, v_p'', T'')^R$ are equal if and only if they are identical, that is, if $v_0' = v_0'', \cdots, v_p' = v_p'', T' = T''$. Thus the complex R is of enormous size as compared with previously used complexes. Let us note that beyond the lack of identifications, R is further increased by the fact that the points $v_0, \cdots, v_p$ occurring in a p-cell $(v_0, \cdots, v_p, T)^R$ are not required to be linearly independent or distinct.

0.3. There arises the question of how the homology groups of R compare with those arising in previous approaches to singular homology theory. In [6], the writer proved that the homology groups of R are isomorphic to those of the so-called total singular complex $S = S(X)$ introduced by Eilenberg [3]. Since this result will be used in the sequel, we shall now give the precise statement of the main theorem established in [6]. For each dimension $p \geq 0$, let us select a fundamental p-simplex, with (linearly independent) vertices $d_0, \cdots, d_p$. For our own purposes, it is convenient to choose $d_0, d_1, d_2, \cdots$ as the points $(1,0,0,0,\cdots)$, $(0,1,0,0,\cdots)$, $(0,0,1,0,\cdots)$, $\cdots$ in E_∞. Given then a sequence $v_0, \cdots, v_p$ of $p+1$ points in E_∞, which need not be linearly independent or distinct, there exists a unique linear map $\alpha: |d_0, \cdots, d_p| \longrightarrow |v_0, \cdots, v_p|$ such that $\alpha(d_i) = v_i$, $i = 0, \cdots, p$. This linear map is denoted by $[v_0, \cdots, v_p]$. The total singular complex $S = S(X)$ of Eilenberg [3] may now be described as follows. For $p \geq 0$, a p-cell of S is an aggregate $(d_0, \cdots, d_p, T)^S$, where T is a continuous mapping from $|d_0, \cdots, d_p|$ into X. The group C_p^S of (integral) p-chains of S is then the free Abelian group with these p-cells as free generators. For $p < 0$, one sets $C_p^S = 0$. The boundary operator $\partial_p^S: C_p^S \longrightarrow C_{p-1}^S$ is defined by

$$\partial_p^S(d_0, \cdots, d_p, T)^S = \sum_{i=0}^{p} (-1)^i (d_0, \cdots, d_{p-1}, T[d_0, \cdots, \hat{d}_i, \cdots, d_p])^S,$$

for $p \geq 1$. For $p \leq 0$, ∂_p^S is the trivial zero-homomorphism. The homology groups of S will be denoted by H_p^S. We have then obvious homomorphisms

$$\sigma_p: C_p^R \longrightarrow C_p^S, \qquad \tau_p: C_p^S \longrightarrow C_p^R,$$

defined as follows for $p \geq 0$:

$$\tau_p(d_0,\cdots,d_p,T)^S = (d_0,\cdots,d_p,T)^R ,$$

$$\sigma_p(v_0,\cdots,v_p,T)^R = (d_0,\cdots,d_p,T[v_0,\cdots,v_p])^S .$$

For $p < 0$, τ_p and σ_p are defined as the trivial zero-homomorphisms. Unfortunately, τ_p is *not* a chain-mapping. On the other hand, σ_p is easily seen to be a chain-mapping, and hence it induces homomorphisms $\sigma_{*p} : H_p^R \longrightarrow H_p^S$. The main result of [6] is contained in the following statement.

THEOREM. *The homomorphism $\sigma_{*p} : H_p^R \longrightarrow H_p^S$ is an isomorphism onto, for every dimension p.*

Since singular homology theory is sometimes thought of only in relation to triangulable spaces, it may be appropriate to note that the preceding theorem is valid for general topological spaces. In particular, the space need not be arc-wise connected.

0.4 In view of the preceding theorem, the complex R appears as an appropriate tool in constructing singular homology theory. It is of interest to note that the various complexes used in previous approaches to singular homology theory may be derived from the complex R by a combination of the following two types of reduction.

(i) The chain groups C_p^R of R are replaced by certain subgroups Γ_p. For example, one may select Γ_p as the group generated by those p-cells $(v_0,\cdots, v_p, T)^R$ for which the points $v_0,\cdots, v_p$ are linearly independent. Another significant choice may be based upon the concept of a minimal complex studied by Eilenberg and Zilber [3].

(ii) One selects in C_p^R, for each p, a certain subgroup G_p, and one replaces C_p^R by the factor group C_p^R/G_p. From the computational point of view, this amounts to an identification of elements of C_p^R which are contained in the same coset relative to G_p. For brevity, we shall refer to this type of process as an *identification scheme.*

In the present paper, we shall study the effect of the various identification schemes, occurring in previous theories, upon the homology structure of the complex R. It is easy to see that these identification schemes may be reduced to three basic types. Our result is that one may apply these basic identification schemes in any desired combination without changing the homology structure of R (see Theorem 1 in §4.7). As a matter of fact, we obtain an identification scheme which appears stronger than those previously used (see Theorem 2 in 4.7 and see

§5). This leads to some interesting questions, formulated in §6, which seem to deserve further study.

0.5. It should be noted that the complex R is semisimplicial in the sense of [3], and therefore can be used to construct a complete homology and cohomology theory.

0.6. Previous relevant literature, as well as further problems arising in this line of thought, will be discussed in §6 when convenient terminology will be available. The writer wishes to express his appreciation of the courtesy extended by S. Eilenberg and N. Steenrod who made available to him the manuscript of their yet unpublished book [2]. Both technically and conceptually, the study of that book proved most valuable in preparing the present paper.

1. Identifications in Mayer Complexes

1.1. A Mayer complex M is a collection of Abelian groups C_p, where the integer p ranges from $-\infty$ to $+\infty$, together with homomorphisms

$$\partial_p : C_p \longrightarrow C_{p-1} ,$$

such that $\partial_{p-1}\partial_p = 0$. Cycles and boundaries are defined in the usual manner. The homology groups H_p of M are defined by $H_p = Z_p/B_p$, where Z_p, B_p are the groups of p-cycles and p-boundaries respectively. If M, M' are Mayer complexes, then a set of homomorphisms

$$f_p : C_p \longrightarrow C'_p$$

is termed a chain mapping if $\partial'_p f_p = f_{p-1}\partial_p$, where primes refer to the complex M'

For clarity, we shall write C_p^M, ∂_p^M, H_p^M, and so on, to identify the complex under consideration. In particular, a p-chain of M (that is, an element of C_p^M) will be denoted by symbols like c_p^M, d_p^M, and so forth.

1.2. We shall now describe the general process of identification in a Mayer complex M. Let $\{G_p\}$ be a collection of Abelian groups such that $G_p \subset C_p^M$ and

$$\partial_p^M G_p \subset G_{p-1} . \tag{1}$$

Explicitly: if $c_p^M \in G_p$, then $\partial_p^M c_p^M \in G_{p-1}$. Set $C_p^m = C_p^M/G_p$. Thus, the elements of C_p^m are cosets relative to G_p. The general element of C_p^m is of the

form $\{c_p^M\}$, where this symbol denotes the coset containing the element c_p^M of C_p^M. In view of (1), we can then define homomorphisms

$$\partial_p^m : \ C_p^m \longrightarrow C_{p-1}^m$$

by the formula $\partial_p^m \{c_p^M\} = \{\partial_p^M c_p^M\}$. Clearly $\partial_{p-1}^m \partial_p^m = 0$. Accordingly, the system of factor groups $\{C_p^m\}$, jointly with the homomorphisms ∂_p^m, constitutes a Mayer complex m. We shall say that m is obtained by identification, with respect to the system $\{G_p\}$, from M. The system $\{G_p\}$, satisfying (1), will be termed an *identifier* for M. We have then natural homomorphisms

$$\pi_p : \ C_p^M \longrightarrow C_p^m$$

defined by $\pi_p c_p^M = \{c_p^M\}$. Clearly

$$\partial_p^m \pi_p c_p^M = \partial_p^m \{c_p^M\} = \{\partial_p^M c_p^M\} = \pi_{p-1} \partial_p^M c_p^M .$$

Thus π_p is a chain mapping, and hence induces homomorphisms

$$\pi_{*p} : \ H_p^M \longrightarrow H_p^m ,$$

defined as follows. If z_p^M is a cycle in M, then we let $[z_p^M]_M$ denote the homology class containing z_p^M. The symbol $[z_p^m]_m$ is defined similarly. Then π_{*p} is given by

$$\pi_{*p}[z_p^M]_M = [\pi_p z_p^M]_m .$$

If π_{*p} is an isomorphism onto for every p, then we shall say that the identifier $\{G_p\}$ is *unessential*. Thus the process of identification with respect to an unessential identifier does not change the homology structure of the complex.

