ITERATES OF ARITHMETIC FUNCTIONS AND A PROPERTY OF THE SEQUENCE OF PRIMES

HAROLD N. SHAPIRO

1. Introduction. In a previous paper [2], the author has investigated certain properties of the iterates of arithmetic functions which are of the following form. For $n = \prod p_i^{\alpha_i}$,

$$f(n) = \prod f(p_i)\,[A(p_i)]^{\alpha_i - 1}, \tag{1.1}$$

where $f(p_i)$ is an integer, $1 < f(p_i) < p_i$, and $A(p_i)$ is an integer $\leq p_i$, for odd primes p_i; whereas $f(2) = 1$, $A(2) = 2$. We shall denote the set of these arithmetic functions by K. These conditions ensure that for $n > 2$, $f(n) < n$, and hence if $f^k(n)$ denotes the k-th iterate of f there is a unique integer k such that

$$f^k(n) = 2. \tag{1.2}$$

For this k we write $k = C_f(n)$. We define

$$C_f(1) = C_f(2) = 0.$$

In this paper we propose to consider the problem of determining a $g \in K$ such that for all odd primes p, and all $f \in K$,

$$C_g(p) \geq C_f(p). \tag{1.3}$$

The solution to this problem produces an interesting property of the sequence of primes in that we shall show that (1.3) is equivalent to having g skip down through the primes. More precisely, if $p_1 = 2$, $p_2 = 3$, $\cdots$, and in general p_i denotes the i-th prime, (1.3) is equivalent to having $g(3) = 2$, $g(5) = 4$ or 3, and

$$g(p_i) = p_{i-1} \qquad \text{for } i > 3. \tag{1.4}$$

2. A theorem concerning functions of K**.** In carrying out the proof of the result

Received September 10, 1952.
Pacific J. Math. 3 (1953), 647-655

stated in the introduction, we shall require a certain property of the iterates of the functions of K, which we now derive.

For $n = \prod p_i^{\alpha_i}$, we define the arithmetic function $A(n)$ as

$$A(n) = \prod [A(p_i)]^{\alpha_i},$$

where the $A(p_i)$ are as given in (1.1). It then follows that, for all integers m and n,

$$A(mn) = A(m)A(n) \text{ and } A(n) \le n.$$

LEMMA 2.1. *For any divisor d of n, we have for $f \in K$,*

$$\frac{A(d)f(n)}{f(d)} \le n, \tag{2.1}$$

where $A(d)f(n)/f(d)$ is an integer.

Proof. We can write

$$\begin{aligned} f(n) &= A(n) \prod_{p \mid n} \frac{f(p)}{A(p)} \\ &= A(n) \prod_{p \mid d} \frac{f(p)}{A(p)} \prod_{\substack{p \mid n \\ p \nmid d}} \frac{f(p)}{A(p)} \\ &= A(n) \frac{f(d)}{A(d)} \cdot \frac{f(d')}{A(d')}, \end{aligned}$$

where

$$d' = \prod_{\substack{p \mid n \\ p \nmid d}} p,$$

so that d' divides n. Since $A(n)$ is completely multiplicative, we have then

$$A(n) = A\left(\frac{n}{d'}\right) A(d'), \quad \text{or} \quad \frac{A(n)}{A(d')} = A\left(\frac{n}{d'}\right).$$

Hence

$$\frac{A(d)f(n)}{f(d)} = A\left(\frac{n}{d'}\right) f(d') \le n,$$

where clearly $A(d)\, f(n)/f(d)$ is an integer.

