# NOTE ON FOURIER ANALYSIS XXXI: CESÀRO SUMMABILITY OF FOURIER SERIES

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1. **Introduction.** M. Jacob [1] proved that if the series

$$\sum_{n=1}^{\infty} \left(a_n^2 + b_n^2\right) n^{2\alpha} \qquad (0 < \alpha < 1)$$

converges then the trigonometrical series $\sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)$ is summable $(C, -\alpha)$ almost everywhere. This result is equivalent to the proposition that if the series $\sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)$ is the Fourier series of a square-integrable function, then the series $\sum_{n=1}^{\infty} (a_n \cos n + b_n \sin nx) n^{-\alpha}$ is summable $(C, -\alpha)$ almost everywhere.

Considering the analogue of this theorem for integrable functions, we shall prove here the following:

THEOREM 1.[1] *If*

$$(1) \qquad \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx) \equiv \frac{A_0(x)}{2} + \sum_{n=1}^{\infty} A_n(x)$$

*is the Fourier series of an integrable function $f(x)$, then the series*

$$(2) \qquad a_0 + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx) n^{-\alpha} \qquad (0 < \alpha < 1)$$

*and its conjugate series*

$$(3) \qquad \sum_{n=1}^{\infty} (a_n \sin nx - b_n \cos nx) n^{-\alpha}$$

[1] Prof. A. Zygmund has pointed out to the author that Theorem 1 can be obtained directly from known results.

Received June 22, 1950, and, in revised form, October 20, 1951.

*Pacific J. Math.* 2 (1952), 419-429

*are summable* $(C, -\alpha)$ *almost everywhere.*

2. **Lemmas.** For the proof we need the following lemmas.

LEMMA 1. *If*

$$(4)\qquad H_n^{(\alpha)}(x) = 1 + \sum_{n=1}^{\infty} k^{-\alpha}\cos kx,\ J_n^{(\alpha)}(x) = \sum_{n=1}^{\infty} k^{-\alpha}\sin kx \qquad (0 < \alpha < 1),$$

*then*

$$(5)\qquad |H_n^{(\alpha)}(x)| \leq A_\alpha x^{\alpha-1},\quad |J_n^{(\alpha)}(x)| \leq B_\alpha x^{\alpha-1} \qquad (0 < x \leq \pi),$$

*where* $A_\alpha$, $B_\alpha$, $\cdots$ *are constants which depend only on* $\alpha$ *and which may be different in different instances.*

The proof of this lemma concerning the cosine series is given by Salem and Zygmund [2], and the part concerning the sine series can be proved similarly.

LEMMA 2. *If* $0 < \alpha < 1$ *and*

$$A_0^{(-\alpha)} = 1,\ A_n^{(-\alpha)} = \binom{n-\alpha}{n} = \frac{(-\alpha+1)(-\alpha+2)\cdots(-\alpha+n)}{n!} \qquad (n \geq 1),$$

*then*

$$(6)\qquad \left|\sum_{k=0}^{n} A_{n-k}^{(-\alpha)} e^{ikx}\right| \leq C_\alpha \left|1 - e^{ix}\right|^{\alpha-1} \qquad (n = 0, 1, 2, \cdots;\ 0 \leq |x| \leq \pi).$$

Lemma 2 is a well-known result of M. Riesz. (Indeed, the constant on the right side of (6) can be replaced by 2; however, the inequality in the above form can easily be derived from Lemma 1.)

Let us denote the $(C, -\alpha)$-means of the series (2) and (3) by $N_n^{(\alpha)}(f;x)$ and $\bar{N}_n^{(\alpha)}(f;x)$, respectively. Then, setting $k^{-\alpha} = 1$ for $k = 0$, we have

$$N_n^{(\alpha)}(f;x) = \frac{1}{A_n^{(-\alpha)}} \sum_{k=0}^{n} A_{n-k}^{(-\alpha)} k^{-\alpha} A_k(x)$$

$$= \frac{1}{\pi} \int_{-\pi}^{\pi} f(x + t) \frac{1}{A_n^{(-\alpha)}} \sum_{k=0}^{n} A_{n-k}^{(-\alpha)} \cos kt \, dt$$

$$= \frac{1}{\pi} \int_{-\pi}^{\pi} f(x + t) \, N_n^{(\alpha)} (t) \, dt,$$

where

$$N_n^{(\alpha)} (t) = \frac{1}{A_n^{(-\alpha)}} \sum_{k=0}^{n} A_{n-k}^{(-\alpha)} \, k^{-\alpha} \cos kt. \tag{7}$$

