CONVOLUTION TRANSFORMS WITH COMPLEX KERNELS

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1. Introduction. In the present paper we shall consider the inversion of a class of convolution transforms with kernel $G(t)$ of the form

$$G(t) = \frac{1}{2\pi i} \int_{-i\infty}^{i\infty} [E(s)]^{-1} e^{st}\, ds \qquad (-\infty < t < \infty), \tag{1.1}$$

$$E(s) = \prod_{1}^{\infty} \left(1 - \frac{s}{a_k}\right) e^{s/b_k}, \tag{1.2}$$

$a_k = b_k + ic_k$ $(k = 1, 2, \cdots)$ being a sequence of complex numbers such that

$$\sum_{k=1}^{\infty} (1/b_k)^2 < \infty, \qquad \sum_{k=1}^{\infty} (c_k/b_k)^2 < \infty. \tag{1.3}$$

This class of kernels is more extensive than that treated previously by the authors, see [4], [5], [6], and [7]; however the results obtained here are slightly less precise than those which it was possible to obtain there. We shall show essentially that if

$$f(x) = \int_{-\infty}^{\infty} G(x - t)\, d\alpha(t), \tag{1.4}$$

and if x_1 and x_2 are points of continuity of $\alpha(t)$, then

$$\lim_{m \to \infty} \int_{x_1}^{x_2} \left[\prod_{k=1}^{m} \left(1 - \frac{D}{a_k}\right) e^{D/b_k} \right] f(x)\, dx = \alpha(x_2) - \alpha(x_1). \tag{1.5}$$

Here D is the operation of differentiation, and $e^{D/a}$ that of translation through the

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distance $1/a$, so that, for example,

$$\left(1-\frac{D}{a_1}\right)e^{D/b_1}\left(1-\frac{D}{a_2}\right)e^{D/b_2}f(x)=f\left(x+\frac{1}{b_1}+\frac{1}{b_2}\right)$$
$$-\left(\frac{1}{a_1}+\frac{1}{a_2}\right)f'\left(x+\frac{1}{b_1}+\frac{1}{b_2}\right)$$
$$+\frac{1}{a_1a_2}f''\left(x+\frac{1}{b_1}+\frac{1}{b_2}\right).$$

If we replace equation (1.2) and inequalities (1.3) by the more special relations

$$E(s)=\prod_{k=1}^{\infty}\left(1-\frac{s^2}{a_k^2}\right), \tag{1.6}$$

$$\lim_{k\to\infty} b_k/k=\Omega>0, \qquad \sum_{k=1}^{\infty}(c_k/b_k)^2<\infty, \tag{1.7}$$

we have in addition the complex inversion formula,

$$\lim_{\lambda\to 1-}\int_{x_1}^{x_2}dx\,\frac{1}{2\pi i}\int_{C_\lambda} f(\lambda w+x)K(w)\,dw=\alpha(x_2)-\alpha(x_1), \tag{1.8}$$

where

$$K(w)=\int_0^{\infty}E(s)\,e^{-sw}\,ds \tag{1.9}$$

and C_λ is a closed rectifiable curve encircling the segment $[-i\Omega, i\Omega]$ and lying in the strip $|\mathcal{I} w| < \Omega/\lambda$. The inner integral in formula (1.8) is to be taken in the counterclockwise direction.

As one example we may take

$$E(s)=\frac{\Gamma(1/2+\nu/2)^2}{\Gamma(1/2+\nu/2-s/2)\,\Gamma(1/2+\nu/2+s/2)},$$

$$G(t)=\frac{(e^t+e^{-t})^{-\nu-1}}{\Gamma(1/2+\nu/2)^2},$$

where $\Re\nu > -1$. If

$$f(x) = \int_{-\infty}^{\infty} [e^{(x-t)} + e^{-(x-t)}]^{-\nu-1}\Gamma(1/2 + \nu/2)^{-2} d\alpha(t) ,$$

