TWO EXISTENCE THEOREMS FOR SYSTEMS OF LINEAR INEQUALITIES

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1. Introduction. In a previous paper [1], the writer initiated the development of the theory of linear inequalities by means of metric methods. This program is continued in the present note to obtain existence theorems for the solutions of two types of (finite) homogeneous systems of inequalities; existence criteria for such systems, different from those established in this paper, are given in the fundamental work of Theodore Motzkin [4]; see also [3].

If A denotes an $m \times n$ matrix of real elements, and x a column matrix of n indeterminates, then the matrix Ax gives rise to the two systems of m homogeneous linear inequalities in n unknowns,

$$Ax \geq 0 \tag{1}$$

and

$$Ax \geqq 0, \tag{2}$$

where the notation $\geqq 0$ is interpreted to demand that at least one of the left members in (1) be positive. In this note necessary and sufficient conditions are found in order that these systems have a solution, which is nontrivial in the case of system (2). These conditions are expressed in terms of the signs of certain minors of the symmetric positive semi-definite matrix of order m formed upon multiplying the matrix A by its transpose A^T. They follow easily from a lemma concerning the distribution of $n+2$ points of the convexly metrized unit n-sphere S_n; this lemma is stated without proof in [2].

2. The Lemma. Let $p_0, p_1, \cdots, p_{n+1}$ be $n+2$ points of the S_n and denote the geodesic distance of p_i, p_j by $p_i p_j$; that is, $p_i p_j$ is the length of a shorter great circle arc that joins p_i and p_j. Denoting the determinant

$$|\cos p_i p_j| \qquad (i, j = 0, 1, \cdots, n+1)$$

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by

$$\Delta(p_0, p_1, \cdots, p_{n+1}),$$

we recall the well-known result that

$$\Delta(p_0, p_1, \cdots, p_{n+1}) = 0,$$

while *each principal minor of the determinant is nonnegative.* If, moreover, a principal minor satisfies

$$\Delta(p_{i_0}, p_{i_1}, \cdots, p_{i_k}) \neq 0,$$

then the points $p_{i_0}, p_{i_1}, \cdots, p_{i_k}$ are contained irreducibly in a k-dimensional (great) hypersphere S_k, and conversely. Clearly each $(m+1)$-tuple of such a set of $k+1$ points is contained irreducibly in an S_m.

LEMMA. *If*

$$p_0, p_1, \cdots, p_n, p_{n+1} \in S_n,$$

with

$$\Delta(p_0, p_1, \cdots, p_n) \neq 0,$$

then (*i*) *the points* $p_0, p_1, \cdots, p_{n-1}$ *determine uniquely an* $(n-1)$*-dimensional great hypersphere* S_{n-1}, *and* (*ii*) *the points* p_n, p_{n+1} *lie on the same or on opposite sides of the hypersphere* S_{n-1} *if and only if the cofactor* $[\cos p_n p_{n+1}]$ *of the element* $\cos p_n p_{n+1}$ *in* $\Delta(p_0, p_1, \cdots, p_{n+1})$ *be negative or positive, respectively.*

Proof. Since $\Delta(p_0, p_1, \cdots, p_n) \neq 0$ (and consequently is positive), the points $p_0, p_1, \cdots, p_n$ are irreducibly contained in S_n, and so $p_0, p_1, \cdots, p_{n-1}$ are irreducibly contained in a great hypersphere $S_{n-1}(p_0, p_1, \cdots, p_{n-1})$ which they determine uniquely.

Let s be any element of S_n, $s \neq p_0, p_1, \cdots, p_n$. Now

$$(1) \quad \Delta(p_0, p_1, \cdots, p_{n-1})\, \Delta(p_0, p_1, \cdots, p_n, s)$$
$$= \Delta(p_0, p_1, \cdots, p_n)\, \Delta(p_0, p_1, \cdots, p_{n-1}, s) - [\cos p_n s]^2,$$

and the vanishing of $\Delta(p_0, p_1, \cdots, p_n, s)$, together with the nonvanishing of $\Delta(p_0, p_1, \cdots, p_n)$, implies that $[\cos p_n s] = 0$ if and only if

$$\Delta(p_0, p_1, \cdots, p_{n-1}, s) = 0;$$

that is, if and only if $s \in S_{n-1}(p_0, p_1, \cdots, p_{n-1})$.

