# NOTE ON SOME TAUBERIAN THEOREMS OF O. SZÁSZ

C. T. RAJAGOPAL

1. **Introduction**. The object of this note is to record extensions of the Tauberian theorems for Abel summability which form the subject of a recent paper of Szász in this journal [6, Theorem 2]. The extensions given as Theorems II, III′, concern a process of summability which may be called the $(\Phi, \lambda)$--process, discussed elsewhere [3] and defined below. Theorems II, III′ include similar results, given as Theorems I, III, which are implicit in [3].

The process of $(\Phi, \lambda)$--summability is defined for any real series $\sum a_n$ as follows.

Let $\phi(u)$ satisfy the following conditions:

C(i) $\phi(u)$ is positive, continuous and monotonic decreasing in $(0, \infty)$;

C(ii) $\phi(0) = 1$, $\int_\epsilon^\infty \{\phi(u)/u\}\, du$ is convergent for every $\epsilon > 0$;

C(iii) $\phi(u)$ has a continuous derivative $-\psi(u)$ in $(0, \infty)$, this derivative being, on account of (i), negative and such that

$$\phi(u) = \int_u^\infty \psi(x)\, dx;$$

C(iv) $\psi(u)$ is monotonic decreasing and has a continuous derivative in $(0, \infty)$;

C(v) $$\int_0^\infty u^{ix}\, \psi(u)\, du \neq 0 \qquad (-\infty < x < \infty).$$

Let

$$\Phi_\lambda(t) = \sum_{n=1}^{\infty} a_n\, \phi(\lambda_n t), \quad t > 0,\ 0 = \lambda_0 < \lambda_1 < \lambda_2 < \cdots, \ \lambda_n \longrightarrow \infty.$$

Then $\sum_{n=1}^\infty a_n$ is said to be $(\Phi, \lambda)$—*summable* if

$$\lim_{t \to +0} \Phi_\lambda(t) \quad \text{exists.}$$

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Examples of $\phi(u)$ which satisfy the conditions (C) are furnished by

$$\phi(u) = e^{-u} \quad \text{[Abel-Laplace kernel]},$$

$$\phi(u) = (1+u)^{-\rho},\ \rho > 0 \quad \text{[Stieltjes kerenel]},$$

$$\phi(u) = u/(e^{u} - 1),\ u > 0,\ \phi(0) = 1 \quad \text{[Lambert kernel]}.$$

The theorems of this note rest primarily on a result proved in my note already referred to [3, Theorem 2] and running as follows:

THEOREM A. *Let $\phi(u)$ fulfil the conditions* C (i)-(v). *Suppose that $A(u)$ is a function of bounded variation in every finite interval of $(0, \infty)$, $A(0) = 0$. If*

$$\liminf_{u\to\infty} \frac{1}{u}\int_0^u u\, d\{A(u)\} > -\infty, \tag{1}$$

*and*

$$\Phi(t) = \int_0^\infty \phi(ut)\, d\{A(u)\} \text{ exists for } t > 0 \text{ and converges as } t \to +0, \tag{2}$$

*then we have*

$$\lim_{u\to\infty} \frac{A_1(u)}{u} = \lim_{t\to+0} \Phi(t), \text{ where } A_1(u) \equiv \int_0^u A(x)\, dx. \tag{3}$$

2. **Extensions of Szász's Tauberian theorems**. It is clear that Theorem A can be restated for a $\lambda$-step function defined thus:

$$\begin{aligned} A(u) &= a_1 + a_2 + \cdots + a_n \quad \text{for} \quad \lambda_n \le u < \lambda_{n+1}, \\ A(u) &= 0 \quad \text{for} \quad 0 \le u < \lambda_1. \end{aligned} \tag{4}$$

For such a step function Szász has proved the following result [5, pp. 126-127].

LEMMA 1. *If $A(u)$ defined by* (4) *is $(R, \lambda, 1)$—summable to $s$, that is, if*

$$\frac{1}{x}\sum_{\lambda_\nu \le x} (x - \lambda_\nu)\, a_\nu \equiv \frac{A_1(x)}{x} \to s \quad \text{as} \quad x \to \infty,$$

*$A_1(x)$ being alternatively defined as in* (3), *and if, as $n \to \infty$, we have*

$$(5)\qquad \sum_{1}^{n} (|a_\nu| - a_\nu)^p \lambda_\nu^p (\lambda_\nu - \lambda_{\nu-1})^{1-p} = O(\lambda_n) \qquad (p > 1),$$

$$\frac{\lambda_{n+1}}{\lambda_n} \longrightarrow 1.$$

*then* $\sum_{n=1}^{\infty} a_n$ *converges to* $s$.