Similarly

$$\overline{N}_n^{(\alpha)} (f; x) = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x + t) \, \overline{N}_n^{(\alpha)} (t) \, dt, \tag{8}$$

where

$$\overline{N}_n^{(\alpha)} (t) = \frac{1}{A_n^{(-\alpha)}} \sum_{k=1}^{n} A_{n-k}^{(-\alpha)} \, k^{-\alpha} \sin kt. \tag{9}$$

LEMMA 3. *For* $0 < \alpha < 1$, *we have*

$$|N_n^{(\alpha)} (t)| \leq A_\alpha t^{\alpha-1}, \; |\overline{N}_n^{(\alpha)} (t)| \leq B_\alpha t^{\alpha-1} \quad (n = 0, 1, 2, \cdots; \; 0 \leq t \leq \pi). \tag{10}$$

*Proof.* From (7), we have

$$N_n^{(\alpha)} (t) = \frac{1}{A_n^{(-\alpha)}} \sum_{k=0}^{[n/2]} A_{n-k}^{(-\alpha)} \, k^{-\alpha} \cos kt + \frac{1}{A_n^{(-\alpha)}} \sum_{k=[n/2]+1}^{n} A_{n-k}^{(-\alpha)} \, k^{-\alpha} \cos kt$$
$$\equiv P_n + Q_n, \tag{11}$$

say. Using Abel's transformation, we get

$$P_n = \frac{1}{A_n^{(-\alpha)}} \left\{ \sum_{k=0}^{[n/2]-1} A_{n-k}^{(-\alpha-1)} H_k^{(\alpha)} (t) + A_{n-[n/2]}^{(-\alpha)} H_{[n/2]}^{(\alpha)} (t) \right\}. \tag{12}$$

By Lemma 1 we obtain

$$
(13)\quad
\begin{aligned}
|P_n| &\le \frac{1}{A_n^{(-\alpha)}}\left\{A_\alpha t^{\alpha-1}\sum_{k=0}^{[n/2]-1}|A_{n-k}^{(-\alpha-1)}| + A_\alpha t^{\alpha-1} A_{n-[n/2]}^{(-\alpha)}\right\}\\
&\le A_\alpha n^\alpha t^{\alpha-1}\left\{\sum_{k=0}^{n} k^{-\alpha-1} + n^{-\alpha}\right\}\\
&\le A_\alpha t^{\alpha-1}.
\end{aligned}
$$

Now, using Abel's transformation again, we have

$$
(14)\quad Q_n = \frac{1}{A_n^{(-\alpha)}}\left\{\sum_{k=[n/2]+1}^{n-1}\Delta k^{-\alpha}\sum_{j=[n/2]+1}^{k} A_{n-j}^{(-\alpha)}\cos jt + n^{-\alpha}\sum_{k=[n/2]+1}^{n} A_{n-k}^{(-\alpha)}\cos kt\right\},
$$

where

$$
\Delta k^{-\alpha} = k^{-\alpha} - (k+1)^{-\alpha} = O(k^{-\alpha-1}).
$$

Thus

$$
(15)\quad
\begin{aligned}
|Q_n| &\le \frac{A_\alpha}{A_n^{(-\alpha)}}\max_{[n/2]+1\le m\le n}\left|\sum_{k=[n/2]+1}^{m} A_{n-k}^{(-\alpha)}\cos kt\right|\cdot\left\{\sum_{k=[n/2]+1}^{n} k^{-\alpha-1} + n^{-\alpha}\right\}\\
&\le A_\alpha \max_{[n/2]+1\le m\le n}\left|\sum_{k=[n/2]+1}^{m} A_{n-k}^{(-\alpha)}\cos kt\right|.
\end{aligned}
$$

Since

$$
\sum_{k=0}^{m} A_{n-k}^{(-\alpha)}\cos kt = \Re\left\{\sum_{k=0}^{m} A_{n-k}^{(-\alpha)} e^{-ikt}\right\} = \Re\left\{e^{-int}\sum_{k=n-m}^{n} A_k^{(-\alpha)} e^{ikt}\right\},
$$

we have, by Lemma 2,

$$\left| \sum_{k=0}^{m} A_{n-k}^{(-\alpha)} \cos kt \right| \leq A_\alpha |1 - e^{it}|^{\alpha-1} \leq C_\alpha t^{\alpha-1}. \tag{16}$$

Therefore, from (15) and (16),

$$|Q_n| \leq A_\alpha t^{\alpha-1}. \tag{17}$$

Combining (11), (13), and (17), we have the first inequality of (10). The second inequality can be proved similarly.

