LENGTH AND AREA OF A CONVEX CURVE UNDER AFFINE TRANSFORMATION

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1. Introduction. We consider in the plane the class of all convex curves into which a given convex curve can be affinely transformed, and seek the minimum of L^2/A, where L denotes perimeter and A the area. This amounts to finding the minimum length for a fixed area, or, what is the same thing, to finding the minimum length under area-preserving affine transformations. In § 2 are found necessary conditions on the supporting function that a given curve yield the minimum of L^2/A, and in § 3 these are shown to be sufficient. In § 4 is derived a property of the minimizing curves; namely that if they are sufficiently smooth, they have at least six vertices. In § 5 is derived an integral representation of the supporting function of a convex curve, and another lemma to be used in § 6. In 6 we study the problem of finding the maximum, over all convex curves, of the minimum over affine transformations of L^2/A; in other words, we seek that curve of given area, which when affinely transformed so as to minimize its length, gives the greatest length. We show that the extreme curve is a polygon of not more than five sides, but fail to show what is extremely likely, that the solution is a triangle.

For general facts about convex figures and their supporting functions which are used, see [3].

2. Necessary conditions. Consider a convex curve K and its area-preserving affine transforms. Since rigid motions can be ignored, any transformation in which we are interested can be written in the form

$$(1) \qquad T: \begin{cases} x = e^{\lambda} x', \\ y = \mu x' + e^{-\lambda} y'. \end{cases}$$

The length $L(\lambda, \mu)$ of the transformed curve $K(\lambda, \mu)$ is a continuous function of λ and μ, and tends to ∞ as $(\lambda^2 + \mu^2)^{1/2}$ becomes large. Thus $L(\lambda, \mu)$ has a minimum value, which we take for the moment to be at $\lambda = \mu = 0$.

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In order to find $L(\lambda, \mu)$ we need the supporting function $p(\lambda, \mu; \theta)$ of $K(\lambda, \mu)$. If $p(\theta) = p(0, 0, \theta)$ is the supporting function of K, then a supporting line to K is

$$x \cos\theta + y \sin\theta = p(\theta). \tag{2}$$

The transformation (1) carries (2) into

$$x'(e^{\lambda} \cos\theta + \mu \sin\theta) + y' e^{-\lambda} \sin\theta = p(\theta), \tag{3}$$

which is a supporting line to $K(\lambda, \mu)$.

To convert (3) into normal form we set

$$\begin{cases} e^{\lambda} \cos\theta + \mu \sin\theta = k \cos\phi, \\ e^{-\lambda} \sin\theta = k \sin\phi, \end{cases} \tag{4}$$

or

$$\begin{aligned} \cot\phi &= e^{2\lambda} \cot\theta + \mu e^{\lambda}, \\ k^2 &= (e^{\lambda} \cos\theta + \mu \sin\theta)^2 + e^{-2\lambda} \sin^2\theta. \end{aligned} \tag{5}$$

The normal form of (3) is then

$$x' \cos\phi + y' \sin\phi = p(\theta)/k,$$

and so

$$p(\lambda, \mu, \phi) = p(\theta)/k.$$

From (5) and (4) we see that

$$\csc^2\phi \, d\phi = e^{2\lambda} \csc^2\theta \, d\theta, \quad e^{2\lambda} k^2 \sin^2\phi = \sin^2\theta,$$

and so $d\phi = d\theta/k^2$. Thus[1]

$$L(\lambda, \mu) = \int p(\lambda, \mu, \phi)\, d\phi = \int p(\theta) \frac{d\theta}{k^3}. \tag{6}$$

Now let λ and μ be functions of a parameter t, with $\lambda(0) = \mu(0) = 0$. Then

$$L(\lambda(t), \mu(t)) = L(t),$$

and direct computation from (6) results in

[1] All integrals go from 0 to 2π unless otherwise noted.

$$(7) \qquad \frac{-L'(0)}{3} = \int p(\theta)\left\{\lambda_0' \cos 2\theta + \frac{1}{2}\mu_0' \sin 2\theta\right\} d\theta = 0.$$

Since λ_0' and μ_0' may be taken at pleasure, it is clear that in order for $t = 0$ to yield a minimum, we must have

$$(8) \qquad \int p(\theta) \cos 2\theta \, d\theta = \int p(\theta) \sin 2\theta \, d\theta = 0.$$

In other words, a necessary condition that K give a minimum length is that the second Fourier coefficients of p be zero.