If, in Theorem A, $A(u)$ is the step function of (4), then (1) is a consequence of

$$(1') \qquad \liminf_{n\to\infty} \sum_{\nu=1}^{n} \frac{\lambda_\nu}{\lambda_n} a_\nu > -\infty,$$

which in turn is a consequence of the first condition in (5), as shown by Szász [5, p. 126], while (2) becomes the hypothesis of $(\Phi, \lambda)$–summability of $\sum a_n$. Thus Theorem A, in conjunction with Lemma 1, yields the following result.

THEOREM I. *If* $\sum_{n=1}^{\infty} a_n$ *is* $(\Phi, \lambda)$– *summable to* $s$, *then* (5) *implies that* $\sum_{n=1}^{\infty} a_n$ *is convergent to* $s$.

Theorem I was proved by Szász in the case $\phi(u) = e^{-u}$ [5, Theorem 4(a)]. It is the case $\lambda_n = n$, $\phi(u) = e^{-u}$ of this theorem which he has recently generalized [6, Theorem 2]. Repeating his arguments, in a slightly more general form, we obtain the following extension of Theorem I.

THEOREM II. *If* $\sum_{n=1}^{\infty} a_n$ *is* $(\Phi, \lambda)$–*summable to* $s$ *and if, as* $n \to \infty$, *we have*

$$(6)\qquad U_n \equiv \sum_{\nu=1}^{n} \lambda_\nu (|a_\nu| - a_\nu) = O(\lambda_n), \qquad \lambda_{n+1}/\lambda_n \longrightarrow 1,$$

*and*

$$(7)\qquad \frac{U_m}{\lambda_m} - \frac{U_n}{\lambda_n} \longrightarrow 0 \text{ whenever } \frac{\lambda_m}{\lambda_n} \longrightarrow 1,$$

*then* $\sum_{n=1}^{\infty} a_n$ *is convergent to* $s$.

Theorem II is an extension of Theorem I in the sense that the hypotheses of the former cover those of the latter. This observation, contained in the next lemma, can be substantiated exactly as in its particular case for which $\lambda_n = n$

[6, p. 119].

LEMMA 2. *The first relation in* (5) *implies the first relation in* (6), *while* (5) *as a whole implies* (7) *through the relation*

$$\frac{U_m - U_n}{\lambda_n} \longrightarrow 0 \quad as \quad \frac{\lambda_m}{\lambda_n} \longrightarrow 1 \qquad (n \longrightarrow \infty).$$

For the proof of Theorem II we require a further lemma which is virtually contained in Szász's proof of the case $\lambda_n = n$, $\phi(u) = e^{-u}$ of Theorem II.

LEMMA 3. *If* $\sum_{n=1}^{\infty} a_n$ *is* $(R, \lambda, 1)$*– summable to* $s$, *and the second condition of* (6) *holds along with* (7), *then* $\sum_{n=1}^{\infty} a_n$ *is convergent to* $s$.

*Proof.* The hypothesis of $(R, \lambda, 1)$–summability of $\sum a_n$ to $s$ implies that

$$\sigma_n \equiv \frac{1}{\lambda_{n+1}} \sum_{\nu=1}^{n} (\lambda_{\nu+1} - \lambda_\nu)\, s_\nu \longrightarrow s, \quad \text{where} \quad s_n = \sum_{\nu=1}^{n} a_\nu .$$

Now, we have the identities

$$s_{n-1} - \sigma_{n+k} = \frac{\lambda_n}{\lambda_{n+k+1} - \lambda_n} (\sigma_{n+k} - \sigma_{n-1}) - \frac{1}{\lambda_{n+k+1} - \lambda_n} \cdot \sum_{\nu=0}^{k} (\lambda_{n+k+1} - \lambda_{n+\nu})\, a_{n+\nu},$$

$$s_n - \sigma_{n-k-1} = \frac{\lambda_{n+1}}{\lambda_{n+1} - \lambda_{n-k}} (\sigma_n - \sigma_{n-k-1}) + \frac{1}{\lambda_{n+1} - \lambda_{n-k}} \cdot \sum_{\nu=0}^{k} (\lambda_{n-\nu} - \lambda_{n-k})\, a_{n-\nu}.$$

By using these identities, exactly as they have been used by Szász in the case $\lambda_n = n$ [6, pp. 118-119, pp. 120-121], we can prove that

$$\limsup_{n \to \infty} s_n \leq s, \quad \liminf_{n \to \infty} s_n \geq s,$$

whence the conclusion of Lemma 3 follows at once.