**3. Proof of Theorem 1.** We proceed now to the proof of Theorem 1. From (7), we have

$$|N_n^{(\alpha)}(f; x)| \leq \frac{1}{\pi} \int_{-\pi}^{\pi} |f(x+t)| \, |N_n^{(\alpha)}(t)| \, dt.$$

By virtue of Lemma 3, we get

$$|N_n^{(\alpha)}(f; x)| \leq A_\alpha \int_{-\pi}^{\pi} |f(x+t)| \, |t|^{\alpha-1} \, dt,$$

whence

$$\begin{aligned} \int_{-\pi}^{\pi} \sup_n |N_n^{(\alpha)}(f; x)| \, dx &\leq A_\alpha \int_{-\pi}^{\pi} dx \int_{-\pi}^{\pi} |f(x+t)| \, |t|^{\alpha-1} \, dt \\ &\leq A_\alpha \int_{-\pi}^{\pi} |t|^{\alpha-1} \, dt \int_{-\pi}^{\pi} |f(x+t)| \, dt \\ &\leq A_\alpha \int_{-\pi}^{\pi} |f(x)| \, dx . \end{aligned} \tag{18}$$

Similarly we have

$$\int_{-\pi}^{\pi} \sup_n |\overline{N}_n^{(\alpha)}(f; x)| \, dx \leq B_\alpha \int_{-\pi}^{\pi} |f(x)| \, dx. \tag{19}$$

From these maximal inequalities we can easily deduce the conclusions of Theorem 1. For example, we can proceed as follows. Let $\epsilon > 0$ be arbitrary, and let $f(x) = f_1(x) + f_2(x)$, where $f_1(x)$ is a trigonometrical polynomial and $\int_{-\pi}^{\pi} |f_2(x)| \, dx < \epsilon/2A_\alpha$, $A_\alpha$ being the constant which appears in the right side of (18). By $f^*(x)$ we denote the function defined by $\sum_{n=0}^{\infty} A_n(x)n^{-\alpha}$ (this series converges almost everywhere); then

$$f^*(x) = f_1^*(x) + f_2^*(x), \tag{20}$$

where $f_1^*(x)$ and $f_2^*(x)$ are determined by $f_1(x)$ and $f_2(x)$, respectively, in the same way as $f^*(x)$ is determined by $f(x)$.

From (18) we have

$$\int_{-\pi}^{\pi} \sup_n |N_n^{(\alpha)}(f_2; x)| \, dx \leq A_\alpha \int_{-\pi}^{\pi} |f_2(x)| \, dx < \epsilon/2, \tag{21}$$

and a fortiori,

$$\int_{-\pi}^{\pi} |f_2^*(x)| \, dx < \epsilon/2. \tag{22}$$

From

$$N_n^{(\alpha)}(f; x) - f^*(x) = N_n^{(\alpha)}(f_1; x) - f_1^*(x) + N_n^{(\alpha)}(f_2; x) - f_2^*(x) \tag{23}$$

we get

$$\limsup_{n\to\infty} |N_n^{(\alpha)}(f, x) - f^*(x)| \leq \sup_n |N_n^{(\alpha)}(f_2; x)| + |f_2^*(x)|, \tag{24}$$

whence, by (21) and (22),

$$\int_{-\pi}^{\pi} \limsup_{n\to\infty} |N_n^{(\alpha)}(f; x) - f^*(x)| \, dx < \epsilon/2 + \epsilon/2 = \epsilon. \tag{25}$$

Therefore it follows that $N_n^{(\alpha)}(f, x) \longrightarrow f^*(x)$ almost everywhere; this implies the validity of the first part of Theorem 1. The proof for conjugate series is analogous.

**4. On multiple Fourier series.** Theorem 1 can be extended to multiple Fourier series. For simplicity we shall state the result for the case of double Fourier series.