then

$$\lim_{m\to\infty} \int_{x_1}^{x_2} \left\{ \prod_{k=1}^{m} \left[1 - \left(\frac{D}{-1/2 + \nu/2 + k} \right)^2 \right] f(x) \right\} dx = \alpha(x_2) - \alpha(x_1) ;$$

and if $\Re\nu > 0$, then

$$\lim_{\lambda\to 1^-} \frac{2^\nu \Gamma(1/2 + \nu/2)^2}{\pi\Gamma(\nu)} \int_{x_1}^{x_2} dx \int_{-\pi/2}^{\pi/2} f(x + i\lambda w) [\cos w]^{\nu-1} dw$$

$$= \alpha(x_2) - \alpha(x_1) .$$

See [7] and [8], and [9]. A second example is

$$E(s) = \pi 2^s \left[\cos \frac{\pi\nu}{2} \Gamma\left(\frac{1}{2} - \frac{\nu}{2} - \frac{s}{2} \right) \Gamma\left(\frac{1}{2} + \frac{\nu}{2} - \frac{s}{2} \right) \right]^{-1} ,$$

$$G(t) = \frac{2}{\pi} \cos \frac{\pi\nu}{2} e^t K_\nu(e^t)$$

for $-1 < \Re\nu < 1$. If

$$f(x) = \int_{-\infty}^{\infty} e^{x-t} K_\nu(e^{x-t}) \left(\frac{2}{\pi} \cos \frac{\nu\pi}{2} \right) d\alpha(t) ,$$

then

$$\lim_{m\to\infty} \int_{x_1}^{x_2} \left\{ \prod_{k=1}^{m} \left(1 - \frac{D}{-1/2 - \nu/2 + k} \right) \left(1 - \frac{D}{-1/2 + \nu/2 + k} \right) f(x) \right\} dx$$

$$= \alpha(x_2) - \alpha(x_1) .$$

See [2].

2. Inversion of a class of convolution transforms. We assume as given throughout this section a sequence, $\{a_k\}_1^\infty$, of complex numbers $a_k = b_k + ic_k$ subject to the restrictions

$$(2.1) \qquad \sum_{k=1}^{\infty} (1/b_k)^2 < \infty, \qquad \sum_{k=1}^{\infty} (c_k/b_k)^2 < \infty.$$

We define the entire functions

$$(2.2) \qquad E_{m,n}(s) = \prod_{k=m+1}^{n} (1 - s/a_k)\, e^{s/b_k},$$

$$E_m(s) = \prod_{k=m+1}^{\infty} (1 - s/a_k)\, e^{s/b_k},$$

$$F_m(s) = \prod_{k=m+1}^{\infty} |b_k/a_k|\,(1 - s/b_k)\, e^{s/b_k}.$$

The definition of $E_m(s)$ is significant because

$$E_m(s) = \left\{\prod_{m+1}^{\infty} \left(1 - \frac{s}{a_k}\right) e^{s/a_k}\right\} \left\{\exp \sum_{m+1}^{\infty} \frac{ic_k s}{b_k(b_k + ic_k)}\right\},$$

and because the series $\sum_{m+1}^{\infty} |a_k|^{-2}$, $\sum_{m+1}^{\infty} c_k/b_k(b_k + ic_k)$ converge as a consequence of (2.1) and Schwarz's inequality. Similarly, $F_m(s)$ is well defined. We define

$$(2.3) \qquad P_m(D) = \prod_{k=1}^{m} (1 - D/a_k)\, e^{D/b_k} \qquad (m = 0, 1, \cdots).$$

We also set

$$(2.4) \qquad \beta_1(m) = \max_{\substack{b_k < 0 \\ k > m}} (b_k, -\infty), \qquad \beta_2(m) = \min_{\substack{b_k > 0 \\ k > m}} (b_k, \infty),$$

