SUBFUNCTIONS OF SEVERAL VARIABLES

E. F. BECKENBACH AND L. K. JACKSON

1. Introduction. Convex functions have been generalized in the following two directions: to subharmonic functions [5] of two (or more) independent variables, by replacing the dominating family $\{F(x)\}$ of linear functions, or solutions of the differential equation

$$\frac{d^2 F}{dx^2} = 0,$$

with a family of harmonic functions $\{F(x, y)\}$, or solutions of the partial differential equation

$$\Delta F \equiv \frac{\partial^2 F}{\partial x^2} + \frac{\partial^2 F}{\partial y^2} = 0; \tag{1}$$

and to subfunctions [1] of one variable, by replacing the dominating family of linear functions with a more general family of functions of one variable having certain geometric features in common with the family of linear functions.

We shall here combine the foregoing considerations, generalizing subharmonic functions by replacing the dominating family of harmonic functions with a more general family of functions of two (or more) independent variables.

Bonsall [2] has recently considered some properties of subfunctions of two independent variables relative to the family of solutions of the second-order elliptic partial differential equation

$$\Delta F + a(x, y)\frac{\partial F}{\partial x} + b(x, y)\frac{\partial F}{\partial y} + c(x, y)F = 0.$$

Received February 17, 1952. This research was supported in part by an Office of Naval Research contract.

Pacific J. Math. 3 (1953), 291-313

Tautz [6] has considered a more general situation; but he restricted the dominating family of functions to being linear, and its members to having no positive maxima or negative minima at interior points of the domain of definition.

After developing some properties of subfunctions from a few postulates for our dominating family of functions (§ 2), we shall introduce further postulates as we need them in studying a Dirichlet problem relative to the dominating family of functions (§§ 3-5). Applications to elliptic partial differential equations will be made in a subsequent publication.

2. **Generalized subharmonic functions.** Let D be a plane domain (nonnull connected open set), and let $\{\gamma\}$ be a family of closed contours γ bounding subdomains Γ of D such that

a) $\overline{\Gamma} \equiv \Gamma + \gamma \subset D$,

b) $\overline{\Gamma}$ is closed,

c) $\{\gamma\}$ includes all circles κ which lie together with their interiors K in D,

$$\overline{K} = K + \kappa \subset D,$$

and have radii less than a fixed $\rho > 0$.

Let $\{F(x, y)\}$ be a family of functions whose domains of definition lie in D and which satisfy the following postulates:

POSTULATE 1. For any member γ of $\{\gamma\}$ and any continuous boundary-value function $f(x, y)$ on γ, there is a unique $F(x, y; f; \gamma) \in \{F(x, y)\}$ such that

a) $F(x, y; f; \gamma) = f(x, y)$ on γ,

b) $F(x, y; f; \gamma)$ is continuous in $\overline{\Gamma}$.

POSTULATE 2. For each constant $M \geq 0$, if

$$F_1(x, y),\ F_2(x, y) \in \{F(x, y)\},$$

and

$$F_1(x, y) \leq F_2(x, y) + M \quad \text{on } \gamma, \tag{2}$$

then

$$F_1(x, y) \leq F_2(x, y) + M \qquad \text{in } \overline{\Gamma};$$

further, if the strict inequality holds at a point of γ then the strict inequality holds throughout Γ.

REMARK. We note that the second part of Postulate 2 might have been restricted to the case $M = 0$ without actual loss of generality. For if the strict inequality in (2) holds at a point of γ then also

$$F_1(x, y) \leq F(x, y; F_2 + M; \gamma) \qquad \text{on } \gamma,$$

with the strict inequality holding at a point of γ. It follows from the second part of Postulate 2, restricted to the case $M = 0$, that

$$F_1(x, y) < F(x, y; F_2 + M; \gamma) \leq F_2(x, y) + M \qquad \text{in } \Gamma;$$

or

$$F_1(x, y) < F_2(x, y) + M \qquad \text{in } \Gamma,$$

for $M \geq 0$.

DEFINITION 1. A function $g(x, y)$ is a *continuous sub-*$\{F(x, y)\}$ *function* in D, or briefly a *subfunction*, provided

a) $g(x, y)$ is continuous in D,

b) the inequality

$$g(x, y) \leq F(x, y) \qquad \text{on } \gamma$$

implies the inequality

$$g(x, y) \leq F(x, y) \qquad \text{in } \Gamma.$$

NOTATION. In the sequel we shall restrict use of the symbols D, γ, Γ, $\overline{\Gamma}$, κ, K, $\overline{K}$, $f(x, y)$, $F(x, y)$, $F(x, y; f; \gamma)$, and $g(x, y)$ to the foregoing designations.

THEOREM 1. *If $g(x, y)$ is a subfunction in D, then either*

$$g(x, y) \equiv F(x, y; g; \gamma) \qquad \textit{in } \overline{\Gamma}, \tag{3}$$

or

$$g(x, y) < F(x, y; g; \gamma) \qquad \textit{throughout } \Gamma.$$

Proof. Suppose that for a point (x_0, y_0) of Γ we have

$$g(x_0, y_0) = F(x_0, y_0; g; \gamma). \tag{4}$$

Let $\kappa \in \{\gamma\}$ be a circle with center at (x_0, y_0) and lying together with its interior K in Γ. Then we have

$$g(x, y) = F(x, y; g; \kappa) \leq F(x, y; g; \gamma) \qquad \text{on } \kappa,$$

and therefore

$$g(x, y) \leq F(x, y; g; \kappa) \leq F(x, y; g; \gamma) \qquad \text{in } \overline{K}. \tag{5}$$

In particular, by (4), we have

$$g(x_0, y_0) \leq F(x_0, y_0; g; \kappa) \leq F(x_0, y_0; g; \gamma) = g(x_0, y_0),$$

so that

$$F(x_0, y_0; g; \kappa) = F(x_0, y_0; g; \gamma),$$

and therefore, by (5) and Postulate 2,

$$g(x, y) = F(x, y; g; \kappa) = F(x, y; g; \gamma) \qquad \text{on } \kappa.$$

Since Γ is a connected open set, for any point (x, y) of Γ there is a finite chain of circles in $\{\gamma\}$, each lying in Γ, each with its center on the circumference of the preceding one, and such that the chain begins with κ and ends with a circle through (x, y). Continued repetition of the foregoing analysis shows that (4) implies (3).