*Proof of Theorem* II. From the first condition in (6) we get

$$-\sum_{\nu=1}^{n} \lambda_\nu a_\nu \le \sum_{\nu=1}^{n} \lambda_\nu (|a_\nu| - a_\nu) \equiv U_n = O(\lambda_n).$$

Therefore, defining $A(u)$ in Theorem A as the $\lambda$-step function of (4), we observe that hypothesis (1) of the theorem obtains as a result of (1′). Since hypothesis (2) also holds, in the form of the $(\Phi, \lambda)$-summability of $\sum a_n$, we are led to conclusion (3) in the form of the $(R, \lambda, 1)$-summability of $\sum a_n$. The desired conclusion now follows from Lemma 3.

3. **A second generalization of Theorem I.** There is a generalization of Theorem I, different from Theorem II and very similar to a theorem of Delange [1, Théorème 8] which may be regarded as one more result of employing the technique of Szász embodied in Theorem A and in certain arguments, already cited, involving $(R, \lambda, 1)$-summability [5, pp. 126-127].

THEOREM I′. *If* $\sum_{n=1}^{\infty} a_n$ *is a series* $(\Phi, \lambda)$*-summable to s, and if*

$$(5') \qquad \sum_{1}^{n} (|a_\nu| - a_\nu)^p \lambda_\nu^p (\lambda_\nu - \lambda_{\nu-1})^{1-p} = O(\lambda_n), \qquad p > 1,\ n \longrightarrow \infty,$$

*then* $\{s_n\}$, *the sequence of partial sums of* $\sum_{n=1}^{\infty} a_n$, *is such that*

$$(8) \qquad \limsup_{n\to\infty} s_n = s, \qquad \liminf_{n\to\infty} s_n = s - l$$

*where*

$$l = \lim_{n\to\infty} \sup (|a_n| - a_n)/2.$$

The proof of Theorem I′ is like that of Theorem I, but uses (in conjunction with Theorem A) the following lemma instead of Lemma 1.

LEMMA 1′. *If* $\sum_{n=1}^{\infty} a_n$ *is summable* $(R, \lambda, 1)$ *to s and satisfies* (5′), *then the sequence* $\{s_n\}$ *of its partial sums behaves as in* (8).

*Proof.* The first half of Szász's arguments [5, p. 126] proving Lemma 1, without any modification, establishes the first conclusion of (8).

To obtain the second conclusion of (8), we note that $l \geq 0$ by definition and $l < \infty$ by (5′) as shown in Lemma 4 which follows this proof. From the fact

$$a_n \geq - \frac{|a_n| - a_n}{2}$$

we then infer that

$$\liminf_{n\to\infty} a_n \geq - l.$$

We thereafter employ the second half of the aforesaid arguments of Szász [5, pp. 126-127] and reach the conclusion

$$\liminf_{n\to\infty} s_n \geq s - l. \tag{9}$$

When $l = 0$, (9) and the universal relation, $\liminf s_n \leq s$, together establish the second conclusion of (8). When $l > 0$, we see that there is an increasing sequence of positive integers $n_1, n_2, \cdots, n_r, \cdots$ such that, as $r \to \infty$, $|a_{n_r}| - a_{n_r} \to 2l$ which implies (since $l > 0$) that $a_{n_r}$ is ultimately negative and $a_{n_r} \to - l$. Hence, when $l > 0$,

$$\begin{aligned} \liminf_{n\to\infty} s_n \leq \limsup_{r\to\infty} s_{n_r} &= \limsup_{r\to\infty} (s_{n_r-1} + a_{n_r}) \\ &= \limsup_{r\to\infty} s_{n_r-1} - l \leq s - l. \end{aligned} \tag{10}$$

Consequently, in the case $l > 0$, the second conclusion of (8) follows from (9) and (10). The proof of Lemma 1′ is now complete.

Theorem I is a special case of Theorem I′ with $l = 0$ as we can see from the following plain statement.

LEMMA 4. *If* $\sum_{n=1}^{\infty} a_n$ *is any (real) series satisfying* (5′), *then*

$$|a_n| - a_n = O\left\{\left(\frac{\lambda_n - \lambda_{n-1}}{\lambda_n}\right)^{(p-1)/p}\right\}.$$

*In particular,* (5′) *and the condition* $\lim \lambda_n/\lambda_{n-1} = 1$ *together imply that* $l \equiv \limsup (|a_n| - a_n)/2 = 0$.