1.3. We shall state presently a convenient condition for the unessential character of an identifier $\{G_p\}$. Let us observe that the condition (1) in 1.2 means that the homomorphisms ∂_p^M, cut down to the subgroups G_p, may be used to turn the system $\{G_p\}$ into a Mayer complex which we call G. The complex m, defined in 1.2, appears then as merely the complex M mod G in the sense of the general relative homology theory of Mayer complexes. From this general theory, the condition for π_{*p} to be an isomorphism onto, for all p, is well known: it is necessary and sufficient that all the homology groups of G be trivial. For convenient application, we shall now state this condition explicitly.

The condition (U). We shall say that the identifier $\{G_p\}$ satisfies the condition

(U) if the following holds: if z_p^M is a cycle in M such that $z_p^M \in G_p$ then there exists a $(p+1)$-chain $c_{p+1}^M \in G_{p+1}$ such that $\partial_{p+1}^M c_{p+1}^M = z_p^M$.

We have then the following criterion.

CRITERION FOR UNESSENTIAL IDENTIFIERS. *An identifier* $\{G_p\}$ *is unessential if and only if it satisfies condition* (U).

Since the elements of G_p represent those elements of C_p^M which are, in a sense, discarded as we pass from the complex M to the complex m, the criterion may be also worded as follows: discarded cycles should bound discarded chains. In a special case, this criterion was used by Tucker [8]. As mentioned above, the general criterion is merely a re-wording of a well-known theorem in the relative homology theory of Mayer complexes (for a comprehensive presentation, see Eilenberg and Steenrod [2]). For the convenience of the reader, we shall now outline a direct proof of the criterion.

1.4. Assume first that the identifier $\{G_p\}$ is unessential. Take a cycle

$$z_p^M \in G_p \,. \tag{1}$$

Then $\pi_p z_p^M = \{z_p^M\} = 0$, and hence $\pi_{*p}[z_p^M]_M = [\pi_p z_p^M]_m = 0$. Since π_* is an isomorphism onto, it follows that z_p^M bounds in M:

$$z_p^M = \partial_{p+1}^M c_{p+1}^M \,. \tag{2}$$

Application of π_p yields, in view of (1), the equation

$$0 = \pi_p z_p^M = \pi_p \partial_{p+1}^M c_{p+1}^M = \partial_{p+1}^m \pi_{p+1} c_{p+1}^M \,.$$

Thus $\pi_{p+1} c_{p+1}^M$ is a cycle of the complex m. Since π_* is an isomorphism onto, we have therefore a cycle z_{p+1}^M such that $\pi_{p+1} z_{p+1}^M$ differs from the cycle $\pi_{p+1} c_{p+1}^M$ only by a boundary. Thus we can write

$$\pi_{p+1} z_{p+1}^M = \pi_{p+1} c_{p+1}^M + \partial_{p+2}^m c_{p+2}^m \,.$$

Now c_{p+2}^m is of the form $\{c_{p+2}^M\} = \pi_{p+2} c_{p+2}^M$. Making this substitution, we obtain

$$\pi_{p+1} z_{p+1}^M = \pi_{p+1} c_{p+1}^M + \partial_{p+2}^m \pi_{p+2} c_{p+2}^M = \pi_{p+1}(c_{p+1}^M + \partial_{p+2}^M c_{p+2}^M) \,.$$

Hence

$$\pi_{p+1}(c_{p+1}^M - z_{p+1}^M + \partial_{p+2}^M c_{p+2}^M) = 0 \,. \tag{3}$$

Now let us consider the $(p+1)$-chain

$$d^M_{p+1} = c^M_{p+1} - z^M_{p+1} + \partial^M_{p+2} c^M_{p+2} .$$

By (3) we have $d^M_{p+1} \in G_{p+1}$, while from (2) we have $z^M_p = \partial^M_{p+1} d^M_{p+1}$. Thus (1) is seen to imply that z^M_p bounds a chain contained in G_{p+1}. In other words, condition (U) holds.

1.5. Assume now, conversely, that condition (U) holds. We have to show that π_{*p} is an isomorphism onto for every p.

(i) Suppose we have

(1) $$\pi_{*p}[z^M_p]_M = 0$$

for a certain cycle z^M_p. The assumption means that $\pi_p z^M_p$ bounds some chain c^m_{p+1}. Since c^m_{p+1} is of the form $\{c^M_{p+1}\} = \pi_{p+1} c^M_{p+1}$, we have

$$\pi_p z^M_p = \partial^m_{p+1} \pi_{p+1} c^M_{p+1} = \pi_p \partial^M_{p+1} c^M_{p+1} ,$$

and hence

$$\pi_p (z^M_p - \partial^M_{p+1} c^M_{p+1}) = 0 .$$

Thus the cycle

(2) $$Z^M_p = z^M_p - \partial^M_{p+1} c^M_{p+1}$$

is contained in G_p. Since condition (U) is now assumed, it follows that Z^M_p is of the form

(3) $$Z^M_p = \partial^M_{p+1} d^M_{p+1} \qquad (d^M_{p+1} \in G_{p+1}) .$$

From (2) and (3) it follows that

$$z^M_p = \partial^M_{p+1} (c^M_{p+1} + d^M_{p+1}) .$$

Thus (1) implies that z^M_p bounds in M, and hence π_{*p} is an isomorphism into.

(ii) Assign now an element $[z^m_p]_m$ of H^m_p. Now z^m_p is of the form

(1) $$z^m_p = \{c^M_p\} = \pi_p c^M_p .$$

Since z^m_p is a cycle, we have

$$0 = \partial^m_p z^m_p = \partial^m_p \pi_p c^M_p = \pi_{p-1} \partial^M_p c^M_p .$$

Hence

$$\partial_p^M c_p^M \in G_{p-1} .$$

Thus $\partial_p^M c_p^M$ is a cycle contained in G_{p-1}. Since condition (U) is now assumed, we have a chain d_p^M such that

$$\partial_p^M c_p^M = \partial_p^M d_p^M \qquad (d_p^M \in G_p) . \tag{2}$$

Thus $c_p^M - d_p^M$ is a cycle:

$$c_p^M - d_p^M = z_p^M .$$

Now we calculate

$$\pi_{*p}[z_p^M]_M = [\pi_p c_p^M - \pi_p d_p^M]_m .$$

By (1), $\pi_p c_p^M = z_p^m$, and by (2), $\pi_p d_p^M = 0$. Thus finally

$$\pi_{*p}[z_p^M]_M = [z_p^m]_m .$$

Thus π_{*p} is onto, and the proof of the criterion is complete.

