CONVEXITY PROPERTIES OF INTEGRAL MEANS OF ANALYTIC FUNCTIONS

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1. Introduction. Let $f = f(z)$ denote an analytic function of the complex variable z in the open circle $|z| < R$. For each positive number t, the mean of order t of the modulus of $f(z)$ is defined as follows:

$$\mathfrak{M}_t(r; f) = \left[\frac{1}{2\pi} \int_0^{2\pi} |f(re^{i\theta})|^t \, d\theta\right]^{1/t}, \qquad (0 \leq r < R).$$

The reader might consult [5, p. 143-144; 3; and 4, p. 134-146] for some of the properties of this mean value function $\mathfrak{M}_t(r; f)$.

We consider the question: does the analyticity in $|z| < R$ of the function f imply the convexity of the mean $\mathfrak{M}_t(r; f)$ as a function of r in the interval $0 \leq r < R$? It is known [1] that:

(A) Unless the function f is suitably restricted, the set of positive values t for which the question may be answered affirmatively has a finite upper bound.

(B) If the number t is of the form $2/k$, with k a positive integer, then, for every analytic function f, the mean of order t is convex.

(C) If the function f vanishes at the origin, then the mean $\mathfrak{M}_t(r; f)$ is convex for every fixed positive number t.

(D) If the function f has no zero in the circle, then its mean of order t is convex, provided that the positive number satisfies $t \leq 2$.

(E) If the function f has at most k zeros, $k \geq 1$, in the circle, then the mean of order t is convex provided that the positive number t satisfies $t \leqq 2/k$.

The main purpose of this paper is to prove that, for every analytic function f, the mean of order four is convex. Moreover, we show by example that if the number t is greater than 5.66, then there is an analytic function whose mean of order t is not convex.

2. Means of nonvanishing functions. Assume that $g(z)$ is analytic in $|z| < R$,

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and that the expansion for $g(z)$ in the given circle is

$$g(z) = \sum_{n=0}^{\infty} a_n z^n.$$

Then the integral

$$h(r; g) = \frac{1}{2\pi} \int_0^{2\pi} |g(re^{i\theta})|^2 \, d\theta$$

has the expansion

$$h(r; g) = \sum_{n=0}^{\infty} |a_n|^2 r^{2n},$$

valid in $r < R$. Let

$$Q(r; g, c) = hh'' - c(h')^2,$$

where primes denote differentiation with respect to r, h is the function $h(r; g)$, and c is a constant independent of the variable r and of the function g. If C is a class of functions $\{g(z)\}$, and if, for all functions g in this class C, for all $r < R$, and for a particular positive value c_0, the inequality

$$Q(r; g, c_0) \geq 0$$

holds, then the inequality

$$Q(r; g, c) \geq 0$$

holds for all $c < c_0$, all $r < R$, and all functions g in the class C. We now specify the class C to be the class of all functions $g(z)$ which are analytic and do not vanish in $|z| < R$. If $f(z)$ is in class C, then any single-valued branch of $[f(z)]^{\alpha}$ where α is an arbitrary real number, is also in class C. Given a function $f_0(z)$ in class C, and a fixed positive number t, let $g_0(z)$ be a single-valued branch of $[f_0(z)]^{t/2}$; and let

$$h_0(r) = \frac{1}{2\pi} \int_0^{2\pi} |g_0(z)|^2 \, d\theta.$$

Then

$$\mathfrak{M}_t(r; f_0) = [h_0]^{1/t};$$

and since h_0 is a nonvanishing function of r, we have

$$\frac{d^2\mathfrak{M}_t(r; f_0)}{dr^2} = P \cdot Q[r; g_0, (1 - 1/t)],$$

where

$$P = \frac{\mathfrak{M}_t(r; f_0)}{th_0^2} > 0.$$

Every function $g(z)$ in class C is a single-valued branch of $[f(z)]^{t/2}$, where $f(z)$ is some appropriate function in class C. Therefore, for positive values t, the mean $\mathfrak{M}_t(r; f)$ is a convex function of r for all functions f in class C if and only if