COROLLARY. *If $g(x, y)$ is a subfunction in D, and, for a fixed γ, $\overline{\Gamma}$ is interior to the domain of definition of $F(x, y; g; \gamma)$, then either*

$$g(x, y) \equiv F(x, y; g; \gamma),$$

or every neighborhood of each point of γ contains points exterior to $\overline{\Gamma}$ for which

$$g(x, y) > F(x, y; g; \gamma).$$

REMARK. For a subfunction $g(x)$ of one variable, relative to a family $\{F(x)\}$ of functions defined on an interval $a < x < b$, the corollary can be strengthened as follows [1]: If $a < x_1 < x_2 < b$, and

$$g(x_1) = F(x_1), \qquad g(x_2) = F(x_2),$$

then either

$$g(x) \equiv F(x)$$

or

$$g(x) > F(x)$$

for $a < x < x_1$ and for $x_2 < x < b$. But, as independently observed in conversation by R. H. Bing and M. H. Heins, the stronger result does not generally hold for subfunctions of more than one variable. Thus the function

$$V(z) \equiv \log \left| z\left(z - \frac{1}{2}\right)\right|, \qquad (z = x + iy)$$

is subharmonic in $|z| < 1$. For large $M > 0$, the set of points where

$$V(z) < -M$$

has exactly two components, one containing the point $z = 0$, the other containing $z = 1/2$. Let $W(z)$ be defined by

$$W(z) \equiv \begin{cases} -M \text{ on the component containing } z = 0, \\ \max[V(z), -2M] \text{ elsewhere in } |z| < 1. \end{cases}$$

Now $W(z)$ is continuous and subharmonic in $|z| < 1$, coincides with the harmonic function $-M$ on small circles with center at the origin, but is strictly dominated by $-M$ in the neighborhood of $z = 1/2$.

THEOREM 2. *If $g_n(x, y)$ is a subfunction in D for $n = 1, 2, \cdots$, and*

$$g_n(x, y) \longrightarrow g_0(x, y)$$

uniformly on each closed and bounded subset of D, then $g_0(x, y)$ is a subfunction in D.

Proof. Clearly, $g_0(x, y)$ is continuous in D. For any $\gamma \in \{\gamma\}$ and any $\epsilon > 0$, there is an $N = N(\epsilon)$ such that for $n \geq N$ we have

$$|g_n(x, y) - g_0(x, y)| < \epsilon \qquad \text{in } \overline{\Gamma}.$$

Then for $n \geq N$ and $(x, y) \in \Gamma$, we have

$$g_0(x, y) \leq g_n(x, y) + \epsilon$$
$$\leq F(x, y; g_n; \gamma) + \epsilon \leq F(x, y; g_0; \gamma) + 2\epsilon.$$

Therefore, since $\epsilon > 0$ is arbitrary, we have

$$g_0(x, y) \leq F(x, y; g_0; \gamma) \qquad \text{in } \overline{\Gamma}.$$

The following result is a generalization of Littlewood's theorem [4, pp. 152-157] concerning subharmonic functions.

THEOREM 3. *A function $g(x, y)$, continuous in D, is a subfunction in D if and only if corresponding to each $(x_0, y_0) \in D$ there exists a sequence of circles $\kappa_n = \kappa_n(x_0, y_0) \in \{\gamma\}$ with centers at (x_0, y_0) and radii $\rho_n(x_0, y_0) \longrightarrow 0$, such that*

$$g(x_0, y_0) \leq F(x_0, y_0; g; \kappa_n).$$

Proof. We shall prove only the sufficiency of the condition, since the necessity is obvious by definition.

Suppose that the condition holds but that $g(x, y)$ is not a subfunction; then there is a $\gamma \in \{\gamma\}$ and an $F(x, y) \in \{F(x, y)\}$ such that

$$g(x, y) \leq F(x, y) \qquad \text{on } \gamma$$

but

$$g(x, y) > F(x, y)$$

at some point of Γ. Then the set of points of Γ on which

$$\max_{(x, y) \in \Gamma} g(x, y) - F(x, y) \equiv M > 0$$

is attained is a closed nonnull interior set E in Γ.

Let (x_0, y_0) be a point of E such that

$$\text{dist}\,[(x_0, y_0), \gamma] = \min_{(x, y) \in E} \text{dist}\,[(x, y), \gamma].$$

By hypothesis, we have

$$g(x_0, y_0) \leq F(x_0, y_0; g; \kappa_n);$$

but, by our selection of (x_0, y_0), for sufficiently large n there is an arc of κ_n on which

$$g(x, y) - F(x, y) < M.$$

Thus on κ_n we have

$$F(x, y; g; \kappa_n) = g(x, y) \leq F(x, y) + M,$$

with the strict inequality holding at some point, so that, by Postulate 2, at each point inside κ_n we have

$$F(x, y; g; \kappa_n) < F(x, y) + M;$$

in particular, we have

$$g(x_0, y_0) \leq F(x_0, y_0; g; \kappa_n) < F(x_0, y_0) + M = g(x_0, y_0),$$

a contradiction.

REMARK. A method similar to that used in proving Theorem 3 can be used to show that Postulate 2, restricted to the case $\gamma = \kappa$, implies the result stated in Postulate 2 for general $\gamma \in \{\gamma\}$. Thus Postulate 2 might have been restricted to the case $\gamma = \kappa$ without actual loss of generality.

THEOREM 4. *If $g_1(x, y)$, $g_2(x, y)$, $\cdots$, $g_n(x, y)$ are subfunctions in D, then $g_0(x, y)$, defined by*

$$g_0(x, y) \equiv \max [g_1(x, y), g_2(x, y), \cdots, g_n(x, y)],$$

is a subfunction in D.

