# ON THE PRIME IDEALS OF THE RING OF ENTIRE FUNCTIONS

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**1. Introduction.** Let $R$ be the ring of entire functions, and let $K$ be the complex field. In an earlier paper [6], the author investigated the ideal structure of $R$, particular attention being paid to the maximal ideals. In 1946, Schilling [9, Lemma 5] stated that every prime ideal of $R$ is maximal. Recently, I. Kaplansky pointed out to the author (in conversation) that this statement is false, and constructed a nonmaximal prime ideal of $R$ (see Theorem 1(a), below). The purpose of the present paper is to investigate these nonmaximal prime ideals and their residue class fields. The author is indebted to Prof. Kaplansky for making this investigation possible.

The nonmaximal prime ideals are characterized within the class of prime ideals, and it is shown that each prime ideal is contained in a unique maximal ideal. The intersection $P^*$ of all powers of a maximal free ideal $M$ is the largest nonmaximal prime ideal contained in $M$. The set $P_M$ of all prime ideals contained in $M$ is linearly ordered under set inclusion, and distinct elements $P$ of $P_M$ correspond in a natural way to distinct rates of growth of the multiplicities of the zeros of functions $f$ in $P$.

It is shown that the residue class ring $R/P$ of a nonmaximal prime ideal $P$ of $R$ is a valuation ring whose unique maximal ideal is principal; $R/P$ is Noetherian if and only if $P = P^*$. The residue class ring $R/P^*$ is isomorphic to the ring $K\{z\}$ of all formal power series over $K$. The structure theory of Cohen [2] of complete local rings is used.

**2. Notation and preliminaries.** A familiarity with the contents of [6] is assumed, but some of it will be reproduced below for the sake of completeness.

DEFINITION 1. If $f \in R$, and $I$ is any nonvoid subset of $R$, let:

(a) $A(f) = [z \in K \mid f(z) = 0]$ (Note that multiple zeros are repeated. Unions and intersections are taken in the same sense.);

(b) $A(I) = [A(f) \mid f \in I]$;

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(c) $A^*(f)$ be the sequence of distinct zeros of $f$, arranged in order of increasing modulus.

In 1940, Helmer showed [5, Theorem 9] that if $A(f) \cap A(g)$ is empty, there exist $s$, $t$ in $R$ such that

$$sf + tg = 1. \tag{2.1}$$

More generally, if $d$ is any element of $R$ such that

$$A(d) = A(f) \cap A(g),$$

then $d$ is a greatest common divisor of $f$ and $g$, unique to within a unit factor, and the ideal $(f, g)$ generated by f and $g$ is the principal ideal $(d)$. It easily follows that every finitely generated ideal of $R$ is principal.

He proved this by showing that if $\{a_n\}$ is any sequence of complex numbers such that

$$\lim_{n\to\infty} a_n = \infty,$$

and $w_{n,k}$ is any set of complex numbers, then there is an $s$ in $R$ such that

$$s^{(k)}(a_n) = w_{n,k}, \quad (n = 1, 2, \cdots; k = 0, \cdots, 1_n). \tag{2.2}$$

The latter was shown independently by Germay [3].

REMARK. In [4], Germay extended (2.2) to the ring of functions analytic in $|z| < r$, where $\lim_{n\to\infty} a_n$ lies on $|z| = r$. Hence (2.1) follows for this ring, as will most of the results in [6] and the present paper, with minor modification.

It follows that if $I$ is an ideal of $R$, then $A(I)$ has the finite intersection property. So we make the following definition.

DEFINITION 2. If $\bigcap_{f\in I} A(f)$ is nonempty, then $I$ is called a *fixed* ideal. Otherwise, $I$ is called a *free* ideal.

DEFINITION 3. (a) If $A^*(f) = \{a_n\}$, let $0_n(f)$ be the multiplicity of $a_n$ as a zero of $f$.

(b) If $A$ is a nonvoid subset of $A^*(f)$, let $0_n(f{:}A)$ be the function $0_n(f)$ with domain restricted to $A$.

(c) Let $m(f) = \sup_{n\geq 1} 0_n(f)$, if $f \neq 0$. Let $m(0) = \infty$.