THEOREM 2a. *Let*

$$G_m(t) = \frac{1}{2\pi i} \int_{-i\infty}^{i\infty} [E_m(s)]^{-1} e^{st}\, ds \qquad (-\infty < t < \infty;\ m = 0, 1, 2, \cdots);$$

then we have

A. $\int_{-\infty}^{\infty} G_m(t)\, e^{-st}\, dt = 1/E_m(s)\,, \qquad \beta_1(m) < \Re s < \beta_2(m)\,;$

B. $\int_{-\infty}^{\infty} |G_m(t)|e^{-\sigma t}\, dt \le 1/F_m(\sigma)\,, \qquad \beta_1(m) < \sigma < \beta_2(m)\,;$

C. $P_m(D)G_0(t) = G_m(t)\,;$

D. $(d/dt)^k G_m(t) = O(e^{\gamma_1 t})\,,\quad t \to +\infty,$

$= O(e^{\gamma_2 t})\,,\quad t \to -\infty \qquad (k = 0, 1, \cdots)\,,$

for $\gamma_1 > \beta_1(m)$ *and* $\gamma_2 < \beta_2(m)$.

Conclusion A is an immediate consequence of Hamburger's theorem; see [4, pp. 141-144]. We define $g(u) = e^{u-1}$ for $-\infty < u \le 1$, and $g(u) = 0$ for $1 < u < \infty$, and we set

$$g_k(t) = a_k \operatorname{sgn} b_k\{\exp[ic_k(t - b_k^{-1})]\}\, g(b_k t)\,.$$

It is immediately verifiable that

$$\int_{-\infty}^{\infty} e^{-st} g_k(t)\, dt = \left[\left(1 - \frac{s}{a_k}\right) e^{s/b_k}\right]^{-1},$$

for $-\infty < \Re s < b_k$ if $b_k > 0$, and for $b_k < \Re s < \infty$ if $b_k < 0$. Let

$$g_1 * g_2(t) = \int_{-\infty}^{\infty} g_1(t-u) g_2(u)\, du\,,$$

and so on; then by the convolution theorem for the bilateral Laplace transform we have

$$\int_{-\infty}^{\infty} g_{m+1} * g_{m+2} * \cdots * g_n(t) e^{-st}\, dt = [E_{m,n}(s)]^{-1}$$

for $\beta_1(m) < \Re s < \beta_2(m)$. From the complex inversion formula for the bilateral Laplace transform we obtain

$$g_{m+1} * g_{m+2} * \cdots * g_n(t) = \frac{1}{2\pi i} \int_{-i\infty}^{i\infty} [E_{m,n}(s)]^{-1} e^{st}\, ds\,.$$

Since

$$\lim_{n\to\infty} \frac{1}{2\pi i} \int_{-i\infty}^{i\infty} \{[E_m(s)]^{-1} - [E_{m,n}(s)]^{-1}\} e^{st}\, ds = 0$$

for $-\infty < t < \infty$, it follows that

$$\lim_{n\to\infty} g_{m+1} * \cdots * g_n(t) = G_m(t) \qquad (-\infty < t < \infty).$$

See [4; pp. 139-145]. It is easily seen that

$$\int_{-\infty}^{\infty} |g_k(t)| e^{-st}\, dt = [(1 - s/b_k) e^{s/b_k} |b_k/a_k|]^{-1},$$

for $-\infty < \Re s < b_k$ if $b_k > 0$, or for $b_k < \Re s < \infty$ if $b_k < 0$. By Fatou's lemma we have

$$\begin{aligned}\int_{-\infty}^{\infty} |G_m(t)| e^{-\sigma t}\, dt &\le \liminf_{n\to\infty} \int_{-\infty}^{\infty} |g_{m+1} * \cdots * g_n(t)| e^{-\sigma t}\, dt, \\ &\le \liminf_{n\to\infty} \int_{-\infty}^{\infty} |g_{m+1}| * \cdots * |g_n(t)| e^{-\sigma t}\, dt \\ &\le [F_m(\sigma)]^{-1},\end{aligned}$$

so that conclusion B is established.