$$Q[r; g, (1 - 1/t)] \geq 0$$

for all functions g in class C. Since the inequality $1 - 1/t < 1 - 1/t_0$ holds for all t and t_0 satisfying $0 < t < t_0$, we conclude from the preceding remarks that, if the positive value t_0 is such that the mean $\mathfrak{M}_{t_0}(r; f)$ is convex for all nonvanishing $f(z)$, then the mean $\mathfrak{M}_t(r; f)$ is convex for all nonvanishing $f(z)$, provided that t is any positive value not exceeding t_0.

For a simple example of a function $\mathfrak{M}_t(r; f)$ which is not convex, consider the mean of order eight of a single-valued branch of

$$f(z) = \sqrt{1 + z} \qquad \text{in } |z| < 1.$$

In this case, we have

$$h(r) = 1 + 4r^2 + r^4;$$

and $[h(r)]^{1/8}$ is not convex in $0 \leq r < 1$.

Since, for every analytic function f, the mean of order two is convex, it now follows that there exists a greatest positive value t_0, in the range $2 \leq t_0 < 8$, such that $\mathfrak{M}_{t_0}(r; t)$ is convex for all nonvanishing analytic functions. It will be a corollary of our result that this greatest value t_0 satisfies the inequalities $4 \leq t_0 < 5.66$.

3. **Preliminary lemmata.** The proof of our main theorem will be based on the following lemmata.

LEMMA 1. *Let* a_i $(i = 1, 2, \cdots)$ *be a sequence of positive numbers such* *that the sum*

$$\sum_{i=1}^{\infty} 1/a_i$$

converges to the finite value M. If the sequence of real variables x_i $(i = 1, 2, \cdots)$ *is restricted to satisfy the inequality*

$$\sum_{i=1}^{\infty} a_i x_i^2 \leq B,$$

then the maximum value of the function

$$f = \sum_{i=1}^{\infty} x_i$$

is $(BM)^{1/2}$.

Proof. We consider first maximizing

$$f_n = \sum_{i=1}^{n} x_i,$$

with the variables subject to the condition

$$\sum_{i=1}^{n} a_i x_i^2 = B.$$

Let

$$M_n = \sum_{i=1}^{n} 1/a_i.$$

The critical points of the function f_n are at the solutions of the simultaneous equations

$$a_i x_i = a_j x_j \qquad (i, j = 1, \cdots, n),$$

which are given by

$$x_i^2 = B(M_n a_i^2), \qquad (i = 1, \cdots, n).$$

Therefore, the maximum f_n is $M_n (B/M_n)^{1/2}$ or $(BM_n)^{1/2}$. Since $M_n < M$, and all the values a_i are positive, it follows that for all n the partial sums f_n are bounded by $(BM)^{1/2}$ and the conclusion of the lemma follows.

LEMMA 2. *Let S be the sum*

$$S = \sum_{n=3}^{\infty} 1/(6n^2 - 9n + 2).$$

Then this sum S is less than 0.09504.

Proof. The function $f(n) = 1/(6n^2 - 9n + 2)$ has the following expansion in powers of $1/(n-1)$:

$$f(n) = \sum_{k=2}^{\infty} a_k/(n-1)^k,$$

with $a_2 = 1/6$, $a_3 = -1/12$, $a_4 = 5/72$. For determining subsequent values of a_k, it is convenient to use the recursion formula:

$$a_{k+2} = (a_k - 3a_{k+1})/6.$$

The coefficients a_2 and a_3 are positive and negative respectively. Therefore it follows directly from the recursion formula that the general coefficients a_k alternate in sign. By another use of the recursion formula, we see that the sum $a_k + a_{k+1}$ is equal to $(a_{k-2} - a_{k-1})/12$, and therefore that the sign of the sum $a_k + a_{k+1}$ is the same as that of the coefficient a_{k-2}, or of the coefficient a_k. Since the inequalities $|a_2| > |a_3| > |a_4|$ hold, it now follows that the numerical values of the coefficients all decrease with increasing k. Let $\zeta(k)$ be the Riemann zeta-function, and let $s(k) = \zeta(k) - 1$. Since the foregoing expansion for $f(n)$ is an absolutely convergent series, the sum S may be expanded in an alternating series of the form