**3. Prime ideals of** $R$. Kaplansky's construction of nonmaximal, prime ideals of $R$ is given in Theorem 1(a), below. The only fallacy in Schilling's demonstration (referred to in the Introduction) is the false assumption that a prime ideal necessarily contains an $f$ such that $m(f) = 1$. Hence a characterization of these nonmaximal prime ideals may be given.

THEOREM 1. (a) *There exist nonmaximal prime ideals of* $R$.

(b) *A necessary and sufficient condition that a prime ideal* $P$ *of* $R$ *be nonmaximal is that* $m(f) = \infty$, *for all* $f \in P$.

*Proof.* (a) Let

$$S = [f \in R \mid m(f) < \infty].$$

Clearly, $S$ is closed under multiplication and does not contain 0. If $g \neq 0$ is in $R - S$, $g$ is contained in a prime ideal $P$ not intersecting $S$ (see [**8**, p. 105]). Since, as noted in [**6**, p. 183], any maximal ideal contains an $f$ such that $m(f) = 1$, $P$ cannot be maximal.

(b) The sufficiency is clear from the above. If $f \in P$ with $m(f) < \infty$, the primality of $P$ ensures that there is a $g \in P$ with $m(g) = 1$. Suppose the maximal ideal $M$ contains $P$, and let $h \in M$. By (2.1), there is a $d \in M$ such that

$$A(d) = A(g) \cap A(h).$$

Now $g = g_1 d$, where $A(g_1) \cap A(d)$ is empty, since $m(g) = 1$. Since $P$ is prime, it follows that either $g_1 \in P$ or $d \in P$. But $M \neq R$, so $g_1$ is not in $P$. It follows that $d$, and hence $h$, is in $P$, whence $P = M$.

COROLLARY. *Any prime, fixed ideal of* $R$ *is maximal.*

THEOREM 2. *Every prime ideal* $P$ *of* $R$ *is contained in a unique maximal (free) ideal* $M$.

*Proof.* By Theorem 1(b) and [**6**, Theorem 4], the ideal $(P, f)$ is maximal if $m(f) = 1$ and $A(f)$ intersects every element of $A(P)$. Let $f_1$, $f_2$ be any two such functions, so that $M_1 = (P, f_1)$ and $M_2 = (P, f_2)$ are maximal ideals containing $P$. If

$$A(d) = A(f_1) \cap A(f_2),$$

then $M = (P, d)$ is a maximal ideal containing $P$, and $M_1 \subset M$, $M_2 \subset M$, so that

$$M_1 = M_2 = M.$$

More concrete constructions of nonmaximal prime ideals are given below in terms of maximal free ideals.

THEOREM 3. *If $M$ is a maximal free ideal of $R$, then*

$$P^* = \bigcap_{k=1}^{\infty} M^k$$

*is a prime ideal, and is the largest nonmaximal prime ideal contained in $M$.*

*Proof.* Since every finitely generated ideal of $R$ is principal, $P^*$ is easily seen to be the set of all $f \in R$ expressible in the form $h_k d_k^k$, with $d_k \in M$, $k = 1, 2, \cdots$. Thus, if $f \in M$, $f \in P^*$ if and only if $m(f/e) = \infty$ whenever $e$ divides $f$ and $e \in R - M$, (whence $f/e \in M$). Suppose $f_1$, $f_2$ are not in $P^*$. Clearly, $f_1 f_2$ is not in $P^*$ except possibly when both $f_1$ and $f_2$ are in $M$. In this case, there exist $e_i$ dividing $f_i$, with $e_i \in R - M$ such that $m(f_i/e_i) < \infty$, $(i = 1, 2)$. Since $M$ is prime, $e_1 e_2 \in R - M$ and $m(f_1 f_2/e_1 e_2) \leq m(f_1/e_1) + m(f_2/e_2) < \infty$. So $f_1 f_2$ is not in $P^*$, whence $P^*$ is a prime ideal.

The second part of the Theorem is a direct consequence of Theorem 1 (b).

We proceed now to identify the remainder of the class $P_M$ of prime ideals contained in $M$. This is done by considering the rates of growth of the functions $0_n(f)$ on the filter $A(M)$. Results of Bourbaki [1] are used without further acknowledgement.