Conclusion C follows from the identity

$$P_m(D) e^{st} = e^{st} \prod_{k=1}^{m} (1 - s a_k^{-1}) e^{s/a_k}.$$

Conclusion D may be established by shifting the line of integration in the integral defining $G_m(t)$ to $\Re s = \gamma_1$ and $\Re s = \gamma_2$. See [4; pp. 152-154].

In what follows we shall write $G(t)$ for $G_0(t)$.

THEOREM 2b. *If*

(a) $G(t)$ *is defined as in Theorem* 2a,

(b) $\beta_1(0) < c < \beta_2(0)$, $c + \gamma_1 > \beta_1(0)$, $c + \gamma_2 < \beta_2(0)$,

(c) *$\alpha(t)$ is of bounded variation on every finite interval,* $\alpha(t) = O(e^{\gamma_1 t})$ *as* $t \longrightarrow -\infty$, $\alpha(t) = O(e^{\gamma_2 t})$ *as* $t \longrightarrow +\infty$,

(d) *$P_m(D)$ is defined as in equation* (2.3),

(e) $f(x) = \int_{-\infty}^{\infty} G(x-t)e^{ct}d\,\alpha(t)$,

(f) *x_1 and x_2 are points of continuity of* $\alpha(t)$,

then

$$\lim_{m\to\infty} \int_{x_1}^{x_2} e^{-cx}[P_m(D)f(x)]\,dx = \alpha(x_2) - \alpha(x_1).$$

From assumption (c) and from conclusion D of Theorem 2a we may show, using integration by parts, that each of the integrals

$$\int_{-\infty}^{\infty} G_m(x-t)e^{ct}d\,\alpha(t)$$

converges uniformly for x in any finite interval. Since $P_m(D)G(t) = G_m(t)$ by conclusion C of Theorem 2a, it follows (see [4; pp. 167-170]) that

$$P_m(D)f(x) = \int_{-\infty}^{\infty} G_m(x-t)e^{ct}d\,\alpha(t) \qquad (-\infty < x < \infty). \tag{2.5}$$

Multiplying by e^{-cx} and integrating by parts, we have

$$\begin{aligned} e^{-cx}P_m(D)f(x) &= -\int_{-\infty}^{\infty}\left\{\frac{\partial}{\partial t}\left[G_m(x-t)e^{-c(x-t)}\right]\right\}\alpha(t)\,dt \\ &= \int_{-\infty}^{\infty}\left\{\frac{\partial}{\partial x}\left[G_m(x-t)e^{-c(x-t)}\right]\right\}\alpha(t)\,dt. \end{aligned}$$

Since this integral converges uniformly for x in any finite interval, we obtain

$$\begin{aligned} &\int_{x_1}^{x_2} e^{-cx}P_m(D)f(x)\,dx \\ &= \int_{x_1}^{x_2} dx \int_{-\infty}^{\infty}\left\{\frac{\partial}{\partial x}\left[G_m(x-t)e^{-c(x-t)}\right]\right\}\alpha(t)\,dt \\ &= \int_{-\infty}^{\infty}\alpha(t)\,dt \int_{x_1}^{x_2}\left\{\frac{\partial}{\partial x}\left[G_m(x-t)e^{-c(x-t)}\right]\right\}dx \end{aligned}$$

$$= \int_{-\infty}^{\infty} \{G_m(x_2 - t)e^{-c(x_2-t)} - G_m(x_1 - t)e^{-c(x_1-t)}\} \alpha(t)\,dt$$

$$= \int_{-\infty}^{\infty} G_m(x_2 - t)e^{-c(x_2-t)} \alpha(t)\,dt - \int_{-\infty}^{\infty} G_m(x_1 - t)e^{-c(x_1-t)} \alpha(t)\,dt .$$

