SOME THEOREMS ON GENERALIZED DEDEKIND SUMS

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1. Introduction. Using a method developed by Rademacher [5], Apostol [1] has proved a transformation formula for the function

$$(1.1)\qquad G_p(x) = \sum_{m,\, n=1}^{\infty} n^{-p} x^{mn} \qquad (|x| < 1),$$

where p is a fixed odd integer > 1. The formula involves the coefficients

$$(1.2)\qquad c_r(h, k) = \sum_{\mu \,(\mathrm{mod}\; k)} P_{p+1-r}\left(\frac{\mu}{k}\right) P_r\left(\frac{h\mu}{k}\right) \qquad (0 \le r \le p+1),$$

where $(h, k) = 1$, the summation is over a complete residue system $(\mathrm{mod}\; k)$, and $P_r(x) = \overline{B}_r(x)$, the Bernoulli function.

We shall show in this note that the transformation formula for (1.1) implies a reciprocity relation involving $c_r(h, k)$, which for $r = p$ reduces to Apostol's reciprocity theorem [1, Th. 1; 2, Th. 2] for the generalized Dedekind sum $c_p(h, k)$. In addition, we prove some formulas for $c_r(h, k)$ which generalize certain results proved by Rademacher and Whiteman [6]. Finally we derive a representation of $c_r(h, k)$ in terms of so-called "Eulerian numbers".

2. Some preliminaries. It will be convenient to recall some properties of the Bernoulli function $P_r(x)$; by definition, $P_r(x) = B_r(x)$ for $0 \le x < 1$, and $P_r(x+1) = P_r(x)$. Also we have the formulas

$$(2.1)\qquad \sum_{r=0}^{k-1} P_r\left(t + \frac{r}{k}\right) = k^{1-m} P_r(kt), \quad P_r(-x) = (-1)^r P_r(x).$$

It follows from the second of (2.1) that $c_r(h, k) = 0$ for p even and $0 \le r \le p+1$. We have also

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$$(2.2)\qquad c_0(h, k) = c_{p+1}(h, k) = k^{-p} B_{p+1}$$

provided $(h, k) = 1$. Further, it is clear from the second of (2.1) that

$$(2.3)\qquad c_r(-h, k) = (-1)^r c_r(h, k).$$

Now as in [5, 321] put $x = e^{2\pi i\tau}$,

$$\tau = \frac{iz + h}{k}, \quad \tau' = \frac{iz^{-1} + h'}{k},$$

so that, on eliminating z, we get

$$(2.4)\qquad \tau' = \frac{h'\tau + k'}{k\tau - h} \qquad (hh' + kk' + 1 = 0);$$

thus (2.4) is a unimodular transformation. Now Apostol's transformation formula [1, Th. 2] reads (in our notation)

$$G_p(e^{2\pi i\tau}) = (iz)^{p-1} G_p(e^{2\pi i\tau'}) - \frac{1}{2}\left(\frac{2\pi z}{k}\right)^p \frac{B_{p+1}}{(p+1)!}$$

$$+ \frac{i^{p-1}}{2z}\left(\frac{2\pi}{k}\right)^p \frac{B_{p+1}}{(p+1)!} + \frac{(2\pi i)^p}{2\cdot p!} c_p(h, k)$$

$$+ \frac{(2\pi)^p z^{p-1}}{2(p+1)!} \sum_{r=0}^{p-2} \binom{p+1}{r+1} e^{\pi i(r-1)/2} z^{-r} \sum_{\mu=1}^{k} P_{p-r}\left(\frac{h'\mu}{k}\right) P_{r+1}\left(\frac{\mu}{k}\right).$$

Making use of (1.2), (2.2), and (2.3), we easily verify that this result can be put in the form

$$(2.5)\qquad G_p(e^{2\pi i\tau}) = (k\tau - h)^{p-1} G_p(e^{2\pi i\tau'}) + \frac{(2\pi i)^p}{2(p+1)!} f(h, k; \tau),$$

where

$$(2.6)\qquad f(h, k; \tau) = \sum_{r=0}^{p+1} \binom{p+1}{r} (k\tau - h)^{p-r} c_r(h, k).$$

We remark that (2.6) can be written in the symbolic form

$$(2.7)\qquad (k\tau-h)\,f(h,k;\tau)=(k\tau-h+c(h,k))^{p+1},$$

where it is understood that after expanding the right member of (2.7) by the binomial theorem, $c^r(h, k)$ is replaced by $c_r(h, k)$.