DEFINITION 4. If $f$, $g \in M$, and there is an $e \in M$ such that

$$A^*(e) \subset A^*(f) \cap A^*(g)$$

with

$$0_n(f\colon A^*(e)) \geq 0_n(g\colon A^*(e)),$$

then $f \geq g$ $(g \leq f)$.

It is easily seen that the relation "$\geq$" is reflexive and transitive. Moreover:

LEMMA 1. *If $f$, $g \in M$, either $f \geq g$ or $g \geq f$.*

*Proof.* Let

$$A(d) = A(f) \cap A(g),$$

and let

$$A_1 = [z \in A^*(d) \mid 0_n(f:\{z\}) \geq 0_n(g:\{z\})],$$
$$A_2 = [z \in A^*(d) \mid 0_n(f:\{z\}) < 0_n(g:\{z\})].$$

Since $A_1 \cap A_2$ is empty, $A_1 \cup A_2 = A^*(d)$; and since $M$ is prime, one and only one of $A_1, A_2 \in M$. Hence $f \geq g$ or $g \geq f$.

DEFINITION 5. Suppose $f, g \in M$.

(a) If there exist positive integers $N_1$, $N_2$ such that $f^{N_1} \geq g$ and $g^{N_2} \geq f$, then $f \sim g$.

(b) If $f \geq g^N$ for all positive integers $N$ or if $f = 0$, then $f >> g$ ($g << f$).

LEMMA 2. (a) *The relation '$\sim$' is an equivalence relation.*

(b) *The relation '$>>$' is transitive.*

(c) *If $f, g \in M$, one and only one of $f \sim g$, $f >> g$, $f << g$ holds.*

*Proof.* The relations (a) and (b) follow easily from the observations that

$$0_n(f^N) = N \cdot 0_n(f), \text{ and if } f \geq g \text{ then } f^N \geq g^N.$$

It is clear that at most one of the relations (c) can hold. By Lemma 1, $f \geq g$ or $g \geq f$. Suppose $f \geq g$ and not $f \sim g$; then $f \geq g^N$ for all $N$, whence $f >> g$. Similarly, if $g \geq f$.

LEMMA 3. *Let $f$ be an element of a prime ideal $P$ of $P_M$. If $g \geq f$, or $g \sim f$, then $g \in P$.*

*Proof.* Suppose first that $g \geq f$. Then, as is evident from the construction in Lemma 1, we can write

$$f = f_1 d_1, \; g = g_1 d_2,$$

where

$$A^*(d_1) = A^*(d_2), \quad 0_n(d_2) \geq 0_n(d_1),$$

and $f_1$, $g_1$ are not in $M$. Hence $d_1 \in P$; and, since $d_2$ is a multiple of $d_1$, $d_2$ and $g \in P$. If $g \sim f$, then $g^N \geq f$, for some $N$. By the above, $g^N \in P$. But $P$ is a prime ideal, so $g \in P$.

THEOREM 4. (a) *Let $\Omega$ be any subset of $M$, and let*

$$P_\Omega = [f \in M \mid f >> g, \;\; \textit{for all } g \in \Omega].$$

*Then $P_\Omega$ is a prime ideal.*

(b) *If $P$ is a prime ideal, then $P = P_\Omega$*, where $\Omega = M - P$.

*Proof.* (a) Note first that if $g_1, g_2 \in M$ and $g_1 g_2 \neq 0$

$$A = A^*(g_1) \cap A^*(g_2),$$

then

$$0_n(g_1 - g_2 : A) = \min \{0_n(g_1 : A),\ 0_n(g_2 : A)\}.$$

If $g_1 \in M$, $g_2 \in R$, $g_1 g_2 \neq 0$, then

$$0_n(g_1 g_2 : A^*(g_1)) = 0_n(g_1 : A^*(g_1)) + 0_n(g_2 : A^*(g_1)).$$

It now follows from the lemmas above that $P$ is an ideal. The primality of $P$ follows from the observation that

$$P_g = [f \in M \mid f >> g]$$

is a prime ideal, and that $P_\Omega$ is an intersection of a descending chain (under set inclusion) of ideals of this form.