Proof. Since the functions $g_j(x, y)$ $(j = 1, 2, \cdots, n)$ are continuous, it follows that $g_0(x, y)$ also is continuous. Let $\gamma \in \{\gamma\}$, and let $(x_0, y_0) \in \Gamma$. Then for some j, with $1 \leq j \leq n$, we have

$$g_0(x_0, y_0) = g_j(x_0, y_0) \leq F(x_0, y_0; g_j; \gamma) \leq F(x_0, y_0; g_0; \gamma).$$

THEOREM 5. *If $g(x, y)$ is a subfunction in D, then, for any fixed $\gamma \in \{\gamma\}$, the function $g(x, y; \gamma)$, defined by*

$$g(x, y; \gamma) \equiv \begin{cases} g(x, y) & \text{for } (x, y) \in D - \Gamma, \\ F(x, y; g; \gamma) & \text{for } (x, y) \in \Gamma, \end{cases}$$

is a subfunction in D.

Proof. It follows from Theorem 3 that we need to test the behavior of $g(x, y; \gamma)$ only relative to small circles $\kappa \in \{\gamma\}$ with centers at points (x_0, y_0) of the given γ. But then we immediately have the desired inequality

$$g(x_0, y_0; \gamma) = g(x_0, y_0) \le F(x_0, y_0; g; \kappa) \le F(x_0, y_0; g(x, y; \gamma); \kappa).$$

DEFINITION 2. *Superfunctions* are defined by reversing the inequalities in the definition (Definition 1) of subfunctions relative to the family $\{F(x, y)\}$.

It is easy to show that results analogous to Theorems 1-5, with suitable alterations, hold for superfunctions: in addition to writing "superfunction" for "subfunction," we reverse the inequality in the last line of Theorem 1 and in the last line of Theorem 3, and replace "max" by "min" in Theorem 4.

3. A Dirichlet problem.

We now introduce some additional symbols.

NOTATION. Let Ω be a bounded connected open subset of D with boundary ω such that

$$\bar{\Omega} \equiv \Omega + \omega \subset D.$$

To distinguish points of Ω from points of ω, we shall often designate points of Ω by capital letters A, B, and points of ω by α, β; and we shall write $f(A)$ for $f(x, y)$, where (x, y) are the coordinates of A, and so on.

Let $h(\alpha)$ be a bounded, but not necessarily continuous, function defined on ω, and define $h_*(\alpha)$ and $h^*(\alpha)$ by

$$h_*(\alpha) \equiv \liminf_{\beta \to \alpha} h(\beta),$$

$$h^*(\alpha) \equiv \limsup_{\beta \to \alpha} h(\beta).$$

DEFINITION 3. By a *solution of the Dirichlet problem* for Ω relative to $\{F(x, y)\}$ and relative to the given bounded boundary-value function $h(\alpha)$ on ω, we shall mean a function $H(x, y)$ which is continuous in Ω, satisfies

$$(6) \qquad h_*(\alpha) \leq \liminf_{A \to \alpha} H(A) \leq \limsup_{A \to \alpha} H(A) \leq h^*(\alpha),$$

and is such that for each $\gamma \in \{\gamma\}$ with $\overline{\Gamma} \subset \Omega$ we have

$$(7) \qquad H(x, y) \equiv F(x, y; H; \gamma) \qquad \text{in } \Gamma.$$

DEFINITION 4. We shall say that a function $H(x, y)$ which is continuous in Ω, and which satisfies (7) for each $\gamma \in \{\gamma\}$ with $\overline{\Gamma} \subset \Omega$, is an $\{F(x, y)\}$-*function in* Ω, though of course in the given family $\{F(x, y)\}$ there might be no member whose domain of definition contains Ω; the given domains of definition might for instance be small circles. Clearly, the only functions which are both subfunctions and superfunctions are the $\{F(x, y)\}$-functions.

DEFINITION 5. The function $\phi(x, y)$ is an *under-function* provided

a) $\phi(x, y)$ is continuous in $\overline{\Omega}$,

b) $\phi(A)$ is a subfunction in Ω,

c) $\phi(\alpha)$ satisfies

$$\phi(\alpha) \leq h_*(\alpha) \qquad \text{on } \omega.$$

DEFINITION 6. The function $\psi(x, y)$ is an *over-function* provided

a) $\psi(x, y)$ is continuous in $\overline{\Omega}$,

b) $\psi(A)$ is a superfunction in Ω,

c) $\psi(\alpha)$ satisfies

$$\psi(\alpha) \geq h^*(\alpha) \qquad \text{on } \omega.$$

THEOREM 6. *If $\phi(x, y)$ is an under-function and $\psi(x, y)$ is an over-function, then*

$$\phi(x, y) \leq \psi(x, y) \qquad \textit{in } \overline{\Omega}.$$

Proof. The result can be established by a method similar to that used in proving Theorem 3.

THEOREM 7. *If* $\phi_1(x, y)$, $\phi_2(x, y), \cdots, \phi_n(x, y)$ *are under-functions, then* $\phi(x, y)$, *defined by*

$$\phi(x, y) \equiv \max [\phi_1(x, y), \phi_2(x, y), \cdots, \phi_n(x, y)],$$

is an under-function.

THEOREM 8. *If* $\psi_1(x, y)$, $\psi_2(x, y), \cdots, \psi_n(x, y)$ *are over-functions, then* $\psi(x, y)$, *defined by*

$$\psi(x, y) \equiv \min [\psi_1(x, y), \psi_2(x, y), \cdots, \psi_n(x, y)],$$

is an over-function.

Proof. Property b) of Definition 5 holds for $\phi(x, y)$ by Theorem 4; the other properties of Definition 5 hold for $\phi(x, y)$ since they hold for each $\phi_j(x, y)$. Thus Theorem 7 is valid; and Theorem 8 can be proved similarly.