LEMMA 2.2. *For $f \in K$, if $e(n) = 0$ or 1 according as n is odd or even,*

$$C_f(2n) \leq C_f(n) + e(n). \tag{2.2}$$

Proof. Since $f \in K$, we have $f(2) = 1$, $A(2) = 2$, and hence

$$f(2n) = 2f(n) \quad \text{or} \quad f(n),$$

where if n is odd $f(2n) = f(n)$ and $C_f(2n) = C_f(n)$. Otherwise, continuing, we have

$$f^2(2n) = 2f^2(n) \quad \text{or} \quad f^2(n)$$
$$\cdots\cdots$$

and in general

$$f^k(2n) = 2f^k(n) \quad \text{or} \quad f^k(n).$$

Then taking $k = C_f(n)$ we get

$$f^k(2n) = 4 \quad \text{or} \quad 2,$$

so that

$$C_f(2n) \leq k + 1 = C_f(n) + 1.$$

THEOREM 2.1. *If x is such that for all $z < x$, $C_f(z) < C_f(x)$, where $f \in K$, then for all y,*

$$C_f(xy) \leq C_f(x) + C_f(y) + e(x). \tag{2.3}$$

Proof. We have

$$f(xy) = \frac{f(x)\, f(y)\, A(d)}{f(d)},$$

where $d = (x, y)$. Letting

$$\beta_1 = \frac{f(x)\, A(d)}{f(d)},$$

we know from Lemma 2.1 that β_1 is an integer less than or equal to x; and

$$f(xy) = \beta_1 f(y).$$

Then similarly

$$f^2(xy) = \beta_2 f^2(y),$$

where

$$\beta_2 = \frac{f(\beta_1) A(\gamma)}{f(\gamma)} \leq \beta_1 \leq x,$$

$$\gamma = (\beta_1, f(y)).$$

Thus in general we have:

$$f^k(xy) = \beta_k f^k(y), \qquad \beta_k \leq \beta_{k-1} \leq \cdots \leq \beta_1 \leq x,$$

so that, letting $k = C_f(y)$, we get

$$f^k(xy) = 2\beta_k, \qquad \beta_k \leq x.$$

Then if $\beta_k < x$ we have via Lemma 2.2, and our hypothesis,

$$\begin{aligned} C_f(xy) &= C_f(y) + C_f(2\beta_k) \\ &\leq C_f(y) + C_f(\beta_k) + 1 \\ &\leq C_f(y) + C_f(x). \end{aligned}$$

On the other hand, if $\beta_k = x$ we have

$$\begin{aligned} C_f(xy) &= C_f(y) + C_f(2x) \\ &\leq C_f(y) + C_f(x) + e(x), \end{aligned}$$

and the theorem is proved.

3. Derivation of the main result. In carrying out the proof of the equivalence of (1.3) and (1.4) we shall need certain estimates from elementary prime number theory. These results are given in the following lemma. As is conventional, we shall write $p_1 = 2$, $p_2 = 3$, $\cdots$, and let p_i denote the i-th prime.

LEMMA 3.1. *Letting $\pi(x)$ = the number of primes $\leq x$, we have*

(a) $2p_{i-2} > p_i$ *for* $i > 5$,

(b) *for all positive integers* $x > 2$,

$$\pi(x) - \pi\left(\frac{x}{7}\right) > \sqrt{x}. \tag{3.1}$$

Proof. Both of the above are deducible from a result of Ramanujan [**1**] which asserts that for $x > 300$,

$$\pi(x) - \pi\left(\frac{x}{2}\right) > \frac{x/6 - 3\sqrt{x}}{\log x} \tag{3.2}$$

Ramanujan gives explicitly the result that for $x \geq 11$,

$$\pi(x) - \pi\left(\frac{x}{2}\right) \geq 2,$$

which implies (a). As for (b), we note that since, for $x \geq 10{,}590$,

$$\frac{1}{\log x}\left(\frac{1}{6}\,x - 3\sqrt{x}\right) > \sqrt{x},$$

we have (3.1) for all $x > 10{,}590$. We can check (3.1) for all $x < 10{,}590$ very quickly. We check up to $x = 17$. Then let

$$\begin{array}{lll} a_0 = 10{,}590, & a_1 = 2{,}309, & a_2 = 653, \\ a_3 = 229, & a_4 = 103, & a_5 = 59, \\ a_6 = 37, & a_7 = 23, & a_8 = 17; \end{array}$$

inspecting tables of primes, we see that these numbers have the property that

$$\pi(a_{i+1}) - \pi\left(\frac{a_i}{7}\right) > \sqrt{a_i},$$

which completes the proof of (b).