(b) If $P$ is a prime ideal, the relations $f \in P$, $g \in M - P$, imply that $f >> g$, by Lemma 3.

COROLLARY. *The ideals of $P_M$ are linearly ordered under set inclusion.*

By the Theorem above, every element of $P_M$ is the upper class of a Dedekind cut (under $<<$). If $P$ contains a least element $f$, then

$$P = P_f^+ = [g \in M \mid g >> f \text{ or } g \sim f].$$

If $M - P$ has a greatest element $g$, then $P = P_g$ as defined in the proof of the theorem. It is clear that $P_M$ contains the greatest lower bound and least upper bound of any set of elements.

Note, moreover that $P_{f_1} = P_{f_2}$ ($P_{f_1}^+ = P_{f_2}^+$) if and only if $f_1 \sim f_2$.

LEMMA 4. *The set $P^* - \{0\}$ has no countable cofinal or coinitial subset. Moreover, if $\{f_{1,n}\}$, $\{f_{2,n}\}$ are two sequences of nonzero elements of $P^*$, such that*

$$f_{1,n+1} >> f_{1,n} >> f_{2,m} >> f_{2,m+1}, \qquad \textit{for all } n, m,$$

*then there is an $f \in P^*$ such that*

$$f_{1,n} \gg f \gg f_{2,m}, \qquad \textit{for all } n, m.$$

*Proof.* See [1, p. 123, exercise 8].

The author is indebted to Dr. P. Erdös and Dr. L. Gillman for the following Theorem.

THEOREM 5. *The set $P_M$ has power at least $2^{\aleph_1}$.*

*Proof.* It is implicit in arguments of Hausdorff and Sierpinski [10, p. 62] that every set satisfying Lemma 4 contains a subset similar to the lexicographically ordered set $S$ of $\omega_1$-sequences of 0's and 1's, each having at most countably many 1's By [11], $S$ is dense in the set of all dyadic $\omega_1$-sequences, which has power $2^{\aleph_1}$. Since the set $P_M$ is complete, card $(P_M) \geq 2^{\aleph_1}$.

Since card $(P_M) \leq 2^c$, where $c$ is the cardinal number of the continuum, we have:

COROLLARY. *If $2^{\aleph_1} = 2^c$, in particular if $\aleph_1 = c$, then* card $(P_M) = 2^c$.

**4. Residue class rings of prime ideals.** We adopt the following definition of Krull [7, p. 110]:

DEFINITION 6. An integral domain $D$ such that if $f, g \in D$, then $f$ divides $g$ or $g$ divides $f$, is called a *valuation ring.*

It is easily seen that a valuation ring possesses a unique maximal ideal, consisting of all its nonunits.

THEOREM 6. *The residue class ring $R/P$ of a prime ideal $P$ of $R$ is a valua-ring whose unique maximal ideal is principal.*

First, we prove a lemma.

LEMMA 5. *If $P \in P_M$, then $f$ is singular modulo $P$ if and only if $f \in M$.*

*Proof.* Consider the equation

$$fX \equiv 1 \pmod{P}.$$

If $f \in M$, the equation clearly has no solution since $A(f) \cap A(p)$ is nonempty for all $p \in P$ (see [6, Theorem 4]).

On the other hand, if $f$ is not in $M$, there is a $p \in P$ such that $A(f) \cap A(p)$ is empty. Let $A^*(p) = \{a_n\}$, with $0_n(p) = l_n$, in which case $f(a_n) \neq 0$. The equation in question has a solution if and only if there exists a $g \in R$ such that

(i) $g(a_n) = \{f(a_n)\}^{-1}$,

and

(ii) $(fg)^{(k)}(a_n) = 0, \; k = 1, \cdots, l_n$.