THEOREM 9. *If* $\phi(x, y)$ *is an under-function, and* $\gamma \in \{\gamma\}$, $\overline{\Gamma} \subset \Omega$, *then* $\phi(x, y; \gamma)$, *defined by*

$$\phi(x, y; \gamma) \equiv \begin{cases} \phi(x, y) & \text{for } (x, y) \text{ in } \overline{\Omega} - \Gamma, \\ F(x, y; \phi, \gamma) & \text{for } (x, y) \text{ in } \Gamma, \end{cases}$$

is an under-function.

THEOREM 10. *If* $\psi(x, y)$ *is an over-function, then* $\psi(x, y; \gamma)$ *is an over-function.*

Proof. Theorem 9 follows immediately from Definition 5 and Theorem 5, and Theorem 10 holds similarly.

POSTULATE 3. For any $\kappa \in \{\gamma\}$, and for any collection $\{f_\nu(x, y)\}$ of functions $f_\nu(x, y)$ which are continuous and uniformly bounded on κ, the functions $F(x, y; f_\nu; \kappa)$ are equicontinuous in K.

We shall use the following well-known and easily established result in connection with Postulate 3.

LEMMA 1. *For any collection* $\{U_\nu(x, y)\}$ *of functions* $U_\nu(x, y)$ *which are uniformly bounded and equicontinuous on a set* E, *the functions* $S(x, y)$ *and* $I(x, y)$, *defined by*

$$S(x, y) \equiv \sup_{U_\nu \in \{U_\nu\}} [U_\nu(x, y)],$$

$$I(x, y) \equiv \inf_{U_\nu \in \{U_\nu\}} [U_\nu(x, y)],$$

are continuous on E.

POSTULATE 4. For any bounded connected open subset Ω of D with boundary ω such that $\overline{\Omega} \subset D$, and for any bounded function $h(\alpha)$ defined on ω, there exists an under-function $\phi(x, y)$, and there exists an over-function $\psi(x, y)$.

DEFINITION 7. By $H_*(x, y)$ and $H^*(x, y)$ we shall denote the functions defined by

$$H_*(x, y) \equiv \sup_{\phi \in \{\phi\}} [\phi(x, y)],$$

$$H^*(x, y) \equiv \inf_{\psi \in \{\psi\}} [\psi(x, y)],$$

where $\{\phi\}$ and $\{\psi\}$ denote the familities of under- and over-functions, respectively.

The existence of the functions $H_*(x, y)$ and $H^*(x, y)$ follows from Postulate 4 and Theorem 6.

THEOREM 11. *The function* $H_*(x, y)$ *is a subfunction in* Ω.

Proof. We shall show first that for each $\kappa \in \{\gamma\}$, with $\overline{K} \subset \Omega$, the function $H_*(x, y)$ is continuous in K, so that $H_*(x, y)$ is continuous in Ω.

Let $\phi_0(x, y)$ and $\psi_0(x, y)$ be fixed members of $\{\phi(x, y)\}$ and $\{\psi(x, y)\}$, respectively; and for each $\phi(x, y) \in \{\phi(x, y)\}$ define

$$\Phi(x, y) \equiv \max [\phi(x, y), \phi_0(x, y)].$$

Using Theorem 4, we readily verify that $\Phi(x, y)$ satisfies the conditions of Definition 5, so that $\Phi(x, y)$ is an under-function; also, by Theorem 9, $\Phi(x, y; \kappa)$ is an under-function.

Since $\Phi(x, y; \kappa)$ is an under-function, and

$$\phi(x, y) \le \Phi(x, y) \le \Phi(x, y; \kappa),$$

we have

$$H_*(x, y) = \sup_{\Phi \in \{\Phi\}} \Phi(x, y; \kappa); \tag{8}$$

further, using Theorem 6, for (x, y) in K we obtain

$$\phi_0(x, y) \le \Phi(x, y; \kappa) = F(x, y; \Phi; \kappa) \le \psi_0(x, y). \tag{9}$$

It now follows from (8), (9), Postulate 3, and Lemma 1 that $H_*(x, y)$ is continuous in K, so that $H_*(x, y)$ is continuous in Ω.

Now, for any circle $\kappa \in \{\gamma\}$ with center (x_0, y_0) and $\overline{K} \subset \Omega$, and for any $\phi \in \{\phi\}$, by the definition of $H_*(x, y)$ we have

$$\phi(x, y) \le H_*(x, y) \qquad \text{on } \kappa;$$

therefore, since $\phi(x, y)$ is a subfunction in Ω, we have

$$\phi(x_0, y_0) \le F(x_0, y_0; \phi, \kappa) \le F(x_0, y_0; H_*; \kappa),$$

whence, again by the definition of $H_*(x, y)$, it follows that

$$H_*(x_0, y_0) \le F(x_0, y_0; H_*; \kappa).$$

Accordingly, by Theorem 3, $H_*(x, y)$ is a subfunction in Ω.

THEOREM 12. *The function $H_*(x, y)$ is a superfunction in Ω.*

Proof. Let there be given any $\epsilon > 0$ and any $\gamma \in \{\gamma\}$ with $\overline{\Gamma} \subset \Omega$. Then, by the definition of $H_*(x, y)$, for any $(x_0, y_0) \in \overline{\Gamma}$ there is a $\phi_0 \in \{\phi\}$ such that

$$\phi_0(x_0, y_0) > H_*(x_0, y_0) - \frac{\epsilon}{2};$$

therefore, by continuity, there is a circle $\kappa_0 \in \{\gamma\}$, with center (x_0, y_0) and $\overline{K}_0 \subset \Omega$, such that

$$\phi_0(x, y) > H_*(x, y) - \epsilon \qquad \text{in } \overline{K}_0. \tag{10}$$