$$S = \sum_{k=2}^{\infty} a_k \, s(k),$$

whose terms decrease in numerical value with increasing k. Using (see [2]) the approximations $s(2) = 0.644935$, $s(4) = 0.082324$, $s(6) = 0.017344$, $s(8) = 0.004078$, $s(10) = 0.000995$, which are too large, and the approximations $s(3) =$

0.202056, $s(5) = 0.036927$, $s(7) = 0.008349$, $s(9) = 0.002008$, which are too small, we obtain the value 0.09504 stated in the lemma by summing this last series up to and including the term for $k = 10$.

LEMMA 3. *Let*

$$y = \sqrt{x} + \sqrt{0.04752}\,\sqrt{9x^2 - 10x + 1}\,,$$

where x lies in the range $0 \le x \le 1/9$. Then the maximum value of y is less than $(\sqrt{2} - 1)$.

Proof. Setting the derivative of y equal to zero, we find that the value of x maximizing y is the solution of the equation

$$0.04752x(10 - 18x)^2 - (9x^2 - 10x + 1) = 0.$$

This critical value of x lies between 0.07 and 0.08. Therefore

$$\max y < \sqrt{0.08} + \sqrt{0.04752\,[9(0.07)^2 - 10(0.07) + 1]}$$

$$< 0.283 + 0.129 = 0.412.$$

Since $(\sqrt{2} - 1)$ is greater than 0.414, the conclusion of the lemma follows.

4. The mean of order four. Let

$$g(z) = [f(z)]^2$$

have the expansion

$$g(z) = \sum_{n=0}^{\infty} a_n z^n,$$

valid in $|z| < R$. Following the ideas developed in § 2, we see that

$$\mathfrak{M}_4(r; f) = [h(r)]^{1/4},$$

with

$$h(r) = \sum_{n=0}^{\infty} |a_n|^2 r^{2n},$$

and that $\mathfrak{M}_4(r; f)$ is convex in $r < R$ if and only if

$$Q(r) \equiv hh'' - \frac{3}{4}(h')^2 = \sum_{i,j=0}^{\infty} Q_{ij}\, p_i\, p_j\, r^{2(i+j)-2},$$

with

$$Q_{ij} = i(2i-1) + j(2j-1) - 3ij \text{ and } p_i = |a_i|^2,$$

is nonnegative in the interval $0 \leqq r < R$. The only coefficient Q_{ij} which is negative is $Q_{11} = -1$. That the mean of order four is convex may be concluded from the following theorem.

THEOREM. *If a function $g(z)$ is analytic in the circle $|z| < R$, and the function*

$$\left[\frac{1}{2\pi}\int_0^{2\pi} |g(re^{i\theta})|^2\, d\theta\right]^{1/4}$$

is not convex as a function of r in the interval $r < R$, then $g(z)$ is not the square of an analytic function in $|z| < R$.

Proof. It is pointed out in the introduction that if $f(0) = 0$, then the mean $\mathfrak{M}_t(r; f)$ is convex for all t. Therefore we may assume that

$$[f(0)]^2 = g(0) = p_0$$

is not zero. The hypothesis of the theorem implies that

$$Q(r) = \sum_{i,j=0}^{\infty} Q_{ij}\, p_i\, p_j\, r^{2(i+j)-2}$$

takes on negative values; since Q_{11} is the only negative coefficient, this is possible only if the value $p_1 = |a_1|^2$ is not zero. Therefore, we may make the normalizations

$$a_0 = 1,\ a_1 = \sqrt{2},\ p_0 = 1, \text{ and } p_1 = 2.$$

Let

$$Q_1(r) = 2p_0 p_1 + (12p_0 p_2 - p_1{}^2)r^2 + 2p_1\, p_2\, r^4$$
$$+ 2\sum_{n=3}^{\infty} (Q_{0n}\, p_0\, p_n\, r^{2n-2} + Q_{1n}\, p_1\, p_n\, r^{2n}),$$