It follows at once that *if* p, q *are any elements of* S_n *which are on the same side of* S_{n-1} $(p_0, p_1, \cdots, p_{n-1})$, *then*

$$\operatorname{sgn}[\cos p_n p] = \operatorname{sgn}[\cos p_n q],$$

where $[\cos p_n p]$, $[\cos p_n q]$ are cofactors of the indicated elements in the determinants $\Delta(p_0, p_1, \cdots, p_n, p)$, $\Delta(p_0, p_1, \cdots, p_n, q)$, respectively. For in the contrary case, the continuous function $[\cos p_n s]$ changes sign for $s = p$ and $s = q$, and consequently it vanishes for some point of the geodesic (shorter) arc joining p and q. But by the above, this point belongs to S_{n-1} $(p_0, p_1, \cdots, p_n)$, and so p, q are on opposite sides of this great hypersphere, contrary to assumption.

If, therefore, p_n and p_{n+1} are on the same side of S_{n-1} $(p_0, p_1, \cdots, p_{n-1})$, then

$$\operatorname{sgn}[\cos p_n p_{n+1}] = \operatorname{sgn}[\cos p_n p_n] = -\operatorname{sgn}\Delta(p_0, p_1, \cdots, p_n),$$

and consequently $[\cos p_n p_{n+1}] < 0$.

Suppose, now, that p_n and p_{n+1} are on opposite sides of

$$S_{n-1}(p_0, p_1, \cdots, p_{n-1}),$$

and denote the reflection of p_n in this hypersphere by p_n^*. Then p_n^*, p_{n+1} are on the same side of the hypersphere, and so

$$\operatorname{sgn}[\cos p_n p_{n+1}] = \operatorname{sgn}[\cos p_n p_n^*].$$

From the vanishing of $\Delta(p_0, p_1, \cdots, p_n, p_n^*)$ and the relations

$$p_i p_n^* = p_i p_n \qquad (i = 0, 1, \cdots, n-1),$$

which follow from the definition of p_n^*, (1) yields

$$[\cos p_n p_n^*] = \pm\Delta(p_0, p_1, \cdots, p_n).$$

To determine the sign, we have, first,

$$[\cos p_n p_n^*] = - \begin{vmatrix} 1 & \cos p_0 p_1 & \cdots & \cos p_0 p_n \\ \cos p_0 p_1 & 1 & \cdots & \cos p_1 p_n \\ \cdot & \cdot & \cdots & \cdot \\ \cos p_0 p_{n-1} & \cos p_1 p_{n-1} & \cdots\ 1 & \cos p_{n-1} p_n \\ \cos p_0 p_n^* & \cos p_1 p_n^* & \cdots & \cos p_n p_n^* \end{vmatrix}.$$

Taking account of the relations $p_i p_n^* = p_i p_n$ $(i = 0, 1, \cdots, n-1)$, and writing the determinant as the sum of two determinants whose last rows are $\cos p_0 p_n$, $\cos p_1 p_n, \cdots, \cos p_{n-1} p_n$, 1 and $0, 0, \cdots, 0, \cos p_n p_n^* - 1$, respectively, we easily obtain

$$(2) \quad [\cos p_n p_n^*]$$
$$= - \Delta(p_0, p_1, \cdots, p_n) + (1 - \cos p_n p_n^*)\, \Delta(p_0, p_1, \cdots, p_{n-1}).$$

Then clearly

$$[\cos p_n p_n^*] = \Delta(p_0, p_1, \cdots, p_n) > 0,$$

for if the negative sign were valid, substitution in (2) would give

$$(1 - \cos p_n p_n^*)\, \Delta(p_0, p_1, \cdots, p_{n-1}) = 0.$$

But

$$\Delta(p_0, p_1, \cdots, p_{n-1}) \neq 0$$

because $p_0, p_1, \cdots, p_{n-1}$ are irreducibly contained in S_{n-1}, while, since p_0, $p_1, \cdots, p_{n-1}, p_n$ are irreducibly contained in S_n, $p_n \notin S_{n-1}(p_0, p_1, \cdots, p_{n-1})$, and so p_n is distinct from its reflection p_n^* in that hypersphere; that is,

$$1 - \cos p_n p_n^* \neq 0.$$

Hence if p_n, p_{n+1} are on opposite sides of $S_{n-1}(p_0, p_1, \cdots, p_{n-1})$, then $[\cos p_n p_{n+1}] > 0$. To complete the proof, it suffices to observe that if

$$[\cos p_n p_{n+1}] \neq 0$$

then $p_{n+1} \notin S_{n-1}(p_0, p_1, \cdots, p_{n-1})$. This is evident upon substituting p_{n+1} for s in (1).