Now, in any (real) series $\sum a_n$,

$$- \frac{|a_n| - a_n}{2} = \min(0, a_n)$$

so that lim inf $a_n \geq 0$ implies $l = 0$ and conversely. Thus, from Theorem I′, we can say that *there is a variant of Theorem* I *with the second condition of* (5) *replaced by* lim inf $a_n \geq 0$; in fact, we can say more as follows.

*For series* $\sum_{n=1}^{\infty} a_n$ *summable* $(\Phi, \lambda)$ *and satisfying the Tauberian condition* (5′), *a necessary and sufficient convergence condition is* lim inf $a_n \geq 0$.

*In the above statement,* (5′) *can be replaced by the following simpler condition which implies* (5′):

$$\liminf_{n\to\infty} \frac{a_n \lambda_n}{\lambda_n - \lambda_{n-1}} > -\infty.$$

**4. A generalization of Theorem II.** Following Szász, we have seen that conditions (6) and (7) of Theorem II together include the corresponding condition (5) of Theorem I. Following Szász further [6, §§4.5], we see that conditions (6) and (7) together can be expressed in the Schmidt form, and the following equivalent of Theorem II obtained.

THEOREM III. *If* $\sum_{n=1}^{\infty} a_n$ *is* $(\Phi, \lambda)$*-summable to s and if, as* $n \longrightarrow \infty$,

$$\text{(11)} \qquad \frac{\lambda_{n+1}}{\lambda_n} \longrightarrow \infty, \qquad \sum_{\nu=n+1}^{m} (|a_\nu| - a_\nu) \longrightarrow 0 \quad \text{whenever} \quad \frac{\lambda_m}{\lambda_n} \longrightarrow 1,$$

*then* $\sum_{n=1}^{\infty} a_n$ *is convergent to sum s.*

A generalization of Theorem III, related to it as Theorem I′ is to Theorem I, may now be stated in a familiar form as under.

THEOREM III′. *If* $\sum_{n=1}^{\infty} a_n$ *is a series* $(\Phi, \lambda)$*-summable to s and such that*

$$\text{(12)} \qquad \begin{aligned} &\text{either} \quad \lim_{\delta\to+0} \limsup_{n\to\infty} \sum_{\lambda_{n+1} \leq \lambda_\nu \leq \lambda_m < (1+\delta)\lambda_n} (|a_\nu| - a_\nu) = 0, \\ &\text{or} \quad \lim_{\delta\to+0} \liminf_{n\to\infty} \min_{\lambda_{n+1} \leq \lambda_\nu \leq \lambda_m < (1+\delta)\lambda_n} \sum a_\nu \geq 0, \end{aligned}$$

*then the sequence* $\{s_n\}$ *of partial sums of the series satisfies* (8), *provided that*

$$l \equiv \limsup_{n\to\infty} (|a_n| - a_n)/2 < \infty.$$

Since $-a_\nu \leq |a_\nu| - a_\nu$, the first alternative of (12) implies the second, and the second is the only alternative that need be considered. Now the second alternative clearly implies that, on the assumption $\lambda_{n+1}/\lambda_n \longrightarrow 1$, we have $\liminf a_n \geq 0$ and hence $l = 0$. Therefore, in Theorem III′, we can drop the explicit assumption $l < \infty$ and assume instead, either

$$\lim \lambda_{n+1}/\lambda_n = 1, \quad \text{or} \quad \liminf a_n \geq 0,$$

getting the two cases of $l = 0$ in the following corollary of which the first case is Theorem III.

COROLLARY III′. *If a series* $\sum_{n=1}^{\infty} a_n$ *is* $(\Phi, \lambda)$*-summable and such that* (12) *holds along with*

$$\textit{either} \quad \lim \lambda_{n+1}/\lambda_n = 1, \quad \textit{or} \quad \liminf a_n \geq 0,$$

*then* $\sum_{n=1}^{\infty} a_n$ *is convergent to s.*

Corollary III′, in the case $\phi(u) = e^{-u}$, is a classic theorem of Szász [4, p. 338]. Theorem III′ in the same case and in an incomplete form has been already obtained by me [2, Theorem F]; its proof readily suggests the following proof of Theorem III′ in its generality.

*Proof of Theorem* III′. Confining ourselves, as we may, to the second alternative of (12), we can show that this alternative and the condition $l < \infty$ (equivalent to $\liminf a_n > -\infty$) together imply (1′), using an argument indicated elsewhere [2, Lemma 1]. After this we can appeal to Theorem A. and infer that $\sum a_n$ is $(R, \lambda, 1)$-summable to $s$. The subsequent completion of the proof is along the lines of the proof of Lemma 1′.

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RAMANUJAN INSTITUTE OF MATHEMATICS
MADRAS, INDIA