1.6. In marked contrast to the general character of the preceding discussion, the unessential identifiers actually employed in the sequel are of a very special and restricted type. There arises the question whether there are general constructions yielding unessential identifiers in Mayer complexes. The following comments may be of interest from this point of view. Let M, L be Mayer complexes and let

$$f_p : C_p^M \longrightarrow C_p^L \tag{1}$$

be a chain-mapping such that the induced homomorphisms $f_{*p} : H_p^M \longrightarrow H_p^L$ are isomorphisms onto. In symbols:

$$f_{*p} : H_p^M \approx H_p^L . \tag{2}$$

Let N_p denote the nucleus of the homomorphism (1). Since f_p is a chain-mapping, it is immediate that the system $\{N_p\}$ is an identifier.

In view of the strong assumption (2) one may be tempted to conjecture that $\{N_p\}$ is unessential. The following simple example shows that this is not the case, even under extremely special and favorable circumstances. Let M be a finite simplicial complex described abstractly as follows. The group C_2^M of (integral)

2-chains of M is the free Abelian group with a single generator t. The 1-chain group C_1^M is the free Abelian group with four generators s_1, s_2, s_3, s_4. The 0-chain group C_0^M is generated by a, b, c, d, e. For $p \neq 0, 1, 2$, the p-chain group C_p^M reduces to a zero-element. The boundary relations are as follows:

$$\partial t = s_1 + s_2 + s_3, \quad \partial s_1 = c - b, \quad \partial s_2 = a - c, \quad \partial s_3 = b - a, \quad \partial s_4 = e - d,$$

$$\partial a = \partial b = \partial c = \partial d = \partial e = 0 .$$

We define first homomorphisms $f_p : C_p^M \longrightarrow C_p^M$ as follows:

$$f_2 t = 0, \qquad f_1 s_1 = f_1 s_2 = f_1 s_3 = 0, \qquad f_1 s_4 = s_1 + s_2 + s_3 ,$$

$$f_0 a = f_0 b = f_0 c = a, \qquad f_0 d = f_0 e = d .$$

For $p \neq 0, 1, 2$, of course f_p is the trivial zero-homomorphism. Next we define homomorphisms $D_p : C_p^M \longrightarrow C_{p+1}^M$ as follows:

$$D_0 a = 0, \qquad D_0 b = -s_3, \qquad D_0 c = s_2, \qquad D_0 d = 0, \qquad D_0 e = -s_4 ,$$

$$D_1 s_1 = -t, \qquad D_1 s_2 = 0, \qquad D_1 s_3 = 0, \qquad D_1 s_4 = t .$$

For $p \neq 0, 1$, of course D_p is the trivial zero-homomorphism. One verifies readily the following facts.

(i) f_p is a chain-mapping.

(ii) $\partial D_p \gamma_p^M + D_{p-1} \partial \gamma_p^M = f_p \gamma_p^M - \gamma_p^M$, for every p-chain γ_p^M of M. Thus $f_p \simeq 1$.

(iii) Let N_p be the nucleus of f_p, and let m be the complex obtained from M by using the identifier $\{N_p\}$, in the sense of 1.2. Then the 1-dimensional homology group H_1^m of m is infinite cyclic.

(iv) The 1-dimensional homology group H_1^M of M is trivial (consists of zero alone).

Thus M and m have different homology structures, and hence $\{N_p\}$ is certainly not unessential. And yet, in view of (i), (ii), the induced homomorphisms $f_{*p} : H_p^M \longrightarrow H_p^M$ are isomorphisms onto. In other words, a very plausible method to obtain unessential identifiers fails even under very special and favorable conditions.

1.7. In dealing with additively written Abelian groups, we shall use certain familiar conventions. Thus we shall write $G = 0$ to state that the Abelian group G

is trivial (consists of a zero-element alone). If $A_1, \cdots, A_n$ are subgroups of G, then $A_1 + \cdots + A_n$ will denote the smallest subgroup containing $A_1, \cdots, A_2$.

2. The Auxiliary Complex K

2.1. The auxiliary complex K, which played an important role in [6] already, is merely the "formal complex," in the sense of [2], of E_∞ taken as a point set. The complex K is defined as follows. For $p \geq 0$, a p-cell of K is a sequence $(v_0, \cdots, v_p)$ of points of E_∞ which are not required to be linearly independent or distinct. Two p-cells $(v_0, \cdots, v_p)$, $(w_0, \cdots, w_p)$ are considered as equal if and only if $v_i = w_i$, $i = 0, \cdots, p$. These p-cells are taken as a base for a free Abelian group, to be denoted by C_p, the group of (finite) p-chains of K. For $p < 0$, one defines $C_p = 0$. For $p \geq 1$, the boundary operator

$$\partial_p : C_p \longrightarrow C_{p-1}$$

is defined by the formula

$$\partial_p(v_0, \cdots, v_p) = \sum_{i=0}^{p} (-1)^i (v_0, \cdots, \hat{v}_i, \cdots, v_p) .$$

Clearly $\partial\partial = 0$. For $p \leq 0$, ∂_p is of course defined as the trivial zero homomorphism.

Let $(v_0, \cdots, v_p)$ be a p-cell of K. Treating the points of E_∞ as vectors in the usual manner, we describe the barycenter $b = b(v_0, \cdots, v_p)$ of the points $v_0, \cdots, v_p$ by the formula

$$b = \frac{v_0 + \cdots + v_p}{p + 1} .$$

2.2. The following homomorphisms will be used.

(i) The homomorphism $\partial_p : C_p \longrightarrow C_{p-1}$, already defined.

(ii) In terms of any assigned point v of E_∞, one defines the *cone homomorphism*

$$h_p^v : C_p \longrightarrow C_{p+1} \qquad (p \geq 0)$$

by the formula

$$h_p^v(v_0, \cdots, v_p) = (-1)^{p+1}(v_0, \cdots, v_p, v) .$$

For $p < 0$, h_p^v is the trivial zero homomorphism.

(iii) The barycentric homomorphism

$$\beta_p : C_p \longrightarrow C_p$$

is defined as follows. For $p < 0$, β_p is the trivial zero homomorphism. For $p = 0$, $\beta_0 = 1$, the identity. For $p > 1$, β_p is defined recursively by the formula

$$\beta_p(v_0, \cdots, v_p) = h_{p-1}^b \beta_{p-1} \partial_p(v_0, \cdots, v_p) ,$$

where b is the barycenter of the points $v_0, \cdots, v_1$.

(iv) The barycentric homotopy operator

$$\rho_p : C_p \longrightarrow C_{p+1}$$

is defined as follows. For $p \leq 0$, $\rho_p(v_0, \cdots, v_p) = 0$. For $p \geq 1$, ρ_p is defined recursively by the formula

$$\rho_p(v_0, \cdots, v_p) = h_p^b(\beta_p - 1 - \rho_{p-1} \partial_p)(v_0, \cdots, v_p) ,$$

where b is the barycenter of the points $(v_0, \cdots, v_p)$.

(v) For $p \geq 1$, $0 \leq j \leq p - 1$, we define the homomorphism.

$$t_{p,j} : C_p \longrightarrow C_p$$

by the formula $t_{p,j}(v_0, \cdots, v_j, v_{j+1}, \cdots, v_p) = (v_0, \cdots, v_{j+1}, v_j, \cdots, v_p)$. The operation $t_{p,j}$ will be referred to as a *transposition*. Thus "transposition" means here a transposition of *adjacent* elements. According to the definition of equality for p-cells (see 2.1), we have $t_{p,j}(v_0, \cdots, v_p) = (v_0, \cdots, v_p)$ if and only if $v_j = v_{j+1}$.

2.3. The following identities hold among these various homomorphisms

(i) $\partial_{p+1} h_p^v + h_{p-1}^v \partial_p = 1$ $\qquad (p \geq 1)$,

(ii) $\partial_p \beta_p = \beta_{p-1} \partial_p$,

(iii) $\partial_{p+1} \rho_p + \rho_{p-1} \partial_p = \beta_p - 1$,

(iv) $\beta_p t_{p,j} = -\beta_p$ $\qquad (p \geq 1, \quad 0 \leq j \leq p - 1)$.

