MULTIPLICATIVE CLOSURE AND THE WALSH FUNCTIONS

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1. **Introduction**. In his memoir [1] *On the Walsh functions*, N. J. Fine utilized the algebraic character of the Walsh functions by interpreting them as the group characters of the group of all infinite sequences of 0's and 1's, the group operation being addition mod 2 of corresponding elements. In this note, we investigate some properties of real orthogonal systems which are multiplicatively closed. We show that any infinite system of the stated type is isomorphic with the group of the Walsh functions. Furthermore, under hypotheses stated in Theorem 3, there is a measurable transformation of the interval $0 \leq x < 1$ into itself, which carries the Walsh functions into the given system of functions.

As is well known, the Walsh functions are linear combinations of the Haar functions. B. R. Gelbaum [2] gave a characterization of the latter functions in which the norm of certain projection operators and the linear closure in L of the set of functions played an essential role. Although the characteristic features of the Walsh functions are, aside from orthogonality, totally distinct from those for the Haar functions, there is some similarity of proof technique in establishing the characterization.

For the sake of completeness, we define the Rademacher functions $\{\phi_n(x)\}$ as follows:

$$(1.1) \qquad \phi_0(x) = \begin{cases} 1 \ (0 \leq x < 1/2) \\ -1 \ (1/2 \leq x < 1) \end{cases} ; \quad \phi_0(x+1) = \phi_0(x).$$

$$(1.2) \qquad \phi_n(x) = \phi_0(2^n n) \qquad (n = 1, 2, \cdots).$$

The Walsh functions, as ordered by Paley [3], are then given by

$$(1.3) \qquad \psi_0(x) = 1,$$

and

$$(1.4) \qquad \text{if } n = 2^{n_1} + 2^{n_2} + \cdots + 2^{n_k} \qquad (n_1 < n_2 < \cdots < n_k), \qquad \text{then}$$

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$$\psi_n(x) = \phi_{n_1}(x)\ \phi_{n_2}(x) \cdots \phi_{n_k}(x).$$

Throughout we denote the Lebesgue measure of a set E by $\mu(E)$. Equalities of functions are in the sense of almost everywhere equalities, and considerations of sets should be interpreted "modulo sets of measure zero".

2. **Algebraic correspondence**. Let us consider any sequence $\{\lambda_n(x)\}$ of functions satisfying the following conditions:

(2.1) $\lambda_n(x)$ is measurable, real valued, and

$$\lambda_n(x+1) = \lambda_n(x) \qquad (n = 0, 1, \cdots),$$

(2.2) $$\int_0^1 \lambda_n(x)\,\lambda_m(x)\,dx = \delta_n^m.$$

(2.3) The sequence $\{\lambda_n(x)\}$ is closed under multiplication.

THEOREM 1. *Any system of functions satisfying* (2.1)-(2.3) *is a commutative multiplicative group.*

Proof. Associativity and commutativity are immediate. Let i, j be given nonnegative integers, and let k, m, and n be defined by the formulas

$$\lambda_i(x)\,\lambda_j(x) = \lambda_k(x),\ \lambda_i^2(x) = \lambda_m(x),\ \lambda_j^2(x) = \lambda_n(x).$$

Then

$$1 = \int_0^1 \lambda_k^2(x)\,dx = \int_0^1 \lambda_m(x)\,\lambda_n(x)\,dx.$$

Hence $m = n$, and by a renumbering we have $\lambda_i^2(x) = \lambda_0(x)$, for all i. In particular, $\lambda_0^2(x) = \lambda_0(x)$, and for almost every x either $\lambda_0(x) = 0$ or $\lambda_0(x) = 1$. Since

$$\int_0^1 \lambda_0^2(x)\,dx = 1, \quad \text{we have } \lambda_0(x) = 1 \text{ almost everywhere.}$$

Thus the set $\{\lambda_n(x)\}$ possesses a multiplicative unit; and since $\lambda_i^2(x) = 1$ for all i, the set is a group.

THEOREM 2. *The multiplicative group* Λ *with elements* $\lambda_n(x)$ *is isomorphic* *with the multiplicative group* Ψ *with elements the Walsh functions,* $\psi_n(x)$.

Proof. We first reorder the sequence $\{\lambda_n(x)\}$. Let $\nu_0(x) = \lambda_0(x)$ and $\nu_1(x) = \lambda_1(x)$. For $k = 1, 2, \cdots$, define $\nu_{2^k}(x)$ as the first set in the sequence $\{\lambda_n(x)\}$ which follows $\nu_{2^k-1}(x)$ in that sequence, and which is not a product of elements of the sequence $\{\lambda_n(x)\}$ which preceed it in that sequence. For

$$n = 2^{k_1} + \cdots + 2^{k_r}, \quad k_1 < k_2 < \cdots < k_r, \quad \text{let } \nu_n(x) = \prod_{j=1}^{r} \nu_{2^{k_j}}(x).$$

By virtue of (2.3), each $\nu_n(x)$ is a $\lambda_j(x)$. Also each $\lambda_j(x)$ is a $\nu_n(x)$, by the method of construction, and the representation is easily seen to be unique. Thus the sequence $\{\nu_n(x)\}$ is a rearrangement of the sequence $\{\lambda_n(x)\}$. The isomorphism between the groups Λ and Ψ is now evident upon pairing $\nu_n(x)$ and $\psi_n(x)$.

We have incidentally established the result that any denumerable group all of whose elements satisfy $a^2 = 1$ is isomorphic to Ψ.