We shall require an explicit formula for $f(0, 1; \tau)$. Since, by (1.2),

$$c_r(0,1)=P_{p+1-r}(0)\,P_r(0)=B_{p+1-r}B_r,$$

it is clear that (2.6) implies

$$(2.8)\qquad f(0,1;\tau)=\frac{1}{\tau}\sum_{r=0}^{p+1}\binom{p+1}{r}B_{p+1-r}B_r\,\tau^{p+1-r}=\frac{1}{\tau}(B+\tau B)^{p+1}.$$

If in (2.4) we replace τ by $-1/\tau$, then τ' becomes

$$(2.9)\qquad \tau^*=\frac{-k'\tau+h'}{h\tau+k},$$

and (2.5) becomes

$$(2.10)\qquad G_p(e^{-2\pi i/\tau})=\left(\frac{h\tau+k}{\tau}\right)^{p-1}G_p(e^{2\pi i\tau^*})+\frac{(2\pi i)^p}{2(p+1)!}\,f\left(h,k;-\frac{1}{\tau}\right).$$

By (2.5) and (2.8) we have

$$(2.11)\qquad G_p(e^{2\pi i\tau})=\tau^{p-1}G_p(e^{-2\pi i/\tau})+\frac{(2\pi i)^p}{2\tau(p+1)!}(B+\tau B)^{p+1},$$

and by (2.5) and (2.9),

$$(2.12)\qquad G_p(e^{2\pi i\tau})=(h\tau+k)^{p-1}G_p(e^{2\pi i\tau^*})+\frac{2\pi i}{2(p+1)!}\,f(-k,h;\tau).$$

Comparison of (2.10), (2.11), (2.12) yields

$$f(-k,h;\tau)=\tau^{p-1}f\left(h,k;-\frac{1}{\tau}\right)+\frac{1}{\tau}(B+\tau B)^{p+1},$$

or with τ replaced by $-1/\tau$,

$$(2.13)\qquad f(h,\,k;\,\tau)=\tau^{p-1}f\left(-k,\,h;\,-\frac{1}{\tau}\right)+\frac{1}{\tau}(B+\tau B)^{p+1}.$$

(For the above, compare [3, pp. 162-163]).

3. The main results. In (2.7) replace $h,\,k,\,\tau$ by $-k,\,h,\,-1/\tau$ respectively; we get

$$\frac{k\tau-h}{\tau}\,f\left(-k,\,h;\,-\frac{1}{\tau}\right)=\left(\frac{k\tau-h}{\tau}+c(-k,\,h)\right)^{p+1}.$$

By (2.3), it is clear that (2.13) becomes

$$(3.1)\qquad \tau(k\tau-h+c(h,\,k))^{p+1}$$
$$=(\tau c(k,\,h)-\tau k+h)^{p+1}+(k\tau-h)(B+\tau B)^{p+1}.$$

Comparison of the coefficients of τ^{r+1} in both members of (3.1) leads immediately to:

THEOREM 1. *For p odd >1, $0\le r\le p$,*

$$(3.2)\qquad \binom{p+1}{r}k^{r}(c(h,\,k)-h)^{p+1-r}=\binom{p+1}{r+1}h^{p-r}(c(k,\,h)-k)^{r+1}$$
$$+\,kB_{p+1-r}B_r-hB_{p-r}B_{r+1}.$$