3. Sufficiency. Suppose now that $\lambda = \mu = 0$ is a critical value of $L(\lambda, \mu)$, not necessarily the minimum. Then, as in § 2, we see that

$$\int p \cos 2\theta \, d\theta = \int p \sin 2\theta \, d\theta = 0.$$

Futher differentiation of (6), with the use of (8) and certain trigonometric identities, results in

$$(9)\; L''(0) = \frac{3}{2}\int p(\theta)\{x^2(1 + 5 \cos 4\theta) + 10xy \sin 4\theta + y^2(1 - 5 \cos 4\theta)\}d\theta,$$

where $x = \lambda_0'$, $2y = \mu_0'$. Setting

$$(10) \qquad K(\theta) = x^2(1 - \frac{1}{3} \cos 4\theta) - \frac{2}{3} xy \sin 4\theta + y^2(1 + \frac{1}{3} \cos 4\theta),$$

we may rewrite (9) as

$$(11) \qquad L''(0) = \frac{3}{2} \int p(\theta)\{K + K''\} \, d\theta.$$

Suppose now that p is twice differentiable, and integrate the K'' term in (11) by parts twice. We get

$$(12) \qquad L''(0) = \frac{3}{2} \int (p + p'') \, K \, d\theta.$$

The discriminant of the quadratic form (10) is equal to $-32/9$, and the form is positive definite. Let M be its minimum value for $x^2 + y^2 = 1$, and all θ. The quantity $p + p''$ is the radius of curvature, $ds/d\theta$, of K, and so

$$(13) \qquad L''(0) \geq \frac{3}{2} \int M \, ds = \frac{3}{2} ML.$$

If p is not twice differentiable, we approximate it uniformly by supporting functions which are. The right member of (9), for these approximating functions, is at least $3ML/2$, where L is computed for the approximating function; thus, passing to the limit, we see that (13) is satisfied in this case also.

Because of (13), we now see that if $\lambda = \mu = 0$ is a critical point for $L(\lambda, \mu)$, then it is a proper relative minimum. Consider now any transformation T_0, corresponding to parameters λ_0, μ_0, which yields a

$$K_0 = K(\lambda_0, \mu_0)$$

for which the second Fourier coefficients of the supporting function vanish. We may write T in the form $(TT_0^{-1})T_0$; that is, in studying the length of the transforms of K as function of T, we may study instead the length of the transforms of K_0 as function of TT_0^{-1}. We may write

$$TT_0^{-1}: \begin{cases} x = e^{(\lambda-\lambda_0)}\, x' & = e^{\xi}\, x', \\ y = (\mu e^{-\lambda_0} - \mu_0 e^{-\lambda})\, x' + e^{-(\lambda-\lambda_0)} y' & = \eta x' + e^{-\xi} y', \end{cases}$$

where

$$(14) \qquad \begin{cases} \xi = \lambda - \lambda_0, \\ \eta = \mu e^{-\lambda_0} - \mu_0\, e^{-\lambda}. \end{cases}$$

Now

$$L(\lambda, \mu) = \mathfrak{L}(\xi, \eta),$$

and, by the foregoing analysis, $\mathfrak{L}(\xi, \eta)$ has a proper relative minimum at $\xi = \eta = 0$. But the transformation (14) is nonsingular, and so $L(\lambda, \mu)$ has a proper relative minimum at λ_0, μ_0. Thus every critical point of $L(\lambda, \mu)$ is a proper relative minimum. But an (analytic) function in the plane which has only minima for critical points and which tends to ∞ at great distance can have only one critical point [6]. Thus $L(\lambda, \mu)$ has only one critical point, and this must be at the minimum.