Let $f(x, y)$ be an integrable function periodic with period $2\pi$ in each variable, and let its Fourier series be

$$f(x, y) \sim \sum_{m,n=0}^{\infty} A_{mn}(x, y), \tag{26}$$

where

$$A_{00}(x,y) = \frac{1}{4} a_{00}, \quad A_{0n}(x,y) = \frac{1}{2}(a_{0n} \cos ny + b_{0n} \sin ny),$$

$$(27) \quad \begin{aligned} A_{m0}(x,y) &= \frac{1}{2}(a_{m0} \cos mx + b_{m0} \sin mx), \\ A_{mn}(x,y) &= a_{mn} \cos mx \cos ny + b_{mn} \cos mx \sin ny \\ &\quad + c_{mn} \sin mx \cos ny + d_{mn} \sin mx \sin ny, \end{aligned}$$

and

$$(28) \quad a_{mn} = \frac{1}{\pi^2} \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} f(x,y) \cos mx \cos ny \, dx\, dy \quad (m, n = 0, 1, 2, \cdots),$$

and similarly for $b_{mn}$, $c_{mn}$ and $d_{mn}$.

We shall say that the double series $\sum_{m,n=0}^{\infty} A_{mn}$ is summable $(C, \alpha, \beta)$ if the $(C, \alpha, \beta)$-means

$$\frac{1}{A_m^{(\alpha)} A_n^{(\beta)}} \sum_{j=0}^{m} \sum_{k=0}^{n} A_{m-j}^{(\alpha)} A_{n-k}^{(\beta)} A_{jk}$$

of $\sum A_{mn}$ converge.

Under these definitions we have:

THEOREM 2. *If $f(x,y)$ is integrable, then the series*

$$(29) \quad \sum_{m,n=0}^{\infty} A_{mn}(x,y) m^{-\alpha} n^{-\beta} \qquad (0 < \alpha < 1, \ 0 < \beta < 1)$$

*is summable $(C, -\alpha, -\beta)$ almost everywhere. A similar result holds for its conjugate series.*

*Proof.* The $(C, -\alpha, -\beta)$-means $N_{mn}^{(\alpha,\beta)}(f; x, y)$ of the series (29) can be written as follows:

$$(30) \quad N_{mn}^{(\alpha,\beta)}(f; x, y) = \frac{1}{\pi^2} \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} f(x+u, y+v) N_m^{(\alpha)}(u) N_n^{(\beta)}(v) \, du\, dv,$$

where $N_n^{(\alpha)}(t)$ is the same as in (7). Following the line of proof of Theorem 1, we obtain the result.

**5. On the capacity of sets.** Generalizing a result of A. Beurling, Salem and Zygmund [2] proved the following theorem:

*If the series*

$$\sum_{n=1}^{\infty} \left(a_n^2 + b_n^2\right) n^{\alpha} \qquad (0 < \alpha < 1)$$

*converges, then the trigonometrical series* $\sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)$ *is convergent except possibly on a set of* $(1 - \alpha)$*- capacity zero.*

We shall here prove the $L_1$ - analogue of this theorem.

THEOREM 3. *Under the same assumption as in Theorem* 1, *the set* $E$ *of the points where the series* (2) *is not summable* $(C, -\alpha)$ *is of* $(1-\alpha)$ - *capacity zero. A similar result hold for the conjugate series* (3).

REMARKS. (a) Theorem 1 is, of course, implied by Theorem 2. (b) For the notion of capacity and other definitions, the reader is referred to the paper [2] of Salem and Zygmund.

To prove Theorem 2 we need the following lemmas.

LEMMA 4. *If*

$$H^{(\alpha)}(x) = \sum_{n=0}^{\infty} n^{-\alpha} \cos nx, \; J^{(\alpha)}(x) = \sum_{n=1}^{\infty} n^{-\alpha} \sin nx \qquad (0 < \alpha < 1),$$

*then*

$$H^{(\alpha)}(x) \sim x^{\alpha-1}, \; J^{(\alpha)}(x) \sim x^{\alpha-1} \qquad (x \longrightarrow +0).$$

This is known (see, for example, [3; p. 116]).

LEMMA 5. *If a set* $E$ *is of positive* $\alpha$*-capacity* $(0 < \alpha < 1)$, *then there exists a positive distribution* $\mu$ *concentrated on* $E$ *such that if the Fourier-Stielties series of* $\mu(x)$ *is denoted by*

$$d\mu(x) \sim \frac{1}{2\pi}, + \sum_{n=1}^{\infty} (\alpha_n \cos nx + \beta_n \sin nx),$$

*then the series*

$$\sum_{n=1}^{\infty} (\alpha_n \cos nx + \beta_n \sin nx)\, n^{-1+\alpha}$$

*and its conjugate series*

$$\sum_{n=1}^{\infty} (\alpha_n \sin nx - \beta_n \cos nx)\, n^{-1+\alpha}$$

*are the Fourier series of bounded functions.*

Lemma 5 is due to Salem and Zygmund [2].