We thus need only show that if x is a point of continuity of $\alpha(t)$ we have

$$\lim_{m\to\infty} \int_{-\infty}^{\infty} G_m(x - t)e^{-c(x-t)} \alpha(t)\,dt = \alpha(x) . \tag{2.6}$$

We shall first show that for any $\epsilon > 0$ we have

$$\lim_{m\to\infty} \int_{|t|\geq\epsilon} G_m(t)e^{-ct} \alpha(x - t)\,dt = 0 . \tag{2.7}$$

Using assumptions (a) and (b) we see that it is enough to prove that for any δ with $\beta_1(0) < \delta < \beta_2(0)$, we have

$$\lim_{m\to\infty} \int_{|t|\geq\epsilon} |G_m(t)|e^{-\delta t}\,dt = 0 . \tag{2.8}$$

Choose $\eta > 0$ so small that $\beta_1(0) < \delta - 2\eta < \delta + 2\eta < \beta_2(0)$. For $|t| \geq \epsilon$ we have

$$e^{-\delta t} \leq \frac{e^{-\delta t}(\sinh \eta t)^2}{(\sinh \epsilon\eta)^2} ,$$

so that it is enough to prove that

$$\lim_{m\to\infty} \int_{-\infty}^{\infty} |G_m(t)|e^{-\delta t}[\sinh \eta t]^2\,dt = 0 .$$

Using conclusions A and B of Theorem 2a we see that

$$\int_{-\infty}^{\infty} |G_m(t)|e^{-\delta t}[\sinh \eta t]^2\,dt$$

$$\leq \frac{1}{4}\left[\frac{1}{F_m(\delta + 2\eta)} + \frac{1}{F_m(\delta - 2\eta)} - \frac{2}{E_m(\delta)}\right] = o(1) \qquad (m \longrightarrow +\infty) ,$$

and equation (2.8) follows from this. We assert that

$$\lim_{m\to\infty} \int_{-\infty}^{\infty} G_m(t)e^{-ct}\,dt = 1 \tag{2.9}$$

(2.10)
$$\limsup_{m\to\infty} \int_{-\infty}^{\infty} |G_m(t)|e^{-ct}\,dt = 1 .$$

These results are immediate consequences of conclusions A and B of Theorem 2a. Now x being fixed and $\eta > 0$ being given, let us choose $\epsilon > 0$ so small that $|\alpha(t) - \alpha(x)| \le \eta$ for $|t-x| \le \epsilon$. We have

$$\int_{-\infty}^{\infty} G_m(x-t)e^{-c(x-t)}\alpha(t)\,dt - \alpha(x) = I_1 + I_2 + I_3 ,$$

where

$$I_1 = \alpha(x)\left[\int_{-\infty}^{\infty} G_m(x-t)e^{-c(x-t)}\,dt - 1\right]$$

$$I_2 = \int_{|t|\ge\epsilon} G_m(t)e^{-ct}[\alpha(x-t) - \alpha(x)]\,dt$$

$$I_3 = \int_{|t|\le\epsilon} G_m(t)e^{-ct}[\alpha(x-t) - \alpha(x)]\,dt .$$

We have $\lim_{m\to\infty} I_1 = 0$ by equation (2.9), $\lim_{m\to\infty} I_2 = 0$ by equation (2.7), and $\limsup_{m\to\infty} |I_3| \le \eta$ by equation (2.10). Since η is arbitrary our demonstration is complete.

