A THEOREM ON RINGS OF OPERATORS

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1. Introduction. The main result (Theorem 1) proved in this paper arose in connection with investigations on the structure of rings of operators. Because of its possible independent interest, it is being published separately.

The proof of Theorem 1 is closely modeled on the discussion in Chapter I of [3]. The connection can be briefly explained as follows. Let N be a factor of type II_1; then in addition to the usual topologies on N, we have the metric defined by $[[A]]^2 = T(A^*A)$, T being the trace on N. Now it is a fact that in any *bounded* subset of N, the $[[\ ]]$-metric coincides with the strong topology—this is the substance of Lemma 1.3.2 of [3]. In the light of this observation, it can be seen that Theorem 1 is essentially a generalization (to arbitrary rings of operators) of the ideas in Chapter I of [3].

Before stating Theorem 1, we collect some definitions for the reader's convenience. Let R be the algebra of all bounded operators on a Hilbert space H (of any dimension). In R we have a natural norm and *-operation. A typical neighborhood of 0 for the strong topology in R is given by specifying $\epsilon > 0, \xi_1, \cdots, \xi_n \in H$, and taking the set of all A in R with $\|A\xi_i\| < \epsilon$; for the weak topology we specify further vectors $\eta_1, \cdots, \eta_n \in H$ and take the set of all A with $|(A\xi_i, \eta_i)| < \epsilon$. By a *-algebra of operators we mean a self-adjoint subalgebra of R, that is, one containing A^* whenever it contains A; unless explicitly stated, it is not assumed to be closed in any particular topology. For convex subsets of R, and in particular for subalgebras, strong and weak closure coincide [2, Th. 5]. An operator A is self-adjoint if $A^* = A$, normal if $AA^* = A^*A$, unitary if $AA^* = A^*A =$ the identity operator I.

2. The main result. We shall establish the following result.

THEOREM 1. *Let M, N be *-algebras of operators on Hilbert space, $M \subset N$, and suppose M is strongly dense in N. Then the unit sphere of M is strongly dense in the unit sphere of N.*

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We shall break up the early part of the proof into a sequence of lemmas. Lemma 1 is well known and is included only for completeness.

LEMMA 1. *In the unit sphere of R, multiplication is strongly continuous, jointly in its variables; and any polynomial in n variables is strongly continuous, jointly in its arguments.*

Proof. It is easy to see that multiplication is strongly continuous separately in its variables, even in all of R. Consequently [1, p. 49] we need only check the continuity of AB at $A = B = 0$. Since $||A|| \leq 1$, this is a consequence of

$$||AB\xi|| \leq ||A|| \ ||B\xi|| \leq ||B\xi|| .$$

Since addition and scalar multiplication are continuous (in all of R), the continuity of polynomials follows.

The precaution taken in the next lemma, in defining the mapping on the pair (A, A^*), is necessary since $A \longrightarrow A^*$ is not strongly continuous.

LEMMA 2. *Let $f(z)$ be a continuous complex-valued function, defined for $|z| \leq 1$. Then the mapping $(A, A^*) \longrightarrow f(A)$ is strongly continuous on the normal operators of the unit sphere of R.*

Proof. We are given a normal operator A_0 with $||A_0|| \leq 1$, a positive ϵ, and vectors ξ_i in H $(i = 1, \cdots, n)$. We have to show that by taking A, A^* to be normal with norm ≤ 1, and in suitable strong neighborhoods of A_0, A_0^*, we can achieve

$$(1) \qquad || [f(A) - f(A_0)] \xi_i || < \epsilon .$$

By the Weierstrass approximation theorem, there exists a polynomial g in two variables such that

$$(2) \qquad |g(z, z^*) - f(z)| < \epsilon/3 ,$$

for $|z| \leq 1$, z^* denoting the conjugate complex of z. By elementary properties of the functional calculus for normal operators, we deduce from (2):

$$(3) \qquad ||g(A, A^*) - f(A)|| < \epsilon/3,$$

$$(4) \qquad ||g(A_0, A_0^*) - f(A_0)|| < \epsilon/3.$$

By Lemma 1, if we take A, A^* in appropriate neighborhoods of A_0, A_0^*, we have

$$(5) \qquad \|[g(A, A^*) - g(A_0, A_0^*)]\xi_i\| < \epsilon/3.$$

By combining (3), (4), and (5) we obtain (1).