Since the circular discs K_0 form an open covering of $\overline{\Gamma}$, by the Heine-Borel Theorem there exists a finite subcovering; let $\phi_1(x, y), \phi_2(x, y), \cdots, \phi_n(x, y)$ be the associated under-functions, and let $\phi(x, y)$ be defined by

$$\phi(x, y) \equiv \max\ [\phi_1(x, y), \phi_2(x, y), \cdots, \phi_n(x, y)].$$

Then, by Theorem 7, $\phi(x, y)$ is an under-function; and, by (10), we have

$$\phi(x, y) > H_*(x, y) - \epsilon \qquad \text{in } \overline{\Gamma}.$$

By Postulate 2, then, we obtain

$$(11) \qquad F(x, y; \phi; y) > F(x, y; H_* - \epsilon; y) \geq F(x, y; H_*; y) - \epsilon \qquad \text{in } \overline{\Gamma}.$$

Since for $(x, y) \in \overline{\Gamma}$ and any $\phi \in \{\phi\}$ we also have

$$(12) \qquad H_*(x, y) \geq \phi(x, y; y) = F(x, y; \phi; y),$$

it follows from (11) and (12) that

$$H_*(x, y) \geq F(x, y; H_*; y) - \epsilon \qquad \text{in } \overline{\Gamma}.$$

Thus since $\epsilon > 0$ is arbitrary, we have

$$H_*(x, y) \geq F(x, y; H_*; y) \qquad \text{in } \overline{\Gamma}.$$

so that $H_*(x, y)$ is a superfunction in Ω.

Since Theorems 11 and 12 hold also for the function $H^*(x, y)$, and since the only functions which are both subfunctions and superfunctions in Ω are $\{F(x, y)\}$-functions in Ω (see Definition 4), we have the following result:

THEOREM 13. *The functions $H_*(x, y)$ and $H^*(x, y)$ are $\{F(x, y)\}$-functions in Ω.*

We now turn our attention to the behavior of the functions $H_*(x, y)$ and $H^*(x, y)$ at the boundary ω of Ω.

4. **Regular boundary points; barrier functions.** We make the following definition.

DEFINITION 8. The point $\alpha_0 \in \omega$ is a *regular* boundary point of Ω relative to $\{F(x, y)\}$ provided that for every bounded function $h(\alpha)$ on ω the functions $H_*(x, y)$ and $H^*(x, y)$ satisfy (6) at α_0:

$$(13a) \qquad h_*(\alpha_0) \leq \liminf_{A \to \alpha_0} H_*(A) \leq \limsup_{A \to \alpha_0} H_*(A) \leq h^*(\alpha_0),$$

(13b) $$h_*(\alpha_0) \leq \liminf_{A \to \alpha_0} H^*(A) \leq \limsup_{A \to \alpha_0} H^*(A) \leq h^*(\alpha_0).$$

THEOREM 14. *If all points of ω are regular boundary points of Ω, and $h(\alpha)$ is continuous on ω, then the Dirichlet problem for Ω, relative to $\{F(x, y)\}$ and $h(\alpha)$, has a unique solution.*

Proof. From (13) and the continuity of $h(\alpha)$ on ω, we see that $H_*(x, y)$ and $H^*(x, y)$ are continuous in $\overline{\Omega}$ and satisfy

$$H_*(\alpha) = H^*(\alpha) = h(\alpha) \qquad \text{on } \omega.$$

Accordingly, by Definitions 5 and 6 and Theorems 11 and 12, $H_*(x, y)$ is both an under-function and an over-function; similarly, $H^*(x, y)$ is both an under-function and an over-function. Therefore, by Theorem 6, we have

$$H_*(x, y) \equiv H^*(x, y) \qquad \text{in } \overline{\Omega}.$$

For the same reason, any other solution of the Dirichlet problem must coincide with $H_*(x, y)$ and $H^*(x, y)$ in $\overline{\Omega}$.

We shall now give local sufficient conditions in terms of barrier functions (see [3, pp. 326-328]) in order that a point $\alpha \in \omega$ be a regular point of Ω; in the next section we shall study conditions under which barrier functions exist.

DEFINITION 9. For a point $\alpha_0 = (x_0, y_0) \in \omega$, a circle κ with center at α_0 and with $\overline{K} \subset D$, and constants $\epsilon > 0$, M, and N, a function

$$s(x, y) \equiv s(x, y; \kappa; \epsilon, M, N)$$

is a *barrier subfunction* provided:

a) $s(x, y)$ is continuous in $\overline{\Omega} \cap \overline{K}$,

b) $s(x, y)$ is a subfunction in $\Omega \cap K$,

c) $s(\alpha_0) \geq N - \epsilon$,

d) $s(x, y) \leq N + 2\epsilon$ on $\omega \cap K$,

e) $s(x, y) \leq M$ on $\overline{\Omega} \cap \kappa$.

DEFINITION 10. With the notation of Definition 9, a function

$$S(x, y) \equiv S(x, y; \kappa; \epsilon, M, N)$$

is a *barrier superfunction* provided

a) $S(x, y)$ is continuous in $\overline{\Omega} \cap \overline{K}$,

b) $S(x, y)$ is a superfunction in $\Omega \cap K$,

c) $S(\alpha_0) \leq N + \epsilon$

d) $S(x, y) \geq N - 2\epsilon$ on $\omega \cap K$,

e) $S(x, y) \geq M$ on $\overline{\Omega} \cap \kappa$.

THEOREM 15. *If for the point $\alpha_0 \in \omega$, and for each set of constants $\epsilon > 0$, M, and N, there exists a sequence of circles $\kappa_n = \kappa_n(\alpha_0)$ with center at α_0 and radii $\rho_n(\alpha_0) \longrightarrow 0$ for which barrier subfunctions $s(x, y; \kappa_n; \epsilon, M, N)$ and barrier superfunctions $S(x, y; \kappa_n; \epsilon, M, N)$ exist, then α_0 is a regular boundary point of Ω relative to $\{F(x, y)\}$.*