3. Complex inversion formulas. In this section we restrict our attention to a much more special class of kernels. We suppose that

(3.1)
$$b_k > 0 , \quad b_k \sim \frac{\Omega k}{\pi} , \quad \sum_{k=1}^{\infty}\left(\frac{c_k}{b_k}\right)^2 < \infty .$$

We define

(3.2)
$$E(s) = \prod_{k=1}^{\infty}\left(1 - \frac{s}{a_k^2}\right)^2 ,$$

(3.3)
$$H(\lambda, s) = \prod_{k=1}^{\infty}\left[\lambda^2 + (1-\lambda^2)\frac{|a_k|b_k}{b_k^2 - s^2}\right] \qquad (0 \le \lambda < 1) .$$

The product (3.3) is defined for $s \neq b_k$ $(k = 1, 2, \cdots)$ since it can be rewritten as

$$H(\lambda, s) = \frac{\prod_{k=1}^{\infty} \left[1 - \frac{\lambda^2 s^2}{b_k^2} + (1-\lambda^2)\frac{|a_k| - b_k}{b_k}\right]}{\prod_{k=1}^{\infty}\left(1 - \frac{s^2}{b_k^2}\right)},$$

and assumption (a) implies that $\Sigma_1^\infty [|a_k| - b_k] b_k^{-1}$ is convergent. We define

$$\beta = \min b_k. \tag{3.4}$$

THEOREM 3a. *If*

$$G(\lambda, w) = \frac{1}{2\pi i} \int_{-i\infty}^{i\infty} \frac{e^{sw} E(\lambda s)}{E(s)}\, ds \qquad (0 \leq \lambda < 1),$$

then

A. $G(\lambda, w)$ *is analytic for* $|\mathfrak{I}\, w| < \Omega(1-\lambda)$;

B. $\int_{-\infty}^{\infty} G(\lambda, t) e^{-st} dt = E(\lambda s)/E(s), \quad -\beta < \mathfrak{R}\, s < \beta$;

C. $$(d/dw)^k G(\lambda, w) = O(e^{\gamma_1 u}) \quad (u \to +\infty)$$
$$= O(e^{\gamma_2 u}) \quad (u \to -\infty) \qquad (k = 0, 1, \cdots),$$

where $\gamma_1 > -\beta$, $\gamma_2 < \beta$, *uniformly for* $|v| < \Omega(1-\lambda) - \epsilon$, $\epsilon > 0$. (*Here* $w = u + iv$.)

D. $\int_{-\infty}^{\infty} |G(\lambda, t)| e^{-\sigma t} dt \leq H(\lambda, \sigma), \qquad -\beta < \sigma < \beta.$

We shall write $G(t)$ for $G(0, t)$.

We assert that

$$\log |E(\sigma + i\tau)| \sim \Omega |\tau| \qquad (\tau \longrightarrow \pm\infty) \tag{3.5}$$

uniformly for σ in any finite interval. We define

$$E*(s) = \prod_{k=1}^{\infty} (1 - s^2 b_k^{-2}) .$$

We have

$$\frac{E(s)}{E*(s)} = \prod_{k=1}^{\infty} \frac{a_k^2}{b_k^2} \prod_{k=1}^{\infty} \left\{ 1 + \frac{a_k^2 - b_k^2}{b_k^2 - s^2} \right\} ,$$

from which it follows that

$$\lim_{s\to\infty} \frac{E(s)}{E*(s)} = \prod_{k=1}^{\infty} \frac{a_k^2}{b_k^2} \tag{3.6}$$

uniformly for $0 < \epsilon \leq |\arg s| \leq \pi - \epsilon$. From [1, pp. 267-279] we have that $\log|E*(\sigma + i\tau) \sim \Omega|\tau|$ as $\tau \longrightarrow \pm\infty$, uniformly for σ in any finite interval. Relation (3.5) now follows.