COROLLARY. *Let* $p_0, p_1, \cdots, p_{n+1}$ *be pairwise distinct points of* S_n, *no* $n+1$ *of which are in a great hypersphere. If* $\epsilon_{ij} = 1$ *or* -1 *according as* p_i *and* p_j *are on opposite sides or on the same side, respectively, of the great hypersphere*

$$S_{n-1}(p_0, \cdots, p_{i-1}, p_{i+1}, \cdots, p_{j-1}, p_{j+1}, \cdots, p_{n+1})$$
$$(i, j = 0, 1, \cdots, n+1;\ i \neq j)$$

and $\epsilon_{ii} = 1$ $(i = 0, 1, \cdots, n+1)$, *then the matrix* (ϵ_{ij}) $(i, j = 0, 1, \cdots, n+1)$ *has rank* 1.

REMARK. In a manner similar to that employed above, companion theorems that characterize in a *purely metric* way the sides of hyperplanes in n-dimensional euclidean and hyperbolic spaces that are determined by a given set of n points may be obtained. We state the euclidean theorem, which may be exploited to obtain existence theorems for systems of linear inequalities in much the same way as the lemma just proved will be used in the next section.

THEOREM 1. *Let* $p_0, p_1, \cdots, p_{n+1}$ *be* $n+2$ *points of euclidean n-space* E_n *with* $p_0, p_1, \cdots, p_n$ *irreducibly contained in* E_n. *Then* $p_0, p_1, \cdots, p_{n-1}$ *determine a unique hyperplane* E_{n-1}, *and* p_n, p_{n+1} *are on the same side or on opposite sides of this hyperplane if and only if*

$$\operatorname{sgn}[p_n p_{n+1}^2] = (-1)^n \quad or \quad (-1)^{n+1},$$

respectively, where $[p_n p_{n+1}^2]$ *denotes the cofactor of* $p_n p_{n+1}^2$ *in the determinant*

$$D(p_0, p_1, \cdots, p_{n+1}) = \begin{vmatrix} 0 & 1 & 1 & \cdots & 1 \\ 1 & 0 & p_0p_1{}^2 & \cdots & p_0p_{n+1}^2 \\ 1 & p_0p_1{}^2 & 0 & \cdots & p_1p_{n+1}^2 \\ \cdot & \cdot & \cdot & \cdots & \cdot \\ 1 & p_0p_{n+1}^2 & p_1p_{n+1}^2 & \cdots & 0 \end{vmatrix}$$

We observe, moreover, that for $p_0, p_1, \cdots, p_n$ irreducibly contained in E_n it may be shown that

$$[p_{n-1}p_n^2]$$
$$= (-1)^{n-1} 2^{n-1} [(n-1)!]^2 V(p_0, \cdots, p_{n-2}, p_{n-1}) V(p_0, p_1, \cdots, p_{n-2}, p_n)$$
$$\times \cos \measuredangle (p_0, p_1, \cdots, p_{n-2} : p_{n-1}, p_n),$$

where $[p_{n-1}p_n{}^2]$ is the cofactor of $p_{n-1}p_n{}^2$ in $D(p_0, p_1, \cdots, p_n)$, V is the volume of the $(n-1)$-dimensional simplex determined by the points indicated, and $\sphericalangle(p_0, p_1, \cdots, p_{n-2}: p_{n-1}, p_n)$ denotes the "dihedral" angle with $(n-2)$-dimensional edge $E_{n-2}(p_0, p_1, \cdots, p_{n-2})$ of the simplex with vertices p_0, $p_1, \cdots, p_{n-1}, p_n$.

Hence $[p_{n-1}p_n{}^2] = 0$ if and only if $\sphericalangle(p_0, p_1, \cdots, p_{n-2}: p_{n-1}, p_n) = \pi/2$, and $\operatorname{sgn}[p_{n-1}p_n{}^2] = (-1)^{n-1}$ if and only if the dihedral angle is acute.

It is, perhaps, worth pointing out that Theorem 1 yields a *purely metric characterization* of a nondegenerate simplex (interior and boundary) of E_n. For if $p_0, p_1, \cdots, p_n$ are the vertices of such a simplex, a point p of E_n evidently belongs to its interior or boundary if and only if p and p_i are not on opposite sides of the hyperplane $E_{n-1}(p_0, \cdots, p_{i-1}, p_{i+1}, \cdots, p_n)$ $(i = 0, 1, \cdots, n)$; that is, according to the theorem, if and only if

$$\operatorname{sgn}[p_i p^2] = (-1)^n \text{ or } 0 \qquad (i = 0, 1, \cdots, n),$$

where $[p_i p^2]$ is the cofactor of $p_i p^2$ in the determinant $D(p_0, p_1, \cdots, p_n, p)$.

Since a point of E_n is contained in the convex extension of a k-tuple of E_n (not of E_{n-1}) if and only if it belongs to the simplex determined by some $n+1$ points of the k-tuple, the above observation yields a metric characterization of such convex extensions.