THEOREM 1. *A necessary and sufficient condition that K have the least length of all curves into which it can be transformed by an area-preserving affine transformation is that*

$$\int p \cos 2\theta \, d\theta = \int p \sin 2\theta \, d\theta = 0.$$

Henceforth we shall refer to such K as *extreme curves*.

4. A six-vertex theorem. A vertex on a convex curve is a point where the radius of curvature has an extremum. It is a theorem of Kneser (see for example [1, p.160]) that every convex curve, if sufficiently smooth, has at least four vertices.

THEOREM 2. *Each extreme curve with a continuous radius of curvature has at least six vertices.*[2]

The radius of curvature ρ is given in terms of the supporting function by $\rho = p + p''$. Now

$$\int \rho \cos\theta \, d\theta = \int \frac{ds}{d\theta} \cos\theta \, d\theta = \int \cos\theta \, ds = \oint dy = 0,$$

and similarly for $\int \rho \sin\theta \, d\theta$. Also

$$\int \rho \cos 2\theta \, d\theta = \int (p + p'') \cos 2\theta \, d\theta = 0,$$

by two integrations by parts. Thus we see that

$$(15) \qquad \rho \sim \frac{L}{2\pi} + \sum_{3}^{\infty} (a_n \cos n\theta + b_n \sin n\theta).$$

It has been known since Liouville ([5, p.264]) that (15) implies that $\rho - L/2\pi$ has at least six alternations in signs, and hence ρ six extrema.

In a very similar manner one can prove the following theorem, which however, will only be stated.

THEOREM 3. *Each extreme curve intersects a certain circle, of radius $L/2\pi$, at least six times.*

5. Some lemmas. If $H(\xi, \mu)$ is the Minkowski Stützfunktion of a convex curve, then

$$p(\theta) = H(\cos\theta, \sin\theta).$$

Now H is a convex function of ξ, η; $p(\theta)$ is not convex, but has the somewhat

[2]Blaschke [2] has already shown that a convex curve K may be affinely transformed until its radius of curvature is in the form (15), and thus that it has six vertices. However, the vanishing of the coefficients a_2 and b_2 was attained in an entirely different way. Namely, he found that ellipse K_1, of area equal to that of K, whose mixed volume with K is a minimum. Transforming affinely so that K_1 becomes a circle, we see that K becomes a curve satisfying (15). We have not been able to discover that Blaschke or others made any application of this result to the present problem.

analogous property of being sub-sine. A function $f(\theta)$ is sub-sine if, provided

$$f(\theta) = A\cos\theta + B\sin\theta \text{ at } \theta_1 \text{ and } \theta_2, \text{ where } \theta_1 < \theta_2 < \theta_1 + \pi,$$

then

$$f(\theta) \leq A\cos\theta + B\sin\theta \text{ for } \theta_1 \leq \theta \leq \theta_2.$$

A necessary and sufficient condition [4] that a periodic function $p(\theta)$ be the supporting function of a convex curve is that it be sub-sine, or, if it is of class C'', that $p + p'' \geq 0$.

LEMMA 1. *A necessary and sufficient condition that a function* $p(\theta)$ *of period* 2π *be the supporting function of a convex curve is that it be expressible in the form*

$$p(\theta) = \int_{\theta_0}^{\theta} \sin(\theta - t)\, d\alpha(t) + A\cos\theta + B\sin\theta, \tag{16}$$

where α *is a nondecreasing function.*

First let a supporting function $p \subset C''$; then

$$p + p'' = g(\theta) \geq 0.$$

The solution of the differential equation $p + p'' = g(\theta)$ is readily verified to be

$$p(\theta) = \int_{\theta_0}^{\theta} \sin(\theta - t)\, g(t)\, dt + p(\theta_0)\cos(\theta - \theta_0) + p'(\theta_0)\sin(\theta - \theta_0), \tag{17}$$

which is of the form (16) with

$$\alpha(\theta) = \int_{\theta_0}^{\theta} g(t)\, dt.$$

Note that

$$\alpha(\theta_0) = 0 \text{ and } \alpha(\theta_0 + 2\pi) = \int (p + p'')\, d\theta = L.$$