Conclusion A follows immediately from (3.5) and the definition of $G(\lambda, w)$. Conclusion B is a consequence of (3.5) and Hamburger's Theorem. The two conclusions C are obtained by shifting the line of integration in the integral defining $G(\lambda,t)$ to $\Re s = \gamma_1$, and to $\Re s = \gamma_2$, respectively. See [6, pp. 688-691]. To establish conclusion D we introduce the functions

$$G_n(\lambda,t) = \frac{1}{2\pi i} \int_{-i\infty}^{i\infty} e^{st} \frac{\prod_{k=1}^{n}\left(1 - \frac{\lambda^2 s^2}{a_k^2}\right)}{\prod_{k=1}^{\infty}\left(1 - \frac{s^2}{a_k^2}\right)} ds .$$

It is immediate that

$$\lim_{n\to\infty} G_n(\lambda,t) = G(\lambda,t) \qquad (-\infty < t < \infty) .$$

We define

$$h_k(\lambda, t) = \lambda^2 j(t) + (1-\lambda^2)\,\frac{a_k}{2}\int_{-\infty}^{t} e^{-a_k|u|}\,du\,,$$

where $j(t) = 0$ for $-\infty < t < 0$; $j(0) = 1/2$; $j(t) = 1$ for $0 < t < \infty$. It is easily verified that for $-b_k < \sigma < b_k$ we have

$$\int_{-\infty}^{\infty} e^{-st}\,dh_k(\lambda,t) = \frac{1-\lambda^2 s^2/a_k^2}{1-s^2/a_k^2}\,.$$

Just as in §2 we may show that

$$G_n(\lambda,t) = \lim_{m\to\infty}\frac{d}{dt}\,[h_1(\lambda,t) * \cdots * h_n(\lambda,t) * h_{n+1}(0,t) * \cdots * h_m(0,t)]\,.$$

Here $h_1 * h_2(t) = \int_{-\infty}^{\infty} h_1(t-u)\,dh_1(u)$. Note that this differs from the convention employed in §2. By Fatou's lemma,

$$\int_{-\infty}^{\infty} e^{-\sigma t}\,|G_n(\lambda,t)|\,dt$$

$$\leq \liminf_{m\to\infty}\int_{-\infty}^{\infty} e^{-\sigma t}\,|dh_1(\lambda,t) * \cdots * h_n(\lambda,t) * h_{n+1}(0,t) * \cdots * h_m(0,t)|$$

$$\leq \liminf_{m\to\infty}\prod_{k=1}^{n}\int_{-\infty}^{\infty} e^{-\sigma t}\,|dh_k(\lambda,t)|\prod_{k=n+1}^{m}\int_{-\infty}^{\infty} e^{-\sigma t}\,|dh_k(0,t)|$$

$$\leq \prod_{k=1}^{n}\int_{-\infty}^{\infty} e^{-\sigma t}\,|dh_k(\lambda,t)|\prod_{k=n+1}^{\infty}\int_{-\infty}^{\infty} e^{-\sigma t}\,|dh_k(0,t)|\,.$$

By Fatou's lemma, again,

$$\int_{-\infty}^{\infty} e^{-\sigma t}\,|G(\lambda,t)|\,dt \leq \liminf_{n\to\infty}\int_{-\infty}^{\infty} e^{-\sigma t}\,|G_n(\lambda,t)|\,dt$$

$$\leq \prod_{k=1}^{\infty}\int_{-\infty}^{\infty} e^{-\sigma t}\,|dh_k(\lambda,t)|$$

$$\leq \prod_{k=1}^{\infty}\left[\lambda^2 + (1-\lambda^2)\,\frac{|a_k|b_k}{b_k^2-\sigma^2}\right]\,.$$

This completes the proof of the theorem.

We define

$$K(w) = \int_0^\infty E(s)e^{-sw}\,ds\,. \tag{3.7}$$

It follows from relations (3.1) that given $\epsilon > 0$, for all sufficiently large r we have $\log|E*(re^{i\theta})| \leq (\epsilon + |\sin\theta|)\Omega r$. See [1, pp. 267-279]. From equation (3.6) it follows that

$$\log|E(re^{i\theta})| \leq (\epsilon + |\sin\theta|)\Omega r$$

for r sufficiently large. Using this inequality and rotating the line of integration in the integral defining $K(w)$ we can show that $K(w)$ is analytic and single valued in the w-plane except on the segment $[-i\Omega, i\Omega]$. It may also be shown, see [1, pp. 295-311], that if C is a closed rectifiable curve encircling $[-i\Omega, i\Omega]$ then

$$E(s) = \frac{1}{2\pi i}\int_C K(w)e^{sw}\,dw\,, \tag{3.8}$$

the integration proceeding in the counterclockwise direction.