The next lemma follows from Lemma 2 as soon as it is admitted that * is strongly continuous on unitary operators. This can, for example, be deduced from two known facts: (a) the strong and weak topologies coincide on the set of unitary operators, and (b) * is weakly continuous.

LEMMA 3. *Let f be a continuous complex-valued function defined on the circumference of the unit circle. Then the mapping $U \longrightarrow f(U)$ is strongly continuous on the set of unitary operators.*

The *Cayley transform* is the mapping $A \longrightarrow (A - i)(A + i)^{-1}$; it is defined for any self-adjoint operator and sends it into a unitary operator.

LEMMA 4. *The Cayley transform is strongly continuous on the set of all self-adjoint operators.*

Proof. We have the identity

$$(6) \quad (A - i)(A + i)^{-1} - (A_0 - i)(A_0 + i)^{-1} = 2i(A + i)^{-1}(A - A_0)(A_0 + i)^{-1}.$$

When A is self-adjoint, we have $\|(A + i)^{-1}\| \leq 1$. In order to make the left side of (6) small on a vector ξ, it therefore suffices to make $A - A_0$ small on the vector $(A_0 + i)^{-1}\xi$.

We shall prove a stronger form of Lemma 5 below (Corollary to Theorem 2).

LEMMA 5. *Let h be a real-valued function defined on the real line, and suppose that h is continuous and vanishes at infinity. Then the mapping $A \longrightarrow h(A)$ is strongly continuous on the set of all self-adjoint operators.*

Proof. Define

$$f(z) = h[-i(z+1)(z-1)^{-1}] \qquad \text{for } |z| = 1,\ z \neq 1,$$
$$= 0 \qquad \text{for } z = 1.$$

Then f is continuous on the circumference of the unit circle. Moreover,

$$h(A) = f[(A - i)(A + i)^{-1}].$$

The mapping $A \longrightarrow h(A)$ is thus the composite of two maps: the Cayley transform, and the mapping on unitary operators given by f. By Lemmas 4 and 3, these latter two maps are strongly continuous. Hence so is $A \longrightarrow h(A)$.

Proof of Theorem 1. There is clearly no loss of generality in assuming M and N to be uniformly closed, for the unit sphere of M is even uniformly dense in the unit sphere of its uniform closure.

Let us write Z for the set of self-adjoint elements in M, and Z_1 for the unit sphere of Z. Let B be a given self-adjoint element in N, $\|B\| \leq 1$. By hypothesis, B is in the strong closure of M. We shall argue in two successive steps that B is actually in the strong closure of Z_1. We begin by remarking that B is in the weak closure of M, since the latter coincides with the strong closure of M. Now * is weakly continuous, and hence so is the mapping $A \longrightarrow (A + A^*)/2$. This mapping leaves B fixed, and sends M onto Z; hence B is in the weak closure of Z. Since Z is convex, this coincides with the strong closure of Z.

Let $h(t)$ be any real-valued function of the real variable t which is continuous and vanishes at infinity, satisfies $|h(t)| \leq 1$ for all t, and satisfies $h(t) = t$ for $|t| \leq 1$. We have that $h(B) = B$. Also h can be meaningfully applied within Z, since we have assumed M to be uniformly closed, and in fact $h(Z) = Z_1$. By Lemma 5, the mapping $A \longrightarrow h(A)$ is strongly continuous on self-adjoint operators. Hence B is in the strong closure of Z_1.

