

ON A CHARACTERISTIC PROPERTY OF FINITE SETS

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1. **Introduction.** There are several equivalent definitions of finite sets [2], [5]. The purpose of this note is to give an equivalent property of finite sets in terms of ramifications of sets.

DEFINITION 1. A partially ordered set $S \equiv (S; \leq)$ is said to be *ramified* or to satisfy the *ramification condition* ([3], pp. 69, 127; cf. 4) provided that for every $x \in S$ the set $(-\infty, x)$ of all $y \in S$ satisfying $y < x$ is totally ordered (that is, contains no distinct noncomparable points). If the points of a ramified set $(S; \leq)$ are the same as these of a set M , one says that $(S; \leq)$ is a *ramification of M* .

DEFINITION 2. A *chain (anti-chain)* of a partially ordered set $(S; \leq)$ is any subset of S containing no distinct incomparable (comparable) points. Every set containing a single point is considered both as chain and as anti-chain.

DEFINITION 3. For a partially ordered set $(S; \leq) = S$, we denote by

$$(1) \quad O(S) \quad \text{or} \quad OS$$

the system of all *maximal* chains contained in S ; analogously,

$$(2) \quad \bar{O}(S) \quad \text{or} \quad \bar{O}S$$

denotes the system of all maximal anti-chains of S .

THEOREM. *In order that a nonvoid set S be finite, it is necessary and sufficient that for every ramification $T(S)$ of S the relations*

$$(3) \quad M \in O T(S), \quad A \in \bar{O} T(S)$$

imply

$$(4) \quad M \cap A \neq \Lambda \quad (\Lambda \equiv \text{vacuous set}).$$

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2. **The condition is necessary.** Otherwise, there would be a finite set S , a ramification $T(S)$, a set $M \in OT(S)$, and a set $A \in \overline{OT}(S)$, such that

$$(5) \quad M \cap A = \Lambda .$$

Now, A is a maximal anti-chain of $T(S)$; consequently, for every $x \in T(S)$ there is a point $a(x) \in A$ such that the set $\{x, a(x)\}$ is a chain of $T(S)$. (Otherwise, the set $A \cup \{x\}$ would be an anti-chain greater than the maximal anti-chain A .)

In particular, for any $x \in M \in OT(S)$, the points $x, a(x)$ are comparable. We say that

$$(6) \quad x < a(x) .$$

Since M is a maximal chain of the ramified set $T(S)$, M is an initial portion of $T(S)$; that is, M , which contains the point x contains also every point of $T(S)$ preceding x . In particular, if (6) did not hold then M would contain also $a(x) \leq x$; consequently, $a(x) \in M \cap A$, contrary to the assumption (5).

Thus if (5) held then for every $x \in M$ one would have (6); but M , as a non-void subset of the finite set $T(S)$, would have a terminal point, say l ; l would be a final point of $T(S)$, too, contrary to the relation (6) for $x = l$. Thus the relation (5) is not possible.

3. **The condition is sufficient.** If for every ramification $T(S)$ the relations (3) imply (4), then the set S is finite. Otherwise, the set S would be infinite; consequently, there would be a one-to-one correspondence ϕ of the set N of all natural numbers into S . Now, let us define the ordering $(S; \leq)$ by transplantation of a certain order of the set N . We shall order N according to the scheme¹,

$$\begin{array}{ccccccc} 1 & \rightarrow & 3 & \rightarrow & 5 & \rightarrow & 7 & \rightarrow & \dots \\ & \searrow & & \searrow & & \searrow & & \searrow & \\ 2 & & 4 & & 6 & & 8 & & \dots \end{array}$$

That is, the set $2N - 1$ of all $2n - 1$ ($n \in N$) is ordered as in the natural order; for every $n \in N$, the set of numbers preceding $2n$ consists of the numbers $2\nu - 1$ ($\nu = 1, 2, \dots, n$); all other couples of natural integers are incomparable, by definition. In the ramified set N_0 so obtained one sees that $2N \in \overline{ON}_0$, that $2N - 1 \in ON_0$, and that the sets $2N, 2N - 1$ are disjoint. Now, the set S being infinite by hypothesis, there is a one-to-one mapping ϕ of $N = N_0$ into S .

¹For the definition of schemes or diagrams of partly ordered sets see Birkhoff [1, p.6].

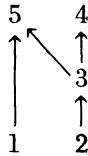
That enables us to define the order in S by transporting the order of N_0 into S so that, on the one hand, the mapping ϕ is a similitude between N_0 and $\phi N_0 \subseteq S$, and so that, on the other hand, no point of ϕN_0 is comparable to any point of $S \setminus \phi N_0$, and so that $S \setminus \phi N_0$ contains no comparable couple of distinct points.

It is obvious that the set $(S; \leq)$ is ramified, that the set $\phi(2N - 1)$ is a maximal chain of $(S; \leq)$, and that the set $A = \phi(2N) \cup (S \setminus \phi N)$ is a maximal anti-chain of $(S; \leq)$.

According to (4), the set $A \cap \phi(2N - 1)$ would be nonvacuous, contrary to the fact that the sets $A, \phi(2N - 1)$ are disjoint.

Thus, the proof of the theorem is completed.

4. Observation. We observe that the condition of ramification in the statement of the theorem is essential. Namely, if we consider the partially ordered set $S_1 = \{1, 2, 3, 4, 5\}$ with the diagram



it is obvious that $\{2, 3, 5\}$ is a maximal chain of S , that $\{1, 4\}$ is a maximal anti-chain of S , and that the set $\{2, 3, 5\}$ does not intersect the set $\{1, 4\}$.

5. Questions. In connection with the statement of the theorem it is interesting to consider the following two questions:

QUESTION 1. Is there a partially ordered nonvacuous set S such that there is no maximal anti-chain $A \in \overline{OS}$ satisfying $A \cap M \neq \Lambda$ for every maximal chain $M \in OS$?

QUESTION 2. Is there a partially ordered nonvacuous set S such that there is no maximal chain $M \in OS$ satisfying $M \cap A \neq \Lambda$ for every maximal anti-chain $A \in \overline{OS}$?

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