2.4. If $(v_0, \cdots, v_p)$ is a p-cell of K, then $|v_0, \cdots, v_p|$ denotes the convex hull of the points $v_0, \cdots, v_p$ (that is, the smallest convex set containing these

points). If c_p is a p-chain of K, and A is a convex set in E_∞, then the inclusion $c_p \subset A$ is defined to mean that c_p can be written in the form

$$c_p = \sum_{j=1}^{n} k_j(v_{0,j}, \cdots, v_{p,j}),$$

where the coefficients k_j are of course integers, so that $|v_{0,j}, \cdots, v_{p,j}| \subset A$, $j = 1, \cdots, n$. One has then the following inclusions:

(i) $\quad \partial_p(v_0, \cdots, v_p) \subset |v_0, \cdots, v_p|,$

(ii) $\quad \beta_p(v_0, \cdots, v_p) \subset |v_0, \cdots, v_p|,$

(iii) $\quad \rho_p(v_0, \cdots, v_p) \subset |v_0, \cdots, v_p|,$

(iv) $\quad t_{p,j}(v_0, \cdots, v_p) \subset |v_0, \cdots, v_p|.$

As a consequence, an inclusion $c_p \subset A$ implies that $\partial_p c_p \subset A$, $\beta_p c_p \subset A$, $\rho_p c_p \subset A$, $t_{p,j} c_p \subset A$. It is understood that the zero chain $c_p = 0$ is agreed to satisfy the inclusion $c_p \subset A$ for every convex set A.

2.5. For $p \geq 1$, an *elementary t-chain* in K is defined as a p-chain c_p which can be written in the form (see 2.2 (v))

$$c_p = (v_0, \cdots, v_p) + t_{p,j}(v_0, \cdots, v_p).$$

LEMMA. *Given an elementary t-chain*

$$c_p = (v_0, \cdots, v_p) + t_{p,j}(v_0, \cdots, v_p) \qquad (p \geq 1),$$

the following statements hold:

(i) *If* $p = 1$, *then* $\partial_p c_p = 0$. *If* $p > 1$, *then* $\partial_p c_p$ *is a linear combination (with integral coefficients) of elementary t-chains* $\subset |v_0, \cdots, v_p|$;

(ii) $\beta_p c_p = 0$;

(iii) $\rho_p c_p$ *is a linear combination (with integral coefficients) of elementary t-chains* $\subset |v_0, \cdots, v_p|$.

Proof. The assertion (ii) is an immediate consequence of 2.4 (iv). The assertions (i) and (iii) are readily verified for $p = 1$. Hence we can assume that $p > 1$.

Proof of (i) *for* $p > 1$. Let us note that $t_{p,j}(v_0, \cdots, v_p)$ is of the form $(w_0, \cdots, w_p)$, where $v_i = w_i$ for $i \neq j, j+1$, and $v_j = w_{j+1}$, $v_{j+1} = w_j$. Now we have

$$\partial_p c_p = \sum_{i=0}^{p} (-1)^i [(v_0, \cdots, \hat{v}_i, \cdots, v_p) + (w_0, \cdots, \hat{w}_i, \cdots, w_p)] .$$

For $i \neq j, j+1$, the quantity in square brackets is clearly an elementary t-chain $\subset |v_0, \cdots, v_p|$. On the other hand, the terms corresponding to $i = j$ and $i = j+1$ cancel. Thus (i) follows.

Proof of (iii) *for* $p > 1$. Since (iii) is verified directly for $p = 1$, we proceed by induction. Assume (iii) to hold for $p - 1$, where $p \geq 2$. Let us write again $t_{p,j} = (w_0, \cdots, w_p)$. Clearly, the points $v_0, \cdots, v_p$ and the points $w_0, \cdots, w_p$ have the same barycenter b. Hence we have (see 2.2 (iv))

$$\rho_p (v_0, \cdots, v_p) = h_p^b [\beta_p (v_0, \cdots, v_p) - (v_0, \cdots, v_p) - \rho_{p-1} \partial_p (v_0, \cdots, v_p)] ,$$

$$\rho_p t_{p,j} (v_0, \cdots, v_p) = h_p^b [\beta_p t_{p,j} (v_0, \cdots, v_p) - t_{p,j} (v_0, \cdots, v_p) - \rho_{p-1} \partial_p t_{p,j} (v_0, \cdots, v_p)] .$$

In view of (ii), addition yields

$$\rho_p c_p = h_p^b (-c_p - \rho_{p-1} \partial_p c_p) . \tag{1}$$

Now, by (i), $\partial_p c_p$ is a linear combination (with integral coefficients) of elementary t-chains $\subset |v_0, \cdots, v_p|$. Hence, by the inductive assumption, the same holds for $\rho_{p-1} \partial_p c_p$, and hence also for the quantity in parentheses in (1), and finally for $\rho_p c_p$ itself, since $b \in |v_0, \cdots, v_p|$.

2.6. For $p \geq 1$, an *elementary d-chain* in K is defined as a p-cell $(v_0, \cdots, v_p)$ such that $v_j = v_{j+1}$ for some j.

LEMMA. *If* $c_p = (v_0, \cdots, v_p)$ *is an elementary d-chain, then the following statements hold.*

(i) *If* $p = 1$, *then* $\partial_p c_p = 0$. *If* $p > 1$, *then* $\partial_p c_p$ *is a linear combination (with integral coefficients) of elementary d-chains* $\subset |v_0, \cdots, v_p|$.

(ii) $\beta_p c_p = 0$.

(iii) $\rho_p c_p$ *is a linear combination (with integral coefficients) of elementary d-chains* $\subset |v_0, \cdots, v_p|$.

The proof is entirely analogous to that in 2.5, except that (ii) requires an additional remark. We have $v_j = v_{j+1}$ for some j by assumption. For this same j,

we have then the relation

$$t_{p,j}(v_0,\cdots,v_p) = (v_0,\cdots,v_p)\,.$$

Hence we have also

$$\beta_p\, t_{p,j}(v_0,\cdots,v_p) = \beta_p(v_0,\cdots,v_p)\,.$$

On the other hand, 2.3 (iv) yields

$$\beta_p\, t_{p,j}(v_0,\cdots,v_p) = -\beta_p(v_0,\cdots,v_p)\,.$$

Hence $2\beta_p(v_0,\cdots,v_p) = 0$. Since $\beta_p(v_0,\cdots,v_p)$ is an element of the free Abelian group C_p, it follows that $\beta_p(v_0,\cdots,v_p) = 0$.

3. The Complex $R = R(X)$

3.1. In working with the complex R (see 0.1), the following device (introduced by Eilenberg and Steenrod in [2] in connection with the complex S; see 0.3) is useful. Let A be a convex subset of E^∞, and let C_p^A denote the subgroup of C_p (see 2.1) generated by those p-cells $(v_0,\cdots,v_p)$ of the complex K which satisfy the inclusion $(v_0,\cdots,v_p) \subset A$ (see 2.4). For $p < 0$, we define $C_p^A = 0$ (see 1.7). Let $T : A \longrightarrow X$ be a continuous mapping. We can define then homomorphisms

$$T_p : C_p^A \longrightarrow C_p^R$$

by the formula

$$T_p(v_0,\cdots,v_p) = (v_0,\cdots,v_p,T)^R \qquad (p \geq 0,\ (v_0,\cdots,v_p) \subset C_p^A)\,.$$