Now if $p \not\subset C''$, it is the uniform limit of supporting functions p_n which are. We put each p_n in the representation (17), and apply the Helly selection theorem and the Bray-Helly theorem ([7, p. 29-31]) to obtain the result immediately. The factors $p_n'(\theta_0)$ offer no difficulty, since one easily shows that they are bounded for all n.

The converse is proved similarly. If a periodic p is given by (16), we can approximate α by a sequence of smooth monotone functions α_n which give periodic functions p_n; these p_n are sub-sine since they satisfy

$$p_n'' + p_n' = \alpha_n' \geq 0.$$

Again using the Bray-Helly theorem, we have that $p = \lim p_n$; that is, p is a limit of sub-sine functions, and so is sub-sine.

LEMMA 2. *If $p(\theta)$ is a supporting function, and if there exist at least six disjoint intervals in $0 \leq \theta \leq 2\pi$, interior to each of which p is not identically of the form $A \cos \theta + B \sin \theta$, then there exists a function $\eta(\theta)$ with the following properties:*

(a) *$p + \lambda\eta$ is a supporting function for small $|\lambda|$,*

(b) $\int \eta \, d\theta = \int \eta \cos 2\theta \, d\theta = \int \eta \sin 2\theta \, d\theta = 0$,

(c) $\eta \not\equiv A \cos \theta + B \sin \theta$.

Let I_j: $a_j < \theta < b_j$, $j = 1, 2, \cdots, 6$, be the disjoint intervals mentioned, and let p be given by (16). We may assume that $\alpha(\theta)$ is continuous at a_j and b_j. Define

$$\beta_j(\theta) = \begin{cases} \alpha(a_j) & \text{for } 0 \leq \theta < a_j, \\ \alpha(\theta) & \text{for } a_j \leq \theta < b_j, \\ \alpha(b_j) & \text{for } b_j \leq \theta \leq 2\pi. \end{cases} \tag{18}$$

while outside $(0, 2\pi)$ we make $d\beta_j$ periodic. Set

$$\beta = \sum \lambda_j \beta_j, \quad \text{where } |\lambda_j| \leq 1.$$

Then $\alpha(\theta) + \lambda\beta(\theta)$ is nondecreasing if $|\lambda| \leq 1$, as simple computation reveals. We set

$$\eta_j = \int_0^\theta \sin(\theta - t) \, d\beta_j(t) \quad \text{and} \quad \eta = \sum \lambda_j \eta_j.$$

Then $p + \lambda\eta$ is of the form (16), with $\alpha + \lambda\beta$ in place of α. In order that η have period 2π, and thus that (a) be satisfied, we demand that

$$\sum \lambda_j \int \sin\theta \, d\beta_j(\theta) = \sum \lambda_j \int \cos\theta \, d\beta_j(\theta) = 0. \tag{19}$$

To satisfy conditions (b) of the lemma, we set

$$(20) \qquad \sum \lambda_j \int \eta_j \, d\theta = \sum \lambda_j \int \eta_j \cos 2\theta \, d\theta = \sum \lambda_i \int \eta_i \sin 2\theta \, d\theta = 0.$$

Equations (19) and (20) comprise five homogeneous equations in the six unknowns λ_j. They always have a nontrivial solution, which we employ for the construction of β. If $\lambda_k \neq 0$, then η is equal in I_k to a nonzero multiple of $p(\theta)$, plus sine and cosine terms, and this by hypothesis is not of the form $A \cos \theta + B \sin \theta$. Thus (c) is satisfied, and the lemma is proved.