3. **Analytic transformation.** We introduce the terminology that the subscripts $i_1, i_2, \cdots, i_r$ are unrelated if no $\lambda_{i_k}(x)$ is a product of the remaining $\lambda_{i_\alpha}(x)$.

In the proof of Theorem 1, we incidentally established the result that $\lambda_n(x)$ takes on only the values ± 1. Let

$$(3.1) \qquad P_k = \{x\colon \lambda_k(x) = 1; \quad 0 \le x < 1\},$$

$$(3.2) \qquad N_k = \{x\colon \lambda_k(x) = -1; \ 0 \le x < 1\} \qquad\qquad k = 1, 2, \cdots.$$

It is convenient to adopt the convention that E_k and F_k represent some ordering of the pair P_k and N_k.

LEMMA 1. *If* $i_1, i_2, \cdots, i_r$ *are unrelated, then*

$$\mu\left(\bigcap_{j=1}^{r} E_{i_j}\right) = 2^{-r}.$$

Proof. Let

$$g(x) = 2^{-r} \prod_{j=1}^{r} [1 + \delta_j \lambda_{i_j}(x)],$$

where

$$\delta_j = 1 \quad \text{if} \quad E_{i_j} = P_{i_j},$$

and

$$\delta_j = -1 \quad \text{if} \quad E_{i_j} = N_{i_j}.$$

Then $g(x)$ is the characteristic function of $\bigcap_{j=1}^{r} E_{i_j}$. Hence

$$\mu\left(\bigcap_{j=1}^{r} E_{i_j}\right) = 2^{-r} \int_0^1 \prod_{j=1}^{r} (1 + \delta_j \lambda_{i_j}(x))\, dx$$

$$= 2^{-r} \int_0^1 \left\{ 1 + \sum_j \delta_j \lambda_{i_j} + \sum_{j<j'} \delta_j \delta_{j'} \lambda_{i_j}(x) \lambda_{i_{j'}}(x) + \cdots \right\} dx = 2^{-r},$$

since the integral of any product of λ's with unrelated indices is zero.

COROLLARY. *Any infinite product of sets E_i, without repetitions, has measure zero.*

Proof. Of the subscripts involved in the infinite product, an infinite number must be unrelated. The product set is thus a subset of a set of measure 2^{-r} for each r.

In the remainder of the discussion it is convenient to revise the notation so that P_k, N_k, E_k are associated with $\nu_{2^k}(x)$ rather than with $\lambda_k(x)$.

THEOREM 3. *If any system of functions satisfy* (2.1)-(2.3) *and for almost every choice of E_k as P_k or N_k, the set $\bigcap_{k=1}^{\infty} E_k$ consists of a single point, then there is a transformation T, defined almost everywhere, of the interval $0 \leq x < 1$ into itself with the following properties:*

(i) *T, where defined, is one-to-one except possibly over a denumerable set over which it is two-to-one;*

(ii) $\nu_k(x) = \psi_k(Tx)$;

(iii) *T is a measurable transformation;*

(iv) *For any measurable set E, $\mu(T^{-1}E) = \mu(E)$.*

Proof. The hypothesis of the theorem implies that almost every x, $0 \leq x < 1$, is in exactly one set $\bigcap_{k=0}^{\infty} E_k$. To any such x, $x \not\subset \bigcap_{k=0}^{\infty} N_k$, associate the number Tx according to the following scheme. Let the kth position of the dyadic expansion of Tx be 0 if $E_{k-1} = P_{k-1}$ and 1 if $E_{k-1} = N_{k-1}$. We thus obtain a transformation, defined almost everywhere, of the interval $0 \leq x < 1$ into itself.

Whenever defined, the transformation is one-to-one except possibly on the inverse image set of the nonzero dyadic rationals where it is exactly two-to-one.

By virtue of Theorem 2, all we need do to establish (ii) is to show that $\nu_{2^k}(x) = \psi_{2^k}(Tx)$ $(k = 0, 1, \cdots)$. For $x \in P_k$, Tx has the $(k+1)$th place of its dyadic expansion 0. Hence

$$\psi_{2^k}(Tx) = \psi_0(2^k x) = 1 = \nu_{2^k}(x).$$

Similarly, if $x \in N_k$, then

$$\psi_{2^k}(Tx) = -1 = \nu_{2^k}(x).$$

In the interval $0 \leq x < 1$, any dyadic interval consists of numbers whose dyadic expansions have their first r digits $a_1 \cdots a_r$ and the remaining digits arbitrary. The inverse image of such an interval is the set $\bigcap_{j=0}^{r-1} E_j$, where $E_j = P_j$ if $a_{j+1} = 0$ and $E_j = N_j$ if $a_{j+1} = 1$. Lemma 1 shows that the measure of the inverse image is 2^{-r}. Hence for dyadic intervals, the inverse image is a measurable set of the same measure. Since any open set is the union of disjoint dyadic intervals, the same statement can be made for any open set, and similarly for any closed, F_σ, or G_δ set. A classical argument then shows the statement to be valid for any measurable set E.

References

1. N. J. Fine, *On the Walsh functions*, Trans. Amer. Math. Soc. 65 (1949), 372-414.

2. B. R. Gelbaum, *On the functions of Haar*, Ann. of Math. 51 (1950), 26-36.

3. R. E. A. C. Paley, *A remarkable class of orthogonal functions*, Proc. London Math. Soc. 34 (1932), 241-279.

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