For $p < 0$, T_p is the trivial zero-homomorphism. For $c_p \in C_p^A$, it will be convenient to use the symbol $(c_p,T)^R$ to denote $T_p c_p$. Among the simple and obvious rules of computation for the symbol $(c_p,T)^R$, we mention the formula

$$\partial_p^R(c_p,T)^R = (\partial_p c_p,T)^R\,.$$

In terms of the preceding notations, we define now homomorphisms

$$\beta_p^R : C_p^R \longrightarrow C_p^R\,,$$

$$\rho_p^R : C_p^R \longrightarrow C_{p+1}^R$$

by the formulas

$$\beta_p^R(v_0,\cdots,v_p,T)^R = (\beta_p(v_0,\cdots,v_p),T)^R\,, \qquad (p \geq 0)\,,$$

$$\rho_p^R(v_0,\cdots,v_p,T)^R = (\rho_p(v_0,\cdots,v_p),T)^R \qquad (p \geq 0)\,.$$

Since $\beta_p(v_0,\cdots,v_p) \subset |v_0,\cdots,v_p|$, $\rho_p(v_0,\cdots,v_p) \subset |v_0,\cdots,v_p|$ by 2.4, the homomorphisms β_p^R, ρ_p^R are well defined. For $p < 0$, β_p^R and ρ_p^R are defined as the trivial zero homomorphisms. In terms of the homomorphisms $t_{p,j}$ defined in 2.2, we define

$$t_{p,j}^R : C_p^R \longrightarrow C_p^R\,, \qquad (p \geq 1,\quad 0 \leq j \leq p-1)\,,$$

by means of the formula

$$t_{p,j}^R(v_0,\cdots,v_p,T)^R = (t_{p,j}(v_0,\cdots,v_p),T)^R\,.$$

We have then the following identities (see [6]):

(1) $$\partial_p^R \beta_p^R = \beta_{p-1}^R \partial_p^R\,;$$

(2) $$\partial_{p+1}^R \rho_p^R + \rho_{p-1}^R \partial_p^R = \beta_p^R - 1\,,$$

where 1 denotes the identity transformation in C_p^R; furthermore (see 0.3)

(3) $$\beta_p^R\, t_{p,j}^R = -\beta_p^R \qquad (0 \leq j < p)\,;$$

(4) $$\sigma_p \tau_p = 1\,;$$

(5) $$\partial_p^S = \sigma_{p-1} \partial_p^R \tau_p\,;$$

(6) $$\sigma_{p-1}\partial_p^R\tau_p\sigma_p = \sigma_{p-1}\partial_p^R\,;$$

(7) $$\sigma_p \beta_p^R \tau_p \sigma_p = \sigma_p \beta_p^R\,;$$

(8) $$\sigma_{p+1} \rho_p^R \tau_p \sigma_p = \sigma_{p+1} \rho_p^R\,;$$

(9) $$\tau_{p-1} \sigma_{p-1} \partial_p^R \beta_p^R = \partial_p^R \tau_p \sigma_p \beta_p^R\,.$$

3.2. For $p \geq 1$, we define an *elementary t-chain* in R as a chain of the form $(v_0,\cdots,v_p,T)^R + (t_{p,j}(v_0,\cdots,v_p),T)^R$ (see 2.2). The subgroup of C_p^R generated by the elementary t-chains will be denoted by T_p^R. For $p \leq 0$, we define $T_p^R = 0$.

LEMMA. *If* $c_p^R \in T_p^R$, *then*

(i) $$\partial_p^R c_p^R \in T_{p-1}^R ,$$

(ii) $$\beta_p^R c_p^R = 0 ,$$

(iii) $$\rho_p^R c_p^R \in T_{p+1}^R .$$

Proof. Clearly, it is sufficient to consider the case where c_p^R is an elementary t-chain:

$$c_p^R = (v_0, \cdots, v_p, T)^R + (t_{p,j}(v_0, \cdots, v_p), T)^R$$
$$= ((v_0, \cdots, v_p) + t_{p,j}(v_0, \cdots, v_p), T)^R .$$

Then we have

$$\partial_p^R c_p^R = (\partial_p[(v_0, \cdots, v_p) + t_{p,j}(v_0, \cdots, v_p)], T)^R .$$

By 2.5 (i), $\partial_p [(v_0, \cdots, v_p) + t_{p,j}(v_0, \cdots, v_p)]$ is either zero or else a linear combination, with integral coefficients, of $(p-1)$-chains of the form $(w_0, \cdots, w_{p-1}) + t_{p-1,n}(w_0, \cdots, w_{p-1})$, all $\subset |v_0, \cdots, v_p|$, and thus (i) is obvious. In a similar manner, (ii) and (iii) follow from 2.5 (ii) and 2.5 (iii).

3.3. For $p \geq 1$, we define an *elementary d-chain* in R as a p-cell $(v_0, \cdots, v_p, T)^R$ such that $v_j = v_{j+1}$ for some j, $0 \leq j \leq p-1$. The subgroup of C_p^R generated by the elementary d-chains is denoted by D_p^R. For $p \leq 0$, we define $D_p^R = 0$.

LEMMA. *If* $c_p^R \in D_p^R$, *then*

(i) $$\partial_p^R c_p^R \in D_{p-1}^R ,$$

(ii) $$\beta_p^R c_p^R = 0 ,$$

(iii) $$\rho_p^R c_p^R \in D_{p+1}^R .$$

These statements are immediate consequences of 2.6 (i), 2.6 (ii), 2.6 (iii).

3.4 Given a p-cell $(v_0, \cdots, v_p, T)^R$, take a sequence $w_0, \cdots, w_p$ of $p+1$ linearly independent points in E_∞. Then we have a linear mapping $\alpha: |w_0, \cdots, w_p| \longrightarrow |v_0, \cdots, v_p|$ such that $\alpha(w_i) = v_i$, $i = 0, \cdots, p$. Then the p-chain

(1) $$c_p^R = (v_0, \cdots, v_p, T)^R - (w_0, \cdots, w_p, T\alpha)^R$$

will be termed an *elementary a-chain.* The subgroup of C_p^R generated by the elementary a-chains will be denoted by A_p^R. For $p < 0$, we define $A_p^R = 0$.

LEMMA. $c_p^R \in A_p^R$ *if and only if* $\sigma_p c_p^R = 0$ (see 0.3).

Proof. Assume $c_p^R \in A_p^R$. Then c_p^R is a linear combination of chains of the form (1), and hence it is sufficient to show that $\sigma_p c_p^R = 0$ for the chain (1). Now we have (see 0.3)

$$\sigma_p c_p^R = (d_0, \cdots, d_p, T[v_0, \cdots, v_p])^S - (d_0, \cdots, d_p, T\alpha[w_0, \cdots, w_p])^S .$$

Clearly $[v_0, \cdots, v_p] = \alpha\,[w_0, \cdots, w_p]$, and thus $\sigma_p c_p^R = 0$.

Assume next that $\sigma_p c_p^R = 0$. Then we also have $\tau_p \sigma_p c_p^R = 0$. The chain c_p^R can be written as a (finite) sum

$$c_p^R = \sum_j n_j (v_{0,j}, \cdots, v_{p,j}, T_j)^R , \tag{2}$$

where the coefficients n_j are integers. We have then

$$0 = \tau_p \sigma_p c_p^R = \sum_j n_j (d_0, \cdots, d_p, T_j[v_{0,j}, \cdots, v_{p,j}])^R . \tag{3}$$

Subtracting (3) from (2), we see that c_p^R appears as a linear combination of elementary a-chains, and thus $c_p^R \in A_p^R$. If $p < 0$, then the lemma is of course obvious.

3.5. LEMMA. *If* $c_p^R \in A_p^R$, *then*

(i) $$\partial_p^R c_p^R \in A_{p-1}^R ,$$

(ii) $$\beta_p^R c_p^R \in A_p^R ,$$

(iii) $$\rho_p^R c_p^R \in A_{p+1}^R .$$

These statements are immediate consequences of the identities (6), (7), (8) in 3.1, in connection with the lemma in 3.4. For example, to prove (iii), we note that by (8) in 3.1, we have

$$\sigma_{p+1} \rho_p^R c_p^R = \sigma_{p+1} \rho_p^R \tau_p \sigma_p c_p^R = 0 , \tag{1}$$

since $c_p^R \in A_p^R$, and hence $\sigma_p c_p^R = 0$ by 3.4. Also by 3.4, the relation (1) implies

that $\rho_p^R c_p^R \in A_{p+1}^R$.