In the next place, if for brevity we put $w=k\tau-h$, then (3.1) becomes

$$(3.3)\qquad k^{p}(w+h)(w+c(h,\,k))^{p+1}$$
$$=((w+h)c(k,\,h)-wk)^{p+1}+w(Bk+(w+h)B)^{p+1}.$$

We now compare coefficients of w^{r+1} in both members of (3.3); a little care is required in connection with the extreme right member. We state the result as:

THEOREM 2. *For p odd >1, $0\le r\le p$,*

$$(3.4)\qquad \binom{p+1}{r+1}hk^{p}c_{p-r}(h,\,k)+\binom{p+1}{r}k^{p}c_{p+1-r}(h,\,k)$$

$$= \binom{p+1}{r+1} h^{p-r}(c(k,h)-k)^{r+1} c^{p-r}(k,h) + \binom{p+1}{r}(Bk+B'h)^{p+1-r} B'^{r},$$

where

$$(Bk+B'h)^{p+1-r} B'^{r} = \sum_{s=0}^{p+1-r} \binom{p+1-r}{s} B_{p+1-r-s} B_{r+s}\, k^{p+1-r-s} h^{s}.$$

For $r = 0$, (3.4) becomes

$$(p+1)hk^p c_p(h,k) + k^p c_{p+1}(h,k)$$

$$= (p+1)h^p \{c_{p+1}(k,h) - kc_p(k,h)\} + (p+1)(Bk+Bh)^{p+1},$$

which reduces to

$$(3.5) \qquad (p+1)\{hk^p c_p(h,k) + k^p hc_p(k,h)\} = (p+1)(Bk+Bh)^{p+1} + pB_{p+1}.$$

This is Apostol's reciprocity theorem.

If we take $r = 1$ in (3.4), we get

$$p\{h^2 k^p c_{p-1}(h,k) - k^2 h^p c_{p-1}(k,h)\}$$

$$= -2\{hk^p c_p(h,k) + pkh^p c_p(h,k)\} + pB_{p+1} + 2(Bk+B'h)^p B'h.$$

If in this formula we interchange h and k and add we again get (3.5), while if we subtract we get

$$(3.6) \qquad p\{h^2 k^p c_{p-1}(h,k) - k^2 h^p c_{p-1}(k,h)\}$$

$$= (p-1)\{hk^p c_p(h,k) - kh^p c_p(k,h)\} - (Bk+Bh)^p (Bk-Bh).$$

In view of (3.6), it does not seem likely that Theorem 2 will yield a simple expression for

$$h^{r+1} k^p c_{p-r}(h,k) + (-1)^r k^{r+1} h^p c_{p-r}(k,h) \qquad (r > 0).$$

We remark that Theorems 1 and 2 are equivalent. Indeed it is evident that

(3.2) is equivalent to (3.1), and (3.4) is equivalent to (3.3); also it is clear that (3.1) and (3.3) are equivalent.

4. Some additional results. We next prove (compare [6, Th. 1]):

THEOREM 3. *For $p, q \geq 1$, $0 \leq r \leq p+1$, we have*

$$c_r(qh, qk) = q^{r-p}\, c_r(h, k). \tag{4.1}$$

Note that we now do not assume p odd, $(h, k) = 1$.

To prove (4.1), we have, using (1.2),

$$\begin{aligned} c_r(qh, qk) &= \sum_{\mu \,(\mathrm{mod}\ qk)} P_{p+1-r}\left(\frac{\mu}{qk}\right) P_r\left(\frac{h\mu}{k}\right) \\ &= \sum_{\substack{\nu\,(\mathrm{mod}\ q) \\ \rho\,(\mathrm{mod}\ k)}} P_{p+1-r}\left(\frac{\nu k + \rho}{qk}\right) P_r\left(\frac{h(\nu k + \rho)}{k}\right) \\ &= \sum_{\rho} P_r\left(\frac{h\rho}{k}\right) \sum_{\nu} P_{p+1-r}\left(\frac{\nu}{q} + \frac{\rho}{qk}\right) \\ &= q^{r-p} \sum_{\rho} P_{p+1-r}\left(\frac{\rho}{k}\right) P_r\left(\frac{h\rho}{k}\right) \\ &= q^{r-p}\, c_r(h, k). \end{aligned}$$