To prove Theorem 3, let us assume that the set $E$ is of positive $(1-\alpha)$-capacity. Then by Lemma 5 we can find a positive distribution $\mu$ concentrated on $E$ such that if

$$d\mu(x) \sim \frac{1}{2\pi} + \sum_{n=1}^{\infty} (\alpha_n \cos nx + \beta_n \sin nx),$$

then the series $\sum_{n=1}^{\infty} (\alpha_n \cos nx + \beta_n \sin nx) n^{-\alpha}$ is the Fourier series of a bounded function.

Since it is easy to verify by Lemmas 1 and 4 that

$$(31) \qquad |N_n^{(\alpha)}(t)| \leq A\, t^{\alpha-1} \leq A_\alpha H^{(\alpha)}(t) + B_\alpha,$$

we have, by (17),

$$(32) \qquad |N_n^{(\alpha)}(f;x)| \leq \frac{1}{\pi} \int_{-\pi}^{\pi} |f(t)|\, |N_n^{(\alpha)}(x-t)|\, dt$$

$$\leq A_\alpha \int_{-\pi}^{\pi} |f(t)|\, H^{(\alpha)}(x-t)\, dt + B_\alpha \int_{-\pi}^{\pi} |f(t)|\, dt.$$

Integrating both sides of (22), we have

$$\int_0^{2\pi} \sup_n |N_n^{(\alpha)}(f;x)|\, d\mu(x)$$

$$\leq A_\alpha \int_0^{2\pi} d\mu(x) \int_{-\pi}^{\pi} |f(t)| H^{(\alpha)}(x-t)\, dt + B_\alpha \int_{-\pi}^{\pi} |f(t)|\, dt \tag{33}$$

$$\leq A_\alpha \int_{-\pi}^{\pi} |f(t)|\, dt \int_0^{2\pi} H^{(\alpha)}(x-t)\, d\mu(x) + B_\alpha \int_{-\pi}^{\pi} |f(t)|\, dt.$$

By a well-known theorem, the Fourier series of

$$\frac{1}{\pi}\int_0^{2\pi} H^{(\alpha)}(x-t)\, d\mu(x)$$

is

$$1 + \sum_{n=1}^{\infty} (\alpha_n \cos nt + \beta_n \sin nt)\, n^{-\alpha},$$

which is the Fourier series of a bounded function. Hence for almost all $t$ we have

$$\left| \int_0^{2\pi} H^{(\alpha)}(x-t)\, d\mu(x) \right| \leq M(\mu),$$

$M(\mu)$ being a constant depending only on the distribution $\mu$. Therefore, from (33), we have

$$\int_0^{2\pi} \sup_n |N_n^{(\alpha)}(f;x)|\, d\mu(x) \leq A_\alpha M(\mu) \int_{-\pi}^{\pi} |f(t)|\, dt + B_\alpha \int_{-\pi}^{\pi} |f(t)|\, dt$$

$$\leq C_\alpha(\mu) \int_{-\pi}^{\pi} |f(x)|\, dx, \tag{34}$$

where $C_\alpha(\mu)$ is a constant depending only on $\alpha$ and on the distribution $\mu$.

Using this maximal inequality (34) and following the proof of Theorem 1, we see[2] that

[2] We must first prove that the set of points where the series $\sum_{n=0}^{\infty} A_n(x)\, n^{-\alpha}$ does not converge is of $\mu$-measure zero. This can be done by the maximal inequality

$$\int_0^{2\pi} \sup_n \left| \sum_{k=0}^{n} A_k(x) k^{-\alpha} \right| d\mu(x) \leq A(\mu) \int_{-\pi}^{\pi} |f(x)|\, dx,$$

which is a simple consequence of (34), by the same argument as in [3, p. 254].

$$\int_0^{2\pi} \limsup_{n\to\infty} \,|N_n^{(\alpha)}(f,x) - f^*(x)|\, d\mu(x) = 0,$$

which implies that $E$ is of $\mu$-measure zero, contrary to the hypothesis that $\mu$ is concentrated on $E$, ( $\int_E d\mu(x) = 1$ ).

For the conjugate series (3) an analogous argument can be applied, and Theorem 3 is proved completely.

## REFERENCES

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