with $Q_{0n} = n(2n-1)$ and $Q_{1n} = 2n^2 - 4n + 1$. Since $Q(r) \geq Q_1(r)$, and $Q_1(r)$ can be negative only for values of r satisfying

$$2p_0\, p_1 - p_1^2 r^2 < 0,$$

we have in the normalized case the result that $Q_1(r)$ is negative for some $r > 1$; and the expression

$$Q_2(r) = 4 + (12p_2 - 4)r^2 + \left[4p_2 + \sum_{n=3}^{\infty} (12n^2 - 18n + 4)p_n\right] r^4$$

also takes on negative values. The discriminant of $Q_2(r)$ as a quadratic form in r^2 must be positive. Therefore we have the inequality

$$\sum_{n=3}^{\infty} (6n^2 - 9n + 2)\, p_n < (9p_2^2 - 10p_2 + 1)/2,$$

and the result that p_2 is less than 1/9. Applying Lemma 1, we see that

$$\sum_{n=3}^{\infty} |a_n| < \sqrt{S(9p_2^2 - 10p_2 + 1)/2},$$

with

$$S = \sum_{n=3}^{\infty} 1/(6n^2 - 9n + 2).$$

By use of Lemma 2, we have

$$\sum_{n=2}^{\infty} |a_n| < \sqrt{p_2} + \sqrt{0.04752}\,\sqrt{9p_2^2 - 10p_2 + 1};$$

and, by use of Lemma 3, we have

$$\sum_{n=2}^{\infty} |a_n| < \sqrt{2} - 1.$$

Applying Rouché's Theorem to the function

$$g(z) = 1 + \sqrt{2}\, z + \sum_{n=2}^{\infty} a_n z^n$$

we see that, if the function $g(z)$ is analytic in the circle $|z| \leq 1$, then $g(z)$ has exactly one zero within this circle, and therefore that $g(z)$ is not the square of an analytic function in this circle. Since the convexity of the mean must break down only for values of r greater than one, we have established the theorem.

5. Examples of nonconvex means. Let $f(z)$ be a single-valued branch of the function $[(1-z)^2/(1-\epsilon z)]^{2/t}$, with $\epsilon = 0.19$. We shall show that if $t \geq 5.66$, then the mean $\mathfrak{M}_t(r; f)$ is not convex in $r < 1$. Since

$$[f(z)]^{t/2} = 1 + (-2+\epsilon) z + [(1-\epsilon)^2 z^2/(1-\epsilon z)],$$

it follows that

$$\mathfrak{M}_t(r; f) = [h(r)]^{1/t},$$

with

$$h(r) = 1 + (4 - 4\epsilon + \epsilon^2) r^2 + [(1-\epsilon)^4 r^4/(1-\epsilon^2 r^2)].$$

By straight-forward calculation, we have

$$(1+\epsilon)\, h(1) = 6 - 2\epsilon = 5.62; \quad (1+\epsilon)^2\, h'(1) = 12 - 4\epsilon^2 = 11.8556;$$

$$(1+\epsilon)^3\, h''(1) = 20 + 4\epsilon - 4\epsilon^2 - 4\epsilon^3 = 20.588164;$$

and

$$\begin{aligned}(1+\epsilon)^4\, Q(r) &= (1+\epsilon)^4\, [hh'' - (1 - 1/t)\,(h')^2] \\ &\leq (1+\epsilon)^4\, [115.71 - (1-1/t)\,(140.55)] \\ &< 0, \text{ if } t > 140.55/24.84, \text{ and therefore if } t \geq 5.66.\end{aligned}$$

Thus we have examples of nonconvex means $\mathfrak{M}_t(r; f)$ for $t \geq 5.66$ even under the restriction that $f(z)$ does not vanish in its circle of analyticity.

References

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