We now give our main result as:

THEOREM 3.1. *If $g(x)$, $g \in K$, is such that $C_g(x)$ is maximal for all primes p, that is $C_g(p) \geq C_f(p)$ for all $f \in K$ and all p, then $g(3) = 2$, $g(5) = 4$ or 3, and, for $i > 3$, $g(p_i) = p_{i-1}$.*

Proof. Since $g \in K$, we clearly have $g(2) = g(1) = 1$; and $g(3) = 2$. Now in choosing $g(5) < 5$, we consider all possible values and choose the one which makes $C_g(5)$ a maximum. Symbolically, we may write

$$g(5) = C^{-1}\{\max[C(j),\ 0 < j < 5]\} = 4 \text{ or } 3.$$

Thus $g(5)$ has two possible values 4 or 3. Similarly proceeding to $p_4 = 7$ and $p_5 = 11$ we have

$$g(7) = C^{-1}\{\max[C(j), 0 < j < 7]\} = 5$$

and

$$g(11) = C^{-1}\{\max[C(j), 0 < j < 11]\} = 7.$$

In general, for the i-th prime we must have

$$(3.3) \qquad g(p_i) = C^{-1}\{\max[C(j), 0 < j < p_i]\}.$$

Now it would seem that the determination of this value $g(p_i)$, since it depends upon the $C(j)$, which in turn may require the values of $g(n)$ for composite n, would remain undetermined so long as nothing is said about the function $A(n)$. However, as we shall see, the *maximum* of these $C(j)$, required in (3.3), will turn out to be completely independent of $A(n)$.

We have noted that the theorem is true for $i = 4, 5$. Proceeding by induction, assume it true for all i', $4 \leq i' < n$, and consider $n > 5$. From (3.3) we see that in order to complete the proof we need only show that for any x such that

$$(3.4) \qquad p_n > x > p_{n-1}$$

we must have

$$(3.5) \qquad C(x) < C(p_{n-1}) = n - 2.$$

Assume that for some x satisfying (3.4), (3.5) is false, and let x be the smallest one satisfying (3.4) for which

$$(3.6) \qquad C(x) \geq n - 2.$$

Then we have also

$$(3.7) \qquad C(g(x)) \geq n - 3.$$

We shall now show that $g(x) \neq p_{n-1}$. For suppose that $g(x) = p_{n-1}$. Then x must have a prime divisor q such that $g(q) = p_{n-1}$. But from (3.4) we see that $q \leq p_{n-1}$, which is impossible.

If $g(x) < p_{n-1}$, by our inductive hypothesis we would have

$$C(g(x)) \leq C(p_{n-2}) = n - 3.$$

Now if $C(g(x)) = n - 3$, it would follow that $g(x) = p_{n-1}$. This in turn implies that p_{n-1} divides x. Since $x \neq p_{n-1}$, this yields

$$x \geq 2p_{n-1} > p_n,$$

which is a contradiction. The only alternative left is that $C(g(x)) < n - 3$, which contradicts (3.7). Thus we conclude that $g(x) > p_{n-1}$ so that we must have

$$p_n > x > g(x) > p_{n-1}.$$

Since x is the smallest integer satisfying (3.4) and not (3.5), we must have $C(g(x)) < n-2$ or $C(x) \leq n-2$; hence

$$C(x) = n - 2. \tag{3.10}$$

Now x is not even, for if it were we would have

$$g(x) \leq \frac{x}{2} < \frac{p_n}{2} < p_{n-1},$$

which is a contradiction. Also x is not divisible by 3 for $n > 5$; for if it were, $g(x)$ would be even and we would get, using Lemma 3.1 (a),