6. The minimax problem. We now restrict our attention to extreme curves, and seek the maximum m of L^2/A. A crude estimate of m can be obtained as follows. If K is any convex curve of area 1, inscribe in K a triangle Δ of maximum area, $A(\Delta)$. Then at each vertex of Δ, K must have a supporting line parallel to the opposite side of Δ, and these three supporting lines form a triangle Δ_1. Transform the plane in an area-preserving affine way so that Δ and Δ_1 are carried into equilateral triangles Δ' and Δ_1', and K into K'. The perimeter $L(\Delta')$ of Δ' is given by

$$L(\Delta') = 6 \sqrt{A(\Delta')/\sqrt{3}}.$$

Then

$$L(K') \leq L(\Delta_1') = 2L(\Delta') = 12\sqrt{A(\Delta')/\sqrt{3}} \leq 12/\sqrt[4]{3}.$$

Thus for the transform K' of K, we have

$$L^2/A \leq 48\sqrt{3}, \text{ and so } m \leq 48\sqrt{3}.$$

On the other hand, the equilateral triangle gives

$$L^2/A = 12\sqrt{3}, \text{ and so } m \geq 12\sqrt{3}.$$

We now normalize our problem by considering extreme curves of length 1, and try to minimize the area. By the usual compactness argument ([2, p. 62]), there does exist a minimizing curve K. Let p be the supporting function of K. Suppose there exists a function $\eta(\theta)$ satisfying conditions (a), (b) of Lemma 2. Consider the area $A(\lambda)$ of the extreme curve, of unit length, whose supporting function is $p + \lambda\eta$. We have

$$(21) \qquad 2A(\lambda) = \int \{(p + \lambda\eta)^2 - (p' + \lambda\eta')^2\}\, d\theta$$
$$= 2A(0) + 2\lambda \int (p\eta - p'\eta')\, d\theta + \lambda^2 \int (\eta^2 - \eta'^2)\, d\theta.$$

Because of the extreme nature of K, the term $\int(p\eta - p'\eta')\, d\theta = 0$. Because of conditions (b) of Lemma 2, the Fourier series of η will be as follows.

$$\eta = a_1 \cos\theta + b_1 \sin\theta + \sum_3^\infty (a_j \cos j\theta + b_j \sin j\theta),$$

and by Parseval's relation,

$$\frac{1}{\pi}\int \eta^2\, d\theta = (a_1^2 + b_1^2) + \sum_3^\infty (a_i^2 + b_i^2).$$

Similarly (η' being bounded),

$$\frac{1}{\pi}\int \eta'^2\, d\theta = (a_1^2 + b_1^2) + \sum_3^\infty j^2 (a_i^2 + b_i^2),$$

and so

$$(22) \qquad \int (\eta^2 - \eta'^2)\, d\theta = \pi \sum_3^\infty (1 - j^2)(a_i^2 + b_i^2).$$

Since $A(\lambda) \geq A(0)$, we see from (21) and (22) that $a_j = b_j = 0$ for $j \geq 2$, so that $\eta \equiv a_1 \cos\theta + b_1 \sin\theta$. Thus it is not possible to satisfy (a), (b), and (c) simultaneously.

Now if K is a polygon, p is piecewise of the form $A\cos\theta + B\sin\theta$, and conversely. If K is not a polygon it is clear that one can find as many intervals as desired in each of which p is not of that form, and Lemma 2 applies. Lemma 2 also applies if K is a polygon of six or more sides. Thus it is not possible for K to be other than a polygon of five or fewer sides.

It appears very likely that K is an equilateral triangle and that $m = 12\sqrt{3}$. To eliminate the cases of four and five sides is just a problem in the calculus, but possibly a very difficult one. In these cases there are not enough sides to construct the variations used above, which consist of sliding sides in and out parallel to themselves, so if a variational method is to be used, a different kind of variation, involving changing the angles, must be found.

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