Proof. For a given bounded function $h(\alpha)$ defined on ω, it follows from Theorem 6 and Definition 7 that

$$H_*(x, y) \leq H^*(x, y) \qquad \text{in } \overline{\Omega},$$

so that

$$\liminf_{A \to \alpha_0} H_*(A) \leq \liminf_{A \to \alpha_0} H^*(A)$$

and

$$\limsup_{A \to \alpha_0} H_*(A) \leq \limsup_{A \to \alpha_0} H^*(A).$$

Accordingly, in order to verify (12) and thus prove the theorem, we need only show that

$$(14) \qquad h_*(\alpha_0) \leq \liminf_{A \to \alpha_0} H_*(A)$$

and

$$(15) \qquad \limsup_{A \to \alpha_0} H^*(A) \leq h^*(\alpha_0).$$

For a given $\epsilon > 0$ there is a circle κ satisfying the hypotheses of the theorem and for which

$$h_*(\alpha_0) - \epsilon \le h_*(\alpha) \le h^*(\alpha) \le h^*(\alpha_0) + \epsilon \qquad \text{on } \omega \cap \overline{K}.$$

For a fixed $\delta > 0$ and for any under-function $\phi(x, y)$, let

$$M = \min_{(x,y) \in \overline{\Omega} \cap \kappa} [\phi(x, y) - \delta],$$

$$N = h_*(\alpha_0) - 3\epsilon,$$

and

$$s(x, y) = s(x, y; \kappa; \epsilon, M, N).$$

Consider the function $\Phi(x, y)$, defined by

$$\Phi(x, y) \equiv \begin{cases} \max[\phi(x, y), s(x, y)] & \text{in } \overline{\Omega} \cap \overline{K}, \\ \phi(x, y) & \text{in } \overline{\Omega} - \overline{K}; \end{cases}$$

we shall show that $\Phi(x, y)$ is an under-function. Since we have

$$s(x, y) \le M < \phi(x, y) \qquad \text{on } \overline{\Omega} \cap \kappa,$$

it follows readily that

$$\Phi(x, y) = \phi(x, y) \qquad \text{on } \overline{\Omega} \cap \kappa,$$

and accordingly that $\Phi(x, y)$ is continuous in $\overline{\Omega}$. Further, $\Phi(x, y)$ is a subfunction in $\Omega - \overline{K}$, since $\Phi(x, y) = \phi(x, y)$ there; $\Phi(x, y)$ is a subfunction in $\Omega \cap K$ by Theorem 4 and Definitions 5 and 9; and for a point $\alpha \in \Omega \cap \kappa$ we have $s(\alpha) < \phi(\alpha) - \delta$, so that there is a circle about α in which $\Phi(x, y) = \phi(x, y)$; thus, by Theorem 3, $\Phi(x, y)$ is a subfunction in Ω; also, by Definition 9 and the choice of κ and N, we have

$$s(\alpha) \le N + 2\epsilon = h_*(\alpha_0) - \epsilon \le h_*(\alpha) \qquad \text{on } \omega \cap \overline{K},$$

and therefore, since

$$\phi(\alpha) \le h_*(\alpha) \qquad \text{on } \omega,$$

we have

$$\Phi(\alpha) \le h_*(\alpha) \qquad \text{on } \omega.$$

By Definition 5 we have thus shown that $\Phi(x, y)$ is an under-function.

By the choice of N and the definitions of $s(x, y)$ and $\Phi(x, y)$, we have

$$h_*(\alpha_0) - 4\epsilon = N - \epsilon \le s(\alpha_0) \le \Phi(\alpha_0),$$

so that by continuity there is a neighborhood of α_0 in whose intersection with $\overline{\Omega} \cap \overline{K}$ we have

$$h_*(\alpha_0) - 5\epsilon \le \Phi(x, y)$$

and consequently

$$h_*(\alpha_0) - 5\epsilon \le H_*(x, y).$$

Since $\epsilon > 0$ is arbitrary, (14) now follows; and (15) can be established similarly.

5. The existence of barrier functions. Relative to the Laplace partial differential equation (1), a criterion of Poincaré [3, p. 329] for α_0 to be a regular boundary point of Ω is that there should exist a circle κ with

$$\overline{\Omega} \cap \overline{K} = \alpha_0. \tag{16}$$

We shall now adjoin postulates concerning the family $\{F(x, y)\}$ under which (16) is a sufficient condition for the existence of barrier sub- and superfunctions at α_0, and therefore, by Theorem 15, for α_0 to be a regular boundary point of Ω relative to $\{F(x, y)\}$.

POSTULATE 5. For any circle $\kappa \in \{\gamma\}$, and any real number M, there exist continuous functions $f_1(x, y)$, $f_2(x, y)$, defined on κ, such that

$$F(x, y; f_1; \kappa) \le M, \quad F(x, y; f_2; \kappa) \ge M \qquad \text{in } \overline{K}.$$

POSTULATE 6. For any circle $\kappa \in \{\gamma\}$, and any real numbers $\epsilon > 0$ and N, there exists a continuous function $f(x, y)$ defined on κ such that

$$|F(x_0, y_0; f; \kappa) - N| \le \epsilon,$$

where (x_0, y_0) is the center of κ.

POSTULATE 7. For any circle $\kappa \in \{\gamma\}$; if the functions $f_j(x, y)$ $(j = 0, 1, \cdots)$, defined on κ, are continuous and uniformly bounded on κ, and

$$\lim_{j\to\infty} f_j(x, y) = f_0(x, y)$$

for all but at most a finite number of points of κ, then

$$\lim_{j\to\infty} F(x, y; f_j; \kappa) = F(x, y; f_0; \kappa)$$

for all points of K.

POSTULATE 8. For any circle $\kappa \in \{\gamma\}$, if the functions $f_j(x, y)$ $(j = 1, 2, \cdots)$, defined on κ, are continuous on κ and equicontinuous at a point $(x_0, y_0) \in \kappa$, then the functions $F(x, y; f_j; \kappa)$ $(j = 1, 2, \cdots)$, defined in $\overline{K}$, are equicontinuous at (x_0, y_0).

THEOREM 16. *If for the point $\alpha_0 \in \omega$ there exists a circle κ, with $\overline{K} \subset D$, such that*

$$\overline{\Omega} \cap \overline{K} = \alpha_0,$$

then α_0 is a regular boundary point of Ω relative to $\{F(x, y)\}$.