LEMMA 3b. *If C_λ is a closed rectifiable curve encircling $[-i\Omega, i\Omega]$ and contained in the strip $|v| < \Omega/\lambda$, then*

$$\frac{1}{2\pi i}\int_{C_\lambda} G(\lambda w + x - t)K(w)\,dw = G(\lambda, x - t)\,,$$

the integration proceeding in the counterclockwise direction.

We have

$$\frac{1}{2\pi i}\int_{C_\lambda} G(\lambda w + x - t)K(w)\,dw$$

$$= \frac{1}{2\pi i}\int_{C_\lambda} K(w)\,dw\,\frac{1}{2\pi i}\int_{-i\infty}^{i\infty}[E(s)]^{-1}e^{s(\lambda w + x - t)}\,ds$$

$$= \frac{1}{2\pi i}\int_{-i\infty}^{i\infty}[E(s)]^{-1}e^{s(x-t)}\,ds\,\frac{1}{2\pi i}\int_{C_\lambda} K(w)e^{\lambda sw}\,dw$$

$$= \frac{1}{2\pi i} \int_{-i\infty}^{i\infty} \left[\frac{E(\lambda s)}{E(s)}\right] e^{s(x-t)}\, ds$$

$$= G(\lambda, x - t)\,.$$

THEOREM 3c. *If*

(a) $G(t)$ *is defined as in Theorem* 3a

(b) $-\beta < c < \beta, \quad -\beta < c + \gamma_1, \quad c + \gamma_2 < \beta$

(c) $\alpha(t)$ *is of bounded variation on every finite interval and*

$$\alpha(t) = (e^{\gamma_1 t}) \quad (t \longrightarrow +\infty)\,, \qquad \alpha(t) = (e^{\gamma_2 t}) \quad (t \longrightarrow -\infty)$$

(d) $f(w) = \int_{-\infty}^{\infty} G(w - t)\, e^{ct}\, d\alpha(t)$

(e) $K(w)$ *is defined as in equation* (3.7)

(f) C_λ *is defined as in Lemma* 3b

(g) x_1 *and* x_2 *are points of continuity of* $\alpha(t)$, *then*

$$\lim_{\lambda \to 1-} \int_{x_1}^{x_2} e^{-cx}\, dx\, \frac{1}{2\pi i} \int_{C_\lambda} f(\lambda w + x) K(w)\, dw = \alpha(x_2) - \alpha(x_1)\,.$$

It follows from assumption (c) and from conclusion C of Theorem 3a that the integral defining $f(w)$ converges uniformly for w in any compact set contained in the strip $|\mathcal{I} w| < \Omega$. Hence

$$\frac{1}{2\pi i} \int_{C_\lambda} f(\lambda w + x) K(w)\, dw$$

$$= \int_{-\infty}^{\infty} e^{ct}\, d\alpha(t)\, \frac{1}{2\pi i} \int_{C_\lambda} G(\lambda w + x - t) K(w)\, dw$$

$$= \int_{-\infty}^{\infty} G(\lambda, x - t) e^{ct}\, d\alpha(t)$$

by Lemma 3b. The proof may now be completed exactly in the manner of Theorem 2b.

4. Remark. If it is assumed that the roots of $E(s)$ occur in conjugate pairs, then equation (1.5) can be established under conditions less restrictive than (1.3). A discussion of this case is given in the Master's thesis of Mr. A. O. Garder [3], written under the direction of one of us.

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