3.6. Let us observe that the chain groups C_p^R, C_p^S are free Abelian groups by their very definition (see 0.3) and hence they do not contain elements of finite order.

4. Unessential Identifications in $R = R(X)$

4.1. Lemma. *Let $\{G_p\}$ be an identifier for R* (see 1.2, 0.1) *such that the following conditions hold*:

(i) $c_p^R \in G_p$ *implies that* $\beta_p^R c_p^R = 0$;

(ii) $c_p^R \in G_p$ *implies that* $\rho_p^R c_p^R \in G_{p+1}$.

Then $\{G_p\}$ is unessential (see 1.2).

Proof. We shall verify that $\{G_p\}$ satisfies condition (U) of 1.3. Take a cycle $z_p^R \in G_p$. In view of (i) and (ii), the homotopy identity

$$\partial_{p+1}^R \rho_p^R z_p^R + \rho_{p-1}^R \partial_p^R z_p^R = \beta_p^R z_p^R - z_p^R \tag{1}$$

yields the relation

$$z_p^R = -\partial_{p+1}^R \rho_p^R z_p^R .$$

Thus z_p^R is the boundary of the (p + 1)-chain $\rho_p^R z_p^R \in G_{p+1}$, and condition (U) is established. By the criterion in 1.3, it follows that $\{G_p\}$ is unessential.

4.2. Lemma. *Let $\{G_p\}$ be an identifier for R, such that the following conditions hold*:

(i) $G_p \supset A_p^R$ (see 3.4);

(ii) $c_p^R \in G_p$ *implies that* $\sigma_p \beta_p^R c_p^R = 0$ (see 0.3);

(iii) $c_p^R \in G_p$ *implies that* $\rho_p^R c_p^R \in G_{p+1}$.

Then $\{G_p\}$ is unessential.

Proof. Again, we verify that $\{G_p\}$ satisfies condition (U). Let us take a cycle $z_p^R \in G_p$; we have to show that it is the boundary of some chain in G_{p+1}. We note that

(1) $$\zeta_p^R = \beta_p^R z_p^R$$

is a cycle, and that by (ii) we have

$$\sigma_p \zeta_p^R = \sigma_p \beta_p^R z_p^R = 0 ,$$

since $z_p^R \in G_p$. Since $\sigma_{*p} : H_p^R \longrightarrow H_p^S$ is an isomorphism onto (see 0.3), it follows that ζ_p^R bounds:

(2) $$\zeta_p^R = \partial_{p+1}^R \gamma_{p+1}^R .$$

Applying σ_p on the left, we get (see 0.3)

$$0 = \sigma_p \zeta_p^R = \sigma_p \partial_{p+1}^R \gamma_{p+1}^R = \partial_{p+1}^S \sigma_{p+1} \gamma_{p+1}^R .$$

Thus $\sigma_{p+1} \gamma_{p+1}^R$ is a cycle:

(3) $$\sigma_{p+1} \gamma_{p+1}^R = z_{p+1}^S .$$

Since σ_* is an isomorphism onto (see 0.3), there exists a cycle z_{p+1}^R such that z_{p+1}^S and $\sigma_{p+1} z_{p+1}^R$ differ only in a boundary:

(4) $$z_{p+1}^S = \sigma_{p+1} z_{p+1}^R + \partial_{p+2}^S c_{p+2}^S .$$

Since $\partial_{p+2}^S = \sigma_{p+1} \partial_{p+2}^R \tau_{p+2}$, the relations (3) and (4) yield

(5) $$\sigma_{p+1}(\gamma_{p+1}^R - z_{p+1}^R - \partial_{p+2}^R \tau_{p+2} c_{p+2}^S) = 0 .$$

On setting

(6) $$d_{p+1}^R = \gamma_{p+1}^R - z_{p+1}^R - \partial_{p+2}^R \tau_{p+2} c_{p+2}^S ,$$

we see that the relations (5), (1), (2), (6) yield

(7) $$\sigma_{p+1} d_{p+1}^R = 0 ,$$

(8) $$\beta_p^R z_p^R = \partial_{p+1}^R d_{p+1}^R .$$

From the homotopy identity 4.1 (1) and from (8) we infer now that

(9) $$z_p^R = \partial_{p+1}^R (d_{p+1}^R - \rho_p^R z_p^R) .$$

By (7), (i), and 3.4, we have $d_{p+1}^R \in G_{p+1}$. Since $\rho_p^R z_p^R \in G_{p+1}$ by (iii), it follows from (9) that z_p^R is the boundary of a chain in G_{p+1}, and the proof of the

lemma is complete.

4.3. LEMMA. *Let* $\{G_p\}$ *be an identifier for* R *which satisfies the assumptions of the lemma in* 4.1. *For each* p, *let* $\hat{G}_p$ *denote the division-hull of* G_p. *Then* $\{\hat{G}_p\}$ *is again an identifier* (see 1.2) *which satisfies the assumptions of Lemma* 4.1.

Proof. Take a chain $c_p^R \in \hat{G}_p$. Then there exists an integer $n \neq 0$, such that $nc_p^R \in G_p$ and hence (since $\{G_p\}$ satisfies the assumptions of Lemma 4.1)

$$n\beta_p^R c_p^R = 0 , \tag{1}$$

$$n\rho_p^R c_p^R \in G_{p+1} . \tag{2}$$

By the definition of $\hat{G}_{p+1}$, (2) implies that $\rho_p^R c_p^R \in \hat{G}_{p+1}$. Since $\beta_p^R c_p^R$ is an element of the free Abelian group C_p^R (see 3.6), (1) implies that $\beta_p^R c_p^R = 0$.

4.4. LEMMA. *Let* $\{G_p\}$ *be an identifier for* R *which satisfies the assumptions of Lemma* 4.2. *Then* $\{\hat{G}_p\}$ *is again an identifier which satisfies the assumptions of the same lemma.*

The proof is the same as in 4.3, except that one uses now the fact that $\sigma_p\beta_p^R c_p^R$ is an element of the free Abelian group C_p^S (see 3.6).

4.5. LEMMA. *Let* $\{G_p^{(1)}\}, \cdots, \{G_p^{(n)}\}$ *be identifiers for* R, *satisfying the assumptions of Lemma* 4.1. *Then* $\{G_p^{(1)} + \cdots + G_p^{(n)}\}$ *is again an identifier which satisfies the assumptions of Lemma* 4.1.

The proof is obvious.

4.6. LEMMA. *Let* Ω' *be a collection* (*perhaps empty*) *of identifiers for* R, *each of which satisfies the assumptions of Lemma* 4.1. *Let* Ω'' *be a nonempty collection of identifiers for* R, *each of which satisfies the assumptions of Lemma* 4.2. *For each* p, *let* G_p *denote the smallest subgroup of* C_p^R *containing the groups, with the same subscript* p, *of the identifiers contained in* Ω' *and* Ω''. *Then* $\{G_p\}$ *is an identifier satisfying the assumptions of Lemma* 4.2.

The proof is obvious.

4.7. The preceding lemmas, combined with the results of §3, yield a number of unessential identifiers for R. In the following two theorems, the symbols A_p^R, D_p^R, T_p^R have the meanings explained in §3.

THEOREM 1. *Each one of the systems* $\{A_p^R\}$, $\{D_p^R\}$, $\{T_p^R\}$, $\{A_p^R + D_p^R\}$, $\{A_p^R + T_p^R\}$, $\{D_p^R + T_p^R\}$, $\{A_p^R + D_p^R + T_p^R\}$ *is an unessential identifier for* R (see 1.2).