3. The theorems. We are now in position to prove the two existence theorems.

THEOREM 2. *Let $Ax \geq 0$ be a system of m linear inequalities in n indeterminates with rank $r+1$, and let B denote the determinant of the matrix AA^T. The system has a solution if and only if a shifting of rows and corresponding columns of A exists such that*

(i) the upper left principal minor M of B of order $r+1$ does not vanish,

(ii) each minor of B formed from M by replacing its last row with that part of the j-th row of B contained in the first $r+1$ columns $(j = r+2, r+3, \cdots, m)$ is positive or zero.

Proof. Each row of A gives, after normalization, a point of the unit n-sphere S_n, and since the rank of A is $r+1$ it follows that the m "row points" are contained irreducibly in an r-dimensional hypersphere S_r of the S_n. Denoting by p_i the point corresponding to the i-th row of A $(i = 1, 2, \cdots, m)$ after a shifting of rows and columns of A in conformity with the hypotheses has been carried out, we see that $p_1, p_2, \cdots, p_r, p_{r+1}$ lie irreducibly in S_r, and that $p_1, p_2, \cdots, p_r$ determine a unique $(r-1)$-dimensional great hypersphere $S_{r-1}(p_1, p_2, \cdots, p_r)$.

Now the cofactor $[\cos p_{r+1} p_j]$ of the element $\cos p_{r+1} p_j$ in the vanishing determinant $\Delta(p_1, p_2, \cdots, p_{r+1}, p_j)$ $(j = r+2, r+3, \cdots, m)$ has the sign *opposite* to that of the minor of that element which, in turn, has the same sign as the minor of B described in hypothesis (ii). Hence

$$[\cos p_{r+1} p_j] \leqq 0 \qquad (j = r+2, \cdots, m)$$

and so, by the lemma, each of the points $p_{r+2}, p_{r+3}, \cdots, p_m$ lies on the same side of $S_{r-1}(p_1, p_2, \cdots, p_r)$ as p_{r+1}, or is contained in that hypersphere.

Hence the m points are contained in a hemi-S_r with at least one of the m points, p_{r+1}, not on the S_{r-1} forming its "rim". The center of this hemi-S_r is evidently a solution of the system of inequalities, and so the conditions stated in the theorem are sufficient. We have, indeed, found that the S_r itself contains a solution of the system.

To extablish the necessity, we remark that if the system has a solution in S_n, then it has a solution in the S_r containing the m points $p_1, p_2, \cdots, p_m$ irreducibly. For if $p \in S_n$ which is a solution of the system, then p is the center of a hemi-S_n which contains $p_1, p_2, \cdots, p_m$ and has at least one of these points, say, p_m, in its interior. But if the S_r has no solution of the system, the (spherical) convex extension of the m-tuple is the whole S_r, which must be contained in the hemi-S_n on which the m points lie. But this is impossible since the point $p_m{}^*$ diametral to p_m lies in the S_r but it clearly does not belong to the hemi-S_n.

If, therefore, the system has a solution, there is a point of the S_r containing $p_1, p_2, \cdots, p_m$ which is the center of a hemi-S_r that contains $p_1, p_2, \cdots, p_m$, *with at least one of these points in its interior.* It is easily seen that any such hemi-S_r may be rotated so as to retain this property and have some r of the m points, say $p_1, p_2, \cdots, p_r$, on the S_{r-1} forming its rim. If p_{r+1} is in the interior of the hemi-S_r so obtained, then clearly each of the remaining points is either on its rim or on the same side of the rim as p_{r+1}. Invoking, now, the lemma, we see that the conditions of the theorem are satisfied.

REMARK. The direction-cosines of the normal to the hyperplane E_r determined by the points $p_1, p_2, \cdots, p_r$ (and the origin) give a solution of the system of inequalities; but since these numbers are found by evaluating determinants, a solution method based on the theorem is probably not suitable for computing machines.

THEOREM 3. *Let $Ax \geqq 0$ be a system of m linear inequalities in n indeterminates. The system possesses a nontrivial solution if and only if whenever the* *rank of A equals n, a shifting of rows and corresponding columns of the determinant B of AA^T exists such that (i) the nth order upper left principal minor M of B is not zero, while (ii) each nth order minor of B obtained from M by replacing the last row of M with that part of the j-th row contained in the first n columns of B is positive or zero for $j = n + 1, n + 2, \cdots, m$.*

Proof. The rank of A is, of course, at most n; and if it is less than n then the m row points lie in an E_{n-1} containing the origin, the coefficients of which annul all the members of the system of inequalities and hence form a solution.

If the rank of A equals n then the row points are not contained in any E_{n-1} passing through the origin, and so the system has a nontrivial solution if and only if the system $Ax \geq 0$ has a solution; that is (by virtue of Theorem 3), if and only if conditions (i) and (ii) are satisfied.

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