This accomplishes our objective as far as self-adjoint operators are concerned. To make the transition to an arbitrary operator, we adopt the device of passing to a matrix algebra.[1] Let N_2 be the algebra of two-by-two matrices over N. In a natural way, N_2 is again a uniformly closed *-algebra of operators on a suitable Hilbert space (compare §2.4 of [3]). It contains in a natural way M_2, the two-by-two matrix algebra over M. The strong topology on N_2 is simply the Cartesian product of the strong topology for the four replicas of N; thus M_2 is again strongly dense in N_2. Now let C be any operator in N, $\|C\| \leq 1$. We form

$$D = \begin{pmatrix} 0 & C \\ C^* & 0 \end{pmatrix}$$

and we note that $D \in N_2$, $D^* = D$, $\|D\| \leq 1$. Let U be any proposed strong neighborhood of D. By what we have proved above, there exists in U a self-adjoint element F,

[1] I am indebted to P. R. Halmos for this device, which considerably shortened my original proof of Theorem 1.

$$F = \begin{pmatrix} G & H \\ H^* & K \end{pmatrix}$$

with $F \in M_2$, $\|F\| \leq 1$. By suitable choice of U we can make H lie in a given strong neighborhood of C. Also $\|F\| \leq 1$ implies $\|H\| \leq 1$. This proves that C lies in the strong closure of the unit sphere of M, and concludes the proof of Theorem 1.

3. Remarks. (a) Since strong and weak closure coincide for convex sets, we can, in the statement of Theorem 1, replace "strongly" by "weakly" at will.

(b) From Theorem 1 we can deduce that portion of [2, Th. 8] that asserts that a *-algebra of operators is strongly closed if its unit sphere is strongly closed; but it does not appear to be possible to reverse the reasoning.

(c) As Dixmier has remarked [2, p. 399], Theorem 1 fails if M is merely assumed to be a subspace (instead of a *-subalgebra).

4. Another result. In concluding the paper we shall return to Lemma 5 and show that the hypothesis can be weakened to the assumption that h is bounded and continuous. It should be noted that we cannot drop the word "bounded," since for example it is known that the mapping $A \longrightarrow A^2$ is not strongly continuous.

Actually we shall prove a still more general result, which may be regarded as a generalization of Lemma 4.2.1 of [3].

THEOREM 2. *Let $h(t)$ be a bounded real-valued Baire function of the real variable t, and A_0 a self-adjoint operator. Let S be the spectrum of A_0, and T the closure of the set of points at which h is discontinuous; suppose S and T are disjoint. Then the mapping on self-adjoint operators, defined by $A \longrightarrow h(A)$, is continuous at $A = A_0$.*

Proof. We may suppose that

$$|h(t)| \leq 1 \tag{7}$$

for all t. Given $\epsilon > 0$, and vectors ξ_i, we have to show that for A in a suitable strong neighborhood of A_0, we have

$$\|[h(A) - h(A_0)]\xi_i\| < \epsilon. \tag{8}$$

Choose a function $k(t)$ which satisfies: (a) k is continuous and vanishes at infinity, (b) $k(t) = 1$ for t in S, (c) $k(t) = 0$ for t in an open set containing T. Define

$p = hk$, $q = 1 - k + hk$. Then $p = q = h$ on S, and so

$$p(A_0) = q(A_0) = h(A_0). \tag{9}$$

Also p and $q - 1$ are continuous and vanish at infinity. Hence Lemma 5 is applicable, and for a certain strong neighborhood of A_0 we have

$$\|[p(A) - p(A_0)]\xi_i\| < \epsilon/4, \qquad \|[q(A) - q(A_0)]\xi_i\| < \epsilon/2. \tag{10}$$

The following is an identity:

$$h = (1 - h)p + hq. \tag{11}$$

From (9) and (11) we get

$$h(A) - h(A_0) = [1 - h(A)][p(A) - p(A_0)] + h(A)[q(A) - q(A_0)]. \tag{12}$$

From (7), (10), and (12), we deduce (8), as desired.

If in particular h is continuous, then T is void and we get a simplified corollary.

COROLLARY. *Let $h(t)$ be a continuous bounded real-valued function of the real variable t. Then the mapping $A \longrightarrow h(A)$ is strongly continuous on the set of all self-adjoint operators.*

REFERENCES

1. N. Bourbaki, *Éléments de Mathématique*, Livre III, *Topologie Générale*, Actualités Sci. Ind., No. 916, Paris, 1942.

2. J. Dixmier, *Les fonctionelles linéaires sur l'ensemble des opérateurs bornés d'un espace de Hilbert*, Ann. of Math. **51** (1950), 387-408.

3. F. J. Murray and J. von Neumann, *On rings of operators IV*, Ann. of Math. **44** (1943), 716-808.

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