For brevity we define

$$b_r(h, k) = (c(h, k) - h)^r = \sum_{s=0}^{r} (-1)^{r-s} \binom{r}{s} h^{r-s} c_s(h, k), \tag{4.2}$$

which occurs in Theorem 1. Clearly

$$c_r(h, k) = (b(h, k) + h)^r.$$

THEOREM 4. *For $p, q \geq 1$, $0 \leq r \leq p+1$, we have*

$$b_r(qh, qk) = q^{r-p}\, b_r(h, k). \tag{4.3}$$

By (4.1) and (4.2) we have

$$b_r(qh, qk) = \sum_{s=0}^{r} (-1)^{r-s} \binom{r}{s} (qh)^{r-s} c_s(qh, qk)$$

$$= \sum_{s=0}^{r} (-1)^{r-s} \binom{r}{s} h^{r-s} q^{r-p} c_s(h, k)$$

$$= q^{r-p} b_r(h, k).$$

If we define

$$a_r(h, k) = (c(h, k) - h)^r c^{p+1-r}(h, k), \tag{4.4}$$

which is suggested by Theorem 2, we get:

THEOREM 5. *For* $p, q \geq 1$, $0 \leq r \leq p+1$,

$$a_r(qh, qk) = q a_r(h, k). \tag{4.5}$$

The proof, which is exactly like the proof of (4.3), will be omitted.

We note that (4.4) implies

$$h^r c^{p+1-r}(h, k) = \sum_{s=0}^{r} (-1)^s \binom{r}{s} a_s(h, k) = (1 - a(h, k))^r. \tag{4.6}$$

Also using (4.2) and (4.6), we get

$$h^{p+1-r} b_r(h, k) = (1 - a(h, k))^{p+1-r} a^r(h, k), \tag{4.7}$$

and reciprocally from (4.4),

$$a_r(h, k) = (b(h, k) + h)^{p+1-r} b^r(h, k). \tag{4.8}$$

Using $a_r(h, k)$ and $b_r(h, k)$, we can state Theorems 1 and 2 somewhat more compactly.

5. Another property of $c_r(h, k)$. For the next theorem compare [6, Th. 2].

THEOREM 6. *For* $p \geq 1$, $0 \leq r \leq p$, *and* q *prime, we have*

$$(5.1)\qquad \sum_{m=0}^{q-1} c_r(h+mk,\ qk) = (q+q^{1-p})\,c_r(h,\ k) - q^{1-r}c_r(ph,\ k).$$

By (1.2), the left member of (5.1) is equal to

$$\sum_{m=0}^{q-1}\ \sum_{\mu=1}^{qk} P_{p+1-r}\left(\frac{\mu}{qk}\right)P_r\left(\frac{(h+mk)\mu}{qk}\right)$$

$$= \sum_{\mu=1}^{qk} P_{p+1-r}\left(\frac{\mu}{qk}\right)\sum_{m=0}^{q-1} P_r\left(\frac{h\mu}{qk}+\frac{m\mu}{q}\right)$$

$$= \sum_{\mu=1}^{qk} P_{p+1-r}\left(\frac{\mu}{qk}\right)P_r\left(\frac{h\mu}{k}\right)q^{1-r}$$

$$+ \sum_{\nu=1}^{k} P_{p+1-r}\left(\frac{\nu}{k}\right)\left\{qP_r\left(\frac{h\nu}{k}\right) - P_r\left(\frac{qh\nu}{k}\right)q^{1-r}\right\}$$

$$= q^{1-r}c_r(qh,\ qk) + qc_r(h,\ k) - q^{1-r}c_r(qh,\ k)$$

$$= (q^{1-p}+q)\,c_r(h,\ k) - q^{1-r}c_r(qh,\ k),$$

by (4.1).