THEOREM 2. *If* $\hat{\Gamma}_p^R$ *denotes the division-hull of the group* $\Gamma_p^R = A_p^R + D_p^R + T_p^R$, *then* $\{\hat{\Gamma}_p^R\}$ *is an unessential identifier for* R.

Proof. By 3.5 and 3.4, the system $\{A_p^R\}$ is an identifier satisfying the assumptions of Lemma 4.2. Similarly, the systems $\{D_p^R\}$, $\{T_p^R\}$ are identifiers satisfying the assumptions of Lemma 4.1, by 3.2 and 3.3 respectively. By 4.5 it follows then that $\{D_p^R + T_p^R\}$ is an identifier satisfying the assumptions of Lemma 4.1. Similarly, by 4.6 it follows that $\{A_p^R + D_p^R + T_p^R\}$, $\{A_p^R + D_p^R\}$, $\{A_p^R + T_p^R\}$ are identifiers satisfying the assumption of Lemma 4.2. Finally, $\{\hat{\Gamma}_p^R\}$ is an identifier satisfying the assumptions of Lemma 4.2, as a consequence of 4.4. The unessential character of all these identifiers is then a direct consequence of 4.1 and 4.2 respectively.

REMARK. The writer was unable to determine whether or not Γ_p^R coincides with $\hat{\Gamma}_p^R$.

5. THE COMPLEX $r = r(X)$

5.1. Theorem 1 in 4.7 shows that any combination of the basic identification schemes, used in previous approaches to singular homology theory, may be applied to the singular complex R without affecting its homology structure. From the point of view of achieving maximum reduction, the identifier $\{\hat{\Gamma}_p^R\}$ is of special interest. We shall therefore go into some detail concerning this particular identifier. By the general remarks made in §1, this identifier leads from the singular complex R to a new and much smaller Mayer complex which we shall denote by $r = r(X)$. Since $\{\hat{\Gamma}_p^R\}$ is unessential, r has the same homology structure as R. We want to examine in some detail the computational facilities and conveniences available in the complex r.

5.2. By the general remarks in §1, the elements of the p-chain group C_p^r of r are of the form $\{c_p^R\}$, where this symbol denotes the coset (relative to $\hat{\Gamma}_p^R$) containing the p-chain c_p^R of R. Let us adopt, in dealing with the complex r, the usual practice of writing c_p^R instead of $\{c_p^R\}$, with the understanding that c_p^R is now considered as a *representative* of the element $\{c_p^R\}$ of C_p^r. For clarity, we shall use the congruence symbol $\equiv$ in writing equations, to remind ourselves of the

fact that we are dealing actually with congruences mod $\hat{\Gamma}_p^R$. We shall presently note some of the computational rules for the complex r.

5.3. Let $(v_0', \cdots, v_p'; T')^R$, $(v_0'', \cdots, v_p'', T'')^R$ be two p-cells of R related as follows. There exists a system of linearly independent points $w_0, \cdots, w_p$ in E_∞ and two linear maps $\alpha'\colon |w_0, \cdots, w_p| \longrightarrow |v_0', \cdots, v_p'|$, $\alpha''\colon |w_0, \cdots, w_p| \longrightarrow |v_0'', \cdots, v_p''|$, such that the following relations hold:

(i) $\qquad \alpha'(w_i) = v_i' , \qquad \alpha''(w_i) = v_i'' \qquad (i = 0, \cdots, p)$,

(ii) $\qquad T'\alpha' = T''\alpha''$.

Then $(v_0', \cdots, v_p', T')^R \equiv (v_0'', \cdots, v_p'', T'')^R$. Indeed, by the definition of A_p^R and $\hat{\Gamma}_p^R$ (see 3.4, 4.7), we have

$$(v_0', \cdots, v_p', T')^R - (w_0, \cdots, w_p, T'\alpha')^R \in A_p^R \subset \hat{\Gamma}_p^R ,$$

and hence

$$(v_0', \cdots, v_p', T')^R \equiv (w_0, \cdots, w_p, T'\alpha')^R .$$

Similarly

$$(v_0'', \cdots, v_p'', T'')^R \equiv (w_0, \cdots, w_p, T''\alpha'')^R .$$

Since $T'\alpha' = T''\alpha''$, the assertion follows.

5.4. Given a sequence $v_0, \cdots, v_p$ of $p + 1$ points in E_∞ (which need not be linearly independent or distinct), by a *transposition* we shall mean (as in §2) the operation of exchanging two *adjacent* elements of the sequence $v_0, \cdots, v_p$. Let then $(v_0', \cdots, v_p', T')^R$, $(v_0'', \cdots, v_p'', T'')^R$ be two p-cells related as follows:

(i) $|v_0', \cdots, v_p'| = |v_0'', \cdots, v_p''|$, and $T' = T''$;

(ii) there exists a sequence of $n \geq 0$ transpositions leading from $(v_0', \cdots, v_p')$ to $(v_0'', \cdots, v_p'')$.

Then $(v_0', \cdots, v_p', T')^R \equiv (v_0'', \cdots, v_p'', T'')^R$ if n is even, and $(v_0', \cdots, v_p', T')^R \equiv -(v_0'', \cdots, v_p'', T'')^R$ if n is odd. Indeed, the assertion is obvious if $n = 0$. If $n = 1$, the assertion follows immediately from the fact that $T_p^R \subset \hat{\Gamma}_p^R$ (see 3.2, 4.7). Repeated application of this remark yields the desired result for a general n.

5.5. Let $(v_0, \cdots, v_p, T)^R$ be a p-cell such that the points $v_0, \cdots, v_p$ are not all distinct. Then $(v_0, \cdots, v_p, T)^R \equiv 0$. Indeed, by a certain number n of transpositions we can obtain a p-cell $(w_0, \cdots, w_p, T)^R$ in which two adjacent points w_j, w_{j+1} coincide. Then (see 3.3, 4.7)

$$(w_0, \cdots, w_p, T)^R \in D_p^R \subset \hat{\Gamma}_p^R ,$$

and hence

$$(w_0, \cdots, w_p, T)^R \equiv 0 .$$

On the other hand, by 5.4,

$$(w_0, \cdots, w_p, T)^R \equiv \pm (v_0, \cdots, v_p, T)^R ,$$

and the assertion follows.

5.6. Let $(v_0, \cdots, v_p, T)^R$ be a p-cell of R. Let $w_0, \cdots, w_q$, where $q > p$, be a system of linearly independent points in E_∞, and let $\alpha: |w_0, \cdots, w_q| \longrightarrow |v_0, \cdots, v_p|$ be a linear map such that the points $\alpha(w_0), \cdots, \alpha(w_q)$ coincide with the points $v_0, \cdots, v_p$ in any order and with any number of repetitions. Then

$$(w_0, \cdots, w_q, T\alpha)^R \equiv 0 .$$

Indeed, by 5.3 we have the relation

$$(w_0, \cdots, w_q, T\alpha)^R \equiv (\alpha(w_0), \cdots, \alpha(w_q), T)^R .$$

On the other hand, since $q > p$, the points $\alpha(w_0), \cdots, \alpha(w_q)$ are not all distinct. Hence, by 5.5, we have

$$(\alpha(w_0), \cdots, \alpha(w_q), T)^R \equiv 0 ,$$

and the assertion follows.