Proof. Since the conclusions of Theorems 15 and 16 are identical, in order to prove Theorem 16 we need only to show that its hypothesis implies that of Theorem 15. Explicitly, we shall give the construction of a barrier subfunction for a suitable circle $\kappa_1(\alpha_0)$ with center at α_0 and inside an arbitrary circle κ_0 with center at α_0, as prescribed in Theorem 15; the existence of barrier superfunctions can be treated similarly.

Let the circle $\kappa_0 \subset D$ be drawn with center at $\alpha_0 = (x_0, y_0)$ and intersecting κ. By Postulate 6 there is a continuous function $f(x, y)$ defined on κ_0 such that

$$|F(x_0, y_0; f; \kappa_0) - N| < \epsilon. \tag{17}$$

By continuity, there is a circle $\kappa_1 \subset K_0$, with center at α_0, such that

$$F(x, y; f; \kappa_0) \le N + 2\epsilon \qquad \text{in } \overline{K}_1. \tag{18}$$

Now we define

$$R = \min_{(x, y) \in \overline{K}_1} F(x, y; f; \kappa_0)$$

and

$$M_* = \min(M, N, R).$$

By Postulate 5, there exists a continuous function $f_1(x, y)$ defined on κ_1 such that

$$(19) \qquad F(x, y; f_1; \kappa_1) \leq M_* \qquad \text{in } \overline{K}_1.$$

Let B be the intersection of the line of centers of κ and κ_1 with the arc of κ_1 lying outside $\overline{K}$, and let B_1', B_1'', B_2', B_2'' be points of κ_1 near B arranged in the order $B_2'\ B_1'\ B\ B_1''\ B_2''$ around κ_1.

We define the function $f_2(x, y)$ on κ_1 as follows:

$$f_2(x, y) = f_1(x, y) - 1 \qquad \text{on arc } B_1'\ B\ B_1'';$$

$$f_2(x, y) = F(x, y; f; \kappa_0) \qquad \text{on long arc } B_2'\ B_2'';$$

$$f_2(x, y) = l'(\theta) \qquad \text{on arc } B_1'\ B_2';$$

$$f_2(x, y) = l''(\theta) \qquad \text{on arc } B_1''\ B_2'';$$

the functions $l'(\theta)$ and $l''(\theta)$ are linear functions of the central angle of κ_1, such that $f_2(x, y)$ is continuous on κ_1.

For (x, y) on κ_1, we set

$$f_3(x, y) = \min[f_2(x, y), F(x, y; f; \kappa_0)].$$

By (17) and Postulate 7, we can take the arc $B_2'\ B\ B_2''$ small enough that

$$(20) \qquad |F(x_0, y_0; f_3; \kappa_1) - N| \leq \epsilon.$$

Further, since

$$f_3(x, y) \leq F(x, y; f; \kappa_0) \qquad \text{on } \kappa_1,$$

by (18) and Postulate 2 we have

$$(21) \qquad F(x, y; f_3; \kappa_1) \leq N + 2\epsilon \qquad \text{in } \overline{K}_1.$$

Let $Q \in K_1$ be a point on the open line-segment $\alpha_0 B$, and sufficiently close to B that

$$F(Q; f_3; \kappa_1) < F(Q; f_1; \kappa_1). \tag{22}$$

Let κ' and κ'' be the two circles through Q and α_0, and tangent to κ_1. Let ρ be the length of the common chord $\alpha_0 Q$ of κ' and κ'', or the length of the common chord of κ' and κ, whichever is less, and choose the constant C so that

$$C\rho > M^*, \tag{23}$$

where

$$M^* = \max_{(x,y)\in \overline{K}_1} |F(x, y; f_1; \kappa_1)| + \max_{(x,y)\in \overline{K}_1} |F(x, y; f_3; \kappa_1)|.$$

We now define continuous functions $h_n'(x, y)$ and $h_n''(x, y)$ on κ' and κ'', respectively, as follows:

$$\begin{aligned} h_1'(x, y) &= F(x, y; f_3; \kappa_1) - C[(x-x_0)^2 + (y-y_0)^2]^{1/2} && \text{on } \kappa', \\ h_1''(x, y) &= F(x, y; f_3; \kappa_1) - C[(x-x_0)^2 + (y-y_0)^2]^{1/2} && \text{on } \kappa'', \end{aligned} \tag{24}$$

and, for $n = 2, 3, \cdots$,

$$\begin{aligned} h_n'(x, y) &= \begin{cases} h_1'(x, y) & \text{on } \kappa' - \overline{K}'', \\ F(x, y; h_{n-1}''; \kappa'') & \text{on } \kappa' \cap \overline{K}'', \end{cases} \\ h_n''(x, y) &= \begin{cases} h_1''(x, y) & \text{on } \kappa'' - \overline{K}' \\ F(x, y; h_n'; \kappa') & \text{on } \kappa'' \cap \overline{K}'. \end{cases} \end{aligned} \tag{25}$$

Let

$$G = \max_{(x,y)\in \overline{K}'} |F(x, y; h_1'; \kappa')| + \max_{(x,y)\in \overline{K}''} |F(x, y; h_1''; \kappa'')|;$$

then by Postulate 5 there is a continuous function $f_4(x, y)$ defined on κ_1 such that

$$F(x, y; f_4; \kappa_1) \leq -G \qquad \text{in } \overline{K}_1.$$

It follows from Postulate 2 and the definitions of the $h_n'(x, y)$ and $h_n''(x, y)$ that for each positive integer n we have

$$F(x, y; f_4; \kappa_1) \leq F(x, y; h_n'; \kappa') \leq F(x, y; f_3; \kappa_1) \qquad \text{in } \overline{K}',$$

and

$$F(x, y; f_4; \kappa_1) \leq F(x, y; h_n''; \kappa'') \leq F(x, y; f_3; \kappa_1) \qquad \text{in } \overline{K}''.$$