It does not seem possible to frame a result like (5.1) for the expressions $b_r(h, k)$ or $a_r(h, k)$ defined by (4.2) and (4.3).

6. Representation by Eulerian numbers. If $k > 1$, $\rho^k = 1$, $\rho \neq 1$, we define the "Eulerian number" $H_m(\rho)$ by means of [4, p. 825]

$$(6.1)\qquad \frac{1-\rho}{e^t-\rho} = \sum_{m=0}^{\infty} H_m(\rho)\frac{t^m}{m!}.$$

Then it is easily verified that [4, p. 825]

$$k^{m-1}\sum_{r=0}^{k-1}\rho^r B_m\left(\frac{r}{k}\right) = \frac{m}{\rho-1}H_{m-1}(\rho^{-1}),$$

which may be put in the more convenient form

$$(6.2)\qquad k^{m-1}\sum_{r\,(\mathrm{mod}\,k)}\rho^r P_m\left(\frac{r}{k}\right)=\frac{m}{\rho-1}\,H_{m-1}(\rho^{-1}).$$

Now consider the representation (finite Fourier series)

$$(6.3)\qquad P_m\left(\frac{r}{k}\right)=\sum_{s=0}^{k-1}A_s\,\zeta^{-rs}\qquad (\zeta=e^{2\pi i/k}).$$

If we multiply both members of (6.3) by ζ^{rt} and sum, we get

$$kA_t=\sum_r \zeta^{rt}P_m\left(\frac{r}{k}\right)=\begin{cases}\dfrac{mk^{1-m}}{\zeta^t-1}\,H_{m-1}(\zeta^{-t}) & (t\neq 0)\\[2ex] k^{1-m}B_m & (t=0),\end{cases}$$

by (6.2) and (2.1). Thus (6.3) becomes

$$(6.4)\qquad P_m\left(\frac{\mu}{k}\right)=k^{-m}B_m+mk^{-m}\sum_{s=1}^{k-1}\frac{H_{m-1}(\zeta^{-s})}{\zeta^s-1}\,\zeta^{-\mu s}.$$

Thus substituting from (6.4) in (1.2), we get after a little reduction

$$(6.5)\qquad c_r(h,k)=\frac{B_{p+1-r}B_r}{k^p}+\frac{r(p+1-r)}{k^p}\sum_{t=1}^{k-1}\frac{H_{p-r}(\zeta^{ht})H_{r-1}(\zeta^{-t})}{(\zeta^{-ht}-1)(\zeta^t-1)}.$$

Thus $c_r(h,k)$ has been explicitly evaluated in terms of the Eulerian numbers. One or two special cases of (6.5) may be mentioned. For $r=p$ we have

$$(6.6)\qquad c_p(h,k)=\frac{p}{k^p}\sum_{t=1}^{k-1}\frac{H_{p-1}(\zeta^{-t})}{(\zeta^{-ht}-1)(\zeta^t-1)}\qquad (p>1),$$

while for $r=p=1$ we have

$$\bar{s}(h,k)=\frac{1}{4k}+\frac{1}{k}\sum_{t=1}^{k-1}\frac{1}{(\zeta^{-ht}-1)(\zeta^t-1)},$$

where $\bar{s}(h,k)=c_1(h,k)$. Note that $\bar{s}(h,k)=s(h,k)+1/4$, where $s(h,k)$ is the ordinary Dedekind sum [6]. We also note that (6.4) becomes, for $m=1$,

$$P_1\left(\frac{\mu}{k}\right) = -\frac{1}{2k} + \frac{1}{k}\sum_{s=1}^{k-1}\frac{\zeta^{-\mu s}}{\zeta^s - 1},$$

which is equivalent to a formula of Eisenstein.

Possibly (6.5) can be used to give a direct proof of Theorem 1 or Theorem 2.

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