$$g^2(x) \leq \frac{g(x)}{2} < \frac{p_n}{2} < p_{n-2}.$$

But then

$$C(g^2(x)) \leq C(p_{n-3}) = n-4.$$

If the inequality sign holds, this implies $C(x) < n-2$ in contradiction to (3.10). On the other hand, if the equality sign holds then $g^2(x) = p_{n-3}$. This in turn implies that p_{n-2} divides $g(x)$. If $g(x) \neq p_{n-2}$, then

$$g(x) \geq 2p_{n-2} > p_n,$$

which is impossible. Finally, $g(x) = p_{n-2}$ implies that x is divisible by p_{n-1}, which is impossible.

Also, if x is not divisible by 5, the argument is the same as for 3. On the other hand if $g(5) = 3$, and x is divisible by 5, it results that $g^2(5)$ is even, and hence

$$g^3(x) \leq \frac{1}{2}\, g^2(x) \leq \frac{1}{2} \cdot \frac{2}{3}\, g(x) \leq \frac{1}{2} \cdot \frac{2}{3} \cdot \frac{3}{5}\, x \leq \frac{p_n}{5} < p_{n-4}.$$

But this again implies that $C(x) < n-2$, which is impossible.

Suppose then that $p \geq 7$ is the smallest prime which divides x. Since x is composite, $1 < x/p < x$ and $p \leq \sqrt{x}$. It is clear from (3.3) and our inductive hypothesis that for $z < p$, $C(z) < C(p)$. Hence via Theorem 2.1 we have

$$n - 2 = C(x) \leq C\left(\frac{x}{p}\right) + C(p).$$

Via our inductive hypothesis we see that, since $p \leq \sqrt{x}$,

$$C(p) \leq \pi(\sqrt{x}),$$

so that

$$C\left(\frac{x}{p}\right) + \pi(\sqrt{x}) \geq n - 2. \tag{3.11}$$

Since

$$x > p_{n-1} > \frac{x}{7} \geq \frac{x}{p},$$

and

$$C(x) = C(p_{n-1}) = n - 2,$$

we have

$$C(x) - C\left(\frac{x}{p}\right) \geq \pi(x) - \pi\left(\frac{x}{7}\right) > \sqrt{x},$$

by Lemma 3.1 (b); and

$$C\left(\frac{x}{p}\right) < n - 2 - \sqrt{x}. \tag{3.12}$$

Combining (3.11) and (3.12) yields $\pi(\sqrt{x}) > \sqrt{x}$, an obvious contradiction: thus the proof of the theorem is completed.

4. Some remarks and generalizations. From the above we note that imposing the condition that the function $C_f(n)$ be maximal at the primes determines uniquely the values of $f(n)$ at the primes without restricting $A(n)$ in any way. This is natural from a certain point of view, since the function $A(n)$ plays a role only in evaluating $f(n)$ for powers of a prime. This might lead one to suspect that requiring that $C_f(n)$ be maximal at the p_i^2 in addition to the p_i would also determine the values of $A(n)$. This is in fact the case, and one may prove (we omit the proof since it is long and very similar to that of § 3):

THEOREM 4.1. *If* $C_g(x)$ *is maximal at the primes and squares of primes, then* $A_g(3) = 2$ *or* 3, $A_g(5) = 5$ *or* 4, *and for* $p_i > 5$, $A_g(p_i) = p_i$ *or* p_{i-1}. *Furthermore this same maximal* $C_g(x)$ *is realized for any admissible choice of the* $A_g(p_i)$ (*that is, as either* p_i *or* p_{i-1}).

REFERENCES

1. S. Ramanujan, *Proof of Bertrand's postulate*, J. Indian Math. Soc. **11** (1919), xxx-xxx.

2. Harold N. Shapiro, *On the iterates of a certain class of arithmetic functions*, Comm. Pure Appl. Math. 3 (1950), 259-272.

NEW YORK UNIVERSITY