Hence, by Postulate 3 and Lemma 1, the functions $u'(x, y)$ and $u''(x, y)$, defined by

$$u'(x, y) = \sup_{h_n' \in \{h_n'\}} F(x, y; h_n'; \kappa')$$

and

$$u''(x, y) = \sup_{h_n'' \in \{h_n''\}} F(x, y; h_n''; \kappa'')$$

are continuous in K' and K'', respectively; indeed, by Postulate 8 and by Lemma 1 applied to the sets $(\overline{K}' - \alpha_0 - Q)$ and $(\overline{K}'' - \alpha_0 - Q)$, the functions $u'(x, y)$ and $u''(x, y)$ are continuous in $\overline{K}'$ and $\overline{K}''$, respectively, except possibly at the points α_0 and Q. As for the behavior of these functions at α_0, since by our construction we have

$$(26) \qquad F(x, y; h_1'; \kappa') \leq u'(x, y) \leq F(x, y; f_3; \kappa_1) \qquad \text{in } \overline{K}',$$

and

$$(27) \qquad F(x, y; h_1''; \kappa'') \leq u''(x, y) \leq F(x, y; f_3; \kappa_1) \qquad \text{in } \overline{K}'',$$

and since the functions at the extremes of these inequalities have equal values at α_0 and are continuous at α_0, it follows that $u'(x, y)$ and $u''(x, y)$ are continuous also at α_0.

As in the last part of the proof of Theorem 11, $u'(x, y)$ and $u''(x, y)$ can easily be shown to be subfunctions in K' and K'', respectively.

Now we define the function $u(x, y)$ in $\overline{K}' \cup \overline{K}''$ as follows:

$$(28)\qquad u(x,y) = \begin{cases} u'(x,y) & \text{in } \overline{K}' - K'' \\ u''(x,y) & \text{in } \overline{K}'' - \overline{K}' \\ \max\,[u'(x,y),\, u''(x,y)] & \text{in } \overline{K}' \cap K''. \end{cases}$$

Since $u'(x, y)$ and $u''(x, y)$ coincide on $\kappa' \cap K''$, on $\kappa'' \cap K'$, and at α_0, and both are continuous at α_0, it follows that $u(x, y)$ is continuous in $\overline{K}' \cup \overline{K}''$ except possibly at Q.

Clearly $u(x, y)$ is a subfunction in $K' - \overline{K}''$ and in $K'' - \overline{K}'$. By Theorem 4, $u(x, y)$ is a subfunction in $K' \cap K''$. Since in addition the hypothesis of Theorem 3 holds for each point of $\kappa' \cap K''$ and for each point of $\kappa'' \cap K'$, it follows that $u(x, y)$ is a subfunction throughout $K' \cup K''$.

To conclude the proof, we shall show that the function

$$(29)\qquad s(x,y) \equiv \begin{cases} F(x, y; f_1; \kappa_1) \text{ for } (x,y) \in \overline{\Omega} \cap [K_1 - (\overline{K}' \cup \overline{K}'')], \\ \max\,[F(x, y; f_1; \kappa_1),\, u(x,y)] \text{ for } (x,y) \in \overline{\Omega} \cap (\overline{K}' \cup \overline{K}'')], \end{cases}$$

satisfies all the conditions of Definition 9 for being a barrier subfunction for $\kappa_1 = \kappa_1(\alpha_0)$ as prescribed in Theorem 15.

Since, by (23), (24), (25), and the definitions of $u'(x, y)$, $u''(x, y)$, and $u(x, y)$, we have

$$(30)\qquad u(x, y) < F(x, y; f_1; \kappa_1)$$

on the part of $\kappa' \cup \kappa''$ which lies in Ω; since $u(x, y)$ is continuous on $\kappa' \cup \kappa''$ except possibly at Q; and since, by (22), (26), and (27), there is a neighborhood of Q in which (30) holds, it follows that $s(x, y)$ is continuous in $\overline{\Omega} \cap \overline{K}_1$.

That $s(x, y)$ is a subfunction follows from exactly the same kind of argument as the one used in discussing $u(x, y)$.

By (20), (24), (25), (26), and (27), we have

$$u(x_0, y_0) = F(x_0, y_0; f_3; \kappa_1) \geq N - \epsilon,$$

whence, by (29), we have also

$$s(x_0, y_0) \geq N - \epsilon.$$

It follows from (19) that

$$F(x, y; f_1; \kappa_1) \leq N + 2\epsilon \qquad \text{in } \overline{K}_1$$

and from (21), (26), (27), and (28) that

$$u(x, y) \leq N + 2\epsilon \qquad \text{in } \overline{K}' \cup \overline{K}'',$$

whence, by (29),

$$s(x, y) \leq N + 2\epsilon \qquad \text{on } \Omega \cap K.$$

Finally, by (19), (23), (24), (25), and the definitions of $u'(x, y)$, $u''(x, y)$, $u(x, y)$, and $s(x, y)$, we have

$$s(x, y) \leq M \qquad \text{on } \overline{\Omega} \cap \kappa_1.$$

Thus $s(x, y)$ satisfies all the conditions of Definition 9, and is a barrier subfunction $s(x, y; \kappa_1; \epsilon, M, N)$ as desired.

REFERENCES

1. E. F. Beckenbach, *Generalized convex functions*, Bull. Amer. Math. Soc. 43 (1937), 363-371.

2. F. F. Bonsall, *On generalized subharmonic functions*, Proc. Cambridge Philos. Soc. 46 (1950), 387-395.

3. O. D. Kellogg, *Potential theory*, Springer, Berlin, 1929.

4. J. E. Littlewood, *Theory of functions*, Oxford University Press, Oxford, 1944.

5. F. Riesz, *Über subharmonische Funktionen und ihre Rolle in der Funktionentheorie und in der Potentialtheorie*, Acta Sci. Math. Szeged 2 (1925), 87-100.

6. Georg Tautz, *Zur Theorie der elliptischen Differentialgleichungen. I.*, Math. Ann. **117** (1939-41), 694-726.

UNIVERSITY OF CALIFORNIA, LOS ANGELES