Since

$$(fg)^{(k)} = fg^{(k)} + \sum_{i=1}^{k} \binom{k}{i} f^{(i)} g^{(k-i)}, \qquad \text{where } \binom{k}{i} = \frac{k!}{i!(k-i)!},$$

(ii) is satisfied if

(iii) $$g^{(k)}(a_n) = -\{f(a_n)\}^{-1} \sum_{i=1}^{k} \binom{k}{i} f^{(i)}(a_n)\, g^{(k-i)}(a_n).$$

Such a $g$ can be constructed by (2.2), whence

$$fg \equiv 1 \pmod{P}.$$

*Proof of Theorem* 6. By Lemma 5, every element of $R - M$ is a unit, so we may assume that $f, g \in M$. Let

$$A(d) = A(f) \cap A(g),$$

so that $A(f/d) \cap A(g/d)$ is empty. Clearly, at least one of $f/d, g/d \in R - M$, and hence is a unit modulo $P$. So $R/P$ is a valuation ring.

If, in particular, $f$ is chosen to be in $M - M^2$, $f/d$ cannot be in $M$, so $g$ is a multiple (modulo $P$) of $f$. Therefore the unique maximal ideal $M/P$ of $R/P$ is generated by $f$, and hence is principal.

If $P \neq P^*$, $R/P$ possesses the nonmaximal prime ideals $P_1/P$, where $P_1$ is a nonmaximal prime ideal of $R$ properly containing $P$. Moreover:

THEOREM 7. *The residue class ring $R/P$ of a nonmaximal prime ideal $P$ is Noetherian if and only if $P = P^*$.*

*Proof.* Every nonzero element of $M - P^*$ is in $M^k - M^{k-1}$, for some unique positive integer $k$. Hence every nonzero ideal of $R/P^*$ is of the form $(f^k)$, where $f \in M - M^2$.

If $f \in P - P^*$, construct $f_k$ such that

$$A^*(f_k) = A^*(f)$$

and

$$0_n(f_k) = \max\{0_n(f) - k,\ 1\}.$$

Then $f_{k+1}$ is a proper divisor (modulo $P$) of $f_k$. Hence the ideal generated by all the $f_k$ does not have finite basis.

The residue class ring $R/P^*$ is concretely identified below by the use of the structure theory of complete local rings [2] of Cohen. First we make a definition.

DEFINITION 7. (a) If the nonunits of a Noetherian ring $D$ with unit form a maximal ideal $M$ such that

$$\bigcap_{k=1}^{\infty} M^k = (0),$$

$D$ is called a *local ring.*

(b) If $f_1, \cdots, f_n$ is a minimal basis for $M$ such that $f_1, \cdots, f_i$ generate a prime ideal ($i = 1, \cdots, n$), $S$ is called a *regular* local ring.

(c) Using the powers of $M$ as a system of neighborhoods of 0, (thereby topologizing $D$), we call $D$ *complete* if every Cauchy sequence in $D$ has a (unique) limit.

THEOREM 8. *The residue class ring $R/P^*$ is isomorphic with the ring $K\{z\}$ of all formal power series over $K$.*

*Proof.* By Theorems 3, 4, 6, $R/P^*$ is a local ring and is trivially regular since $M/P^*$ is principal. Cohen [2, Theorem 15] has shown that every regular, complete, local ring, whose unique maximal ideal is principal, and such that $D/M$ is isomorphic to $K$, is isomorphic to $K\{z\}$. By [6, Theorem 6],

$$(R/P^*)/(M/P^*) \cong R/M \cong K.$$

The proof is completed by the following Lemma.

LEMMA 6. *The residue class ring $R/P^*$ is complete.*

*Proof.* Let $\{f_k\}$ be any Cauchy sequence in $R/P^*$. We may assume without loss of generality that $f_{k+1} - f_k \in M^k$, since a Cauchy sequence has at most one limit. Let

$$A_k = \{a_k, a_{k+1}, \cdots\} \in A(M),$$

with all $a_k$ distinct. Let

$$B_k = A_k \cap A(f_{k+1} - f_k).$$

Clearly, $B_k \in A(M)$, and $\bigcap_{k=1}^{\infty} B_k$ is empty. Hence, we may construct by (2.2) an $f \in R$ such that

$$f(z) = f_1(z) \qquad \text{for } z \in B_1,$$

and

$$f^{(k)}(z) = f_k^{(k)}(z) \qquad \text{for } z \in B_{k+1}.$$

Then

$$f_k \equiv f \pmod{M^k},$$

whence

$$L_{k \to \infty} f_k = f.$$

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