5.7. Let $(v_0, \cdots, v_p, T)^R$ be a p-cell of R, such that the points $v_0, \cdots, v_p$ are linearly independent. Suppose this p-cell possesses the following type of symmetry. There exists a linear map $\alpha: |v_0, \cdots, v_p| \longrightarrow |v_0, \cdots, v_p|$, such that (i) the points $\alpha(v_0), \cdots, \alpha(v_p)$ form an odd permutation of the points $v_0, \cdots, v_p$ (taken in the indicated order) and (ii) $T\alpha = T$. Then $(v_0, \cdots, v_p, T)^R \equiv 0$. Indeed by 5.4 and 5.3 we have

$$(v_0,\cdots,v_p',T)^R \equiv -(\alpha(v_0),\cdots,\alpha(v_p),T)^R,$$

$$(v_0,\cdots,v_p,T\alpha)^R \equiv (\alpha(v_0),\cdots,\alpha(v_p),T)^R .$$

Since $T = T\alpha$, it follows that $2(v_0,\cdots,v_p,T)^R \equiv 0$, or equivalently

$$2(v_0,\cdots,v_p,T)^R \in \hat{\Gamma}_p^R .$$

Now since $\hat{\Gamma}_p^R$ is the division-hull of Γ_p^R (see 4.7), the last relation implies the existence of an integer $k \neq 0$ such that $2k(v_0,\cdots,v_p,T)^R \in \Gamma_p^R$, and hence (by the definition of the division-hull) $(v_0,\cdots,v_p,T)^R \in \hat{\Gamma}_p^R$. Thus $(v_0,\cdots,v_p,T)^R \equiv 0$.

5.8. The argument just used yields obviously the general result: if $nc_p^R \equiv 0$, where n is an integer $\neq 0$, then $c_p^R \equiv 0$. In other words, *the p-chain group C_p^r of the complex r has no elements of finite order.* Of course, this is *a priori* obvious from the remark that a division-hull is closed under division. It may be of interest to determine whether or not C_p^r is in fact a free Abelian group. The writer was unable to answer this question.

5.9. *The homomorphisms ∂_p^R, β_p^R, ρ_p^R, $\tau_p\sigma_p$ apply to congruences.* In detail: if $c_p^R \equiv \bar{c}_p^R$, then

$$\partial_p^R c_p^R \equiv \partial_p^R \bar{c}_p^R, \quad \beta_p^R c_p^R \equiv \beta_p^R \bar{c}_p^R, \quad \rho_p^R c_p^R \equiv \rho_p^R \bar{c}_p^R, \quad \tau_p\sigma_p c_p^R \equiv \tau_p\sigma_p \bar{c}_p^R .$$

The first one of these asserted congruences is of course merely a restatement of the fact that $\{\hat{\Gamma}_p^R\}$ is an identifier. The last one may be verified as follows. In view of the identity 3.1 (4) we have

$$\sigma_p(\tau_p\sigma_p c_p^R - c_p^R) = \sigma_p c_p^R - \sigma_p c_p^R = 0 ,$$

and hence, by 3.4,

$$\tau_p\sigma_p c_p^R - c_p^R \in A_p^R .$$

Since $A_p^R \subset \hat{\Gamma}_p^R$, it follows that

$$\tau_p\sigma_p c_p^R \equiv c_p^R . \tag{1}$$

Similarly, $\tau_p\sigma_p \bar{c}_p^R \equiv \bar{c}_p^R$. Since $c_p^R \equiv \bar{c}_p^R$, it follows that $\tau_p\sigma_p c_p^R \equiv \tau_p\sigma_p \bar{c}_p^R$. Now let us recall that $\{\hat{\Gamma}_p^R\}$ satisfies the assumptions of Lemma 4.2, as we observed in the course of the proof in 4.7. Accordingly, the assumption $c_p^R \equiv \bar{c}_p^R$, which is equivalent to $c_p^R - \bar{c}_p^R \in \hat{\Gamma}_p^R$, implies that

$$\sigma_p \beta_p^R (c_p^R - \bar{c}_p^R) = 0 , \tag{2}$$

$$\rho_p^R (c_p^R - \bar{c}_p^R) \in \hat{\Gamma}_{p+1}^R . \tag{3}$$

The relation (3) is equivalent to $\rho_p^R c_p^R \equiv \rho_p^R \bar{c}_p^R$. On the other hand, (2) implies, by 3.4, that $\beta_p^R (c_p^R - \bar{c}_p^R) \in A_p^R \subset \hat{\Gamma}_p^R$, and hence that $\beta_p^R c_p^R \equiv \beta_p^R \bar{c}_p^R$.

5.10. In terms of familiar terminology, the preceding results may be summarized as follows. In the complex r, *affine-equivalent* p-cells of R become equal to each other (see 5.3). The *permutation rule* (or the *orientation convention*) holds in r (see 5.4). *Degenerate p-cells* of R may be discarded in r (see 5.5, 5.6), as well as *affine-symmetric* p-cells (see 5.7). The operators ∂_p^R, β_p^R, ρ_p^R continue to apply in r (see 5.9). Furthermore, the operation $\tau_p \sigma_p$ is also applicable in r (see 5.9). The effect of this operation is to replace a general p-cell $(v_0, \cdots, v_p, T)^R$ by a p-cell of the form $(d_0, \cdots, d_p, T^*)$ (see 0.3). Accordingly, one can avoid entirely the use of p-cells $(v_0, \cdots, v_p, T)^R$ where the points $v_0, \cdots, v_p$ are not linearly independent (it is not obvious, however, that this practice, if followed consistently, contributes to clarity and simplicity of calculations). Finally, let us note that the complex r offers the advantage that *its chain-groups do not have elements of finite order* (see 5.8). In the light of comments made in previous literature, this may represent a desirable feature.

5.11. In the course of a correspondence on these subjects, Professor S. MacLane communicated to the writer a simple and ingenious proof of the fact that the chain-groups of the complex r are indeed free Abelian groups (cf. 5.8).

6. Conclusion

6.1. One may raise the question whether the singular complex R admits of further reductions, in terms of identifications, without affecting its homology structure. In particular, one may ask whether there exists a *maximal identification scheme*, in some natural and appropriate sense. A plausible approach may be obtained by setting up the principle that only those identifications are admitted for which the computational rules set forth in 5.3–5.9 hold. The problem consists then of determining whether among all unessential identifiers $\{G_p\}$, conforming to this principle, there exists one, say $\{\bar{G}_p\}$, such that $G_p \subset \bar{G}_p$ for all identifiers $\{G_p\}$ satisfying the requirements just stated. The writer was unable to settle various interesting questions upon which the answer to this problem seems to depend.

6.2. From a heuristic point of view, one may conjecture that, in view of the intensive study and manifold applications of singular homology theory, it is unlikely that any relevant identification scheme escaped the attention of the many

workers in this field. For example, one may assume, as a heuristic working hypothesis, that by applying simultaneously all the identification schemes used in the papers listed in the References of the present paper one obtains a maximal identification scheme in the sense of 6.1. The writer was unable to find a proof for the theorem suggested by these remarks.

6.3. As regards previous literature concerned with the unessential character of identification schemes, precise comparisons would lead to excessive detail, particularly because our complex R has not been considered explicitly in the literature, as far as the writer is aware. The following comments are meant to indicate the origin of certain questions rather than the exact formulation of definitions occurring in other theories. The initial motivation for the present study, as well as for the previous paper [6] of the writer, came from the important paper of Eilenberg [1]. In that paper, Eilenberg shows, in effect, that (in our terminology) the identifier $\{T_p^R\}$ is unessential (see 3.2). In his previous paper [6], the writer showed then that the identifier $\{A_p^R\}$ is also unessential. However, the unessential character of certain identifications has been recognized by various authors. Thus Seifert-Threlfall [7] and Lefschetz [5] contain remarks suggesting that the "affine symmetric" p-cells may be discarded without affecting the homology structure. Tucker [8] showed, in effect, that the system $\{D_p^R\}$ is unessential, at least in relation to the identifier $\{T_p^R\}$. In a sense, our complex R appears thus as the *singular complex in unreduced form*, alternative theories being derivable by various types of reduction. The problems we stated in 6.1 and 6.2 amount merely to the question whether there is some end to this process of reduction without changing the homology structure.

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