ON SUMS OF SERIES OF COMPLEX NUMBERS

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1. Introduction. We recall certain facts about the convergence of series.

1.1. Let $\sum_{i=1}^{\infty} a_i$ be a series of real numbers, $a_i \longrightarrow 0$. Then it is obvious that a sequence of signs $\epsilon_i = \pm 1$ $(i = 1, 2, \cdots)$ may be chosen so that $\sum_{i=1}^{\infty} \epsilon_i a_i$ is convergent. It is, furthermore, well known that all the possible sums so obtained form a perfect set, and if $\sum_{i=1}^{\infty} |a_i| = \infty$ then any preassigned sum may be obtained.

1.2. The first statement remains true also for complex numbers. Aryeh Dvoretzky and the author [2] proved that if $\sum_{i=1}^{\infty} c_i$ is a series of complex numbers with $c_i \longrightarrow 0$, then a sequence of signs $\epsilon_i = \pm 1$ $(i = 1, 2, \cdots)$ may be chosen so that $\sum_{i=1}^{\infty} \epsilon_i c_i$ converges and

$$\left| \sum_{i=1}^{n} \epsilon_i c_i \right| \leq \sqrt{3} \cdot \max |c_i| \qquad (n = 1, 2, \cdots).$$

1.3. The object of the present paper is to determine the sets of points which may be sums of the series $\sum_{i=1}^{\infty} \epsilon_i c_i$ when suitable sequences ϵ_i are chosen.

2. Notation and definitions. In this paper the following notations and definitions will be used.

2.1. NOTATION.

$c = a + ib$ denotes a term of a (finite or infinite) series of complex numbers, a being its real and ib its imaginary part;

$C = A + iB$ also denotes a complex number;

$\gamma = \alpha + i\beta$ denotes a direction in the plane of complex numbers, and also a unit vector in the same direction;

(C, C') is the scalar product of the vectors C and C'; that is $(C, C') = AA' + BB'$;

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γ' denotes a direction perpendicular to γ; that is, $(\gamma, \gamma') = 0$;

ϵ denotes ± 1;

$\sum$ without summation limits denotes summation by the summation index from 1 to ∞. In any other cases the summation limits will be indicated.

2.2. DEFINITION. C will be called an *attainable point* of the series $\sum c_i$ if a sequence ϵ_i $(i = 1, 2, \cdots)$ exists, such that $\sum \epsilon_i c_i = C$.

2.3. DEFINITION. Let $\sum c_i$ be a series of complex numbers with $c_i \longrightarrow 0$ and $\sum |c_i| = \infty$. We say that γ is a *direction of divergence* of the series $\sum c_i$ if a subseries $\sum c_{i*}$ of $\sum c_i$ exists such that

$$\sum |c_{i*}| = \infty$$

and

$$\frac{(c_{i*}, \gamma')}{(c_{i*}, \gamma)} \longrightarrow 0.$$

If γ is a direction of divergence, then clearly also the inverse direction $-\gamma$ is such. The directions γ and $-\gamma$ form an *axis of divergence*. It can easily be seen [3, p. 93] that if $\sum |c_i| = \infty$, then $\sum c_i$ has at least one axis of divergence.

2.4. DEFINITION. Let $\sum c_i$ be a series of complex numbers with $c_i \longrightarrow 0$ and $\sum |c_i| = \infty$. We define the *convergence strip* of $\sum c_i$ as follows:

If $\sum c_i$ has at least two axes of divergence, the convergence strip is the whole plane.

If $\sum c_i$ has exactly one axis of divergence, then the convergence strip is composed of all the lines parallel to this axis which contain attainable points of the series $\sum [\gamma'(c_i, \gamma')]$, where γ' is a unit vector perpendicular to the axis of divergence.

According to 1.1, the convergence strip is either i) a cartesian product of a perfect set by a straight line or ii) the whole plane. It is obvious that every attainable point of $\sum c_i$ is a point of the convergence strip.

3. Theorem. We shall establish the following result.

3.1. THEOREM. *Let $\sum c_i$ be a series of complex numbers which tend to* *zero, and let* $\sum |c_i| = \infty$; *then the attainable points of* $\sum c_i$ *form a set which is dense in the convergence strip of* $\sum c_i$, *and within this strip is dense on every straight line not parallel to the axis of divergence of* $\sum c_i$.

Proof. We may, without restricting the generality of the theorem, suppose the axis of divergence to be the real axis.

The following statement is clearly equivalent to our theorem: Let $C = A + iB$ be any point of the convergence strip, δ any real number, and η any positive number however small; then there exists an attainable point $C' = A' + iB'$ of $\sum c_i$ such that $|C - C'| < \eta$ and $A - A' = \delta(B - B')$. This will now be proved.

Put

$$\eta' = \frac{1}{4\sqrt{1+\delta^2}}\,\eta.$$

Let N_1 be such that $|c_i| < \eta'$ for every $i > N_1$. According to 1.1 , there exist $N_2 \geq N_1$ and a sequence $\epsilon_{i'}$ $(i' = 1, 2, \cdots, N_2)$ such that

$$\left| B - \sum_{i'=1}^{N_2} \epsilon_{i'}\, b_{i'} \right| < \eta'.$$

We put

$$C_1 = \sum_{i'=1}^{N_2} \epsilon_{i'}\, c_{i'}.$$

It is evident [3] that the series $\sum_{i=N_2+1}^{\infty} c_i$ can be separated into two subseries $\sum c_{i_k''}$ and $\sum c_{i_k'''}$, so that for $\sum c_{i_k''}$ we have

$$\sum |a_{i_k''}| = \infty \text{ and } \sum |b_{i_k''}| < \eta'.$$

According to 1.2, there exists a sequence $\epsilon_{i_k'''}$ $(k = 1, 2, \cdots)$ such that the series $\sum \epsilon_{i_k'''}\, c_{i_k'''}$ converges and

$$\left|\sum \epsilon_{i_k'''}\, c_{i_k'''}\right| < \sqrt{3}\,\eta'.$$

Let us put $C_2 = C_1 + \sum \epsilon_{i_k'''}\, c_{i_k'''}$. Now, according to 1.1 there exists a sequence $\epsilon_{i_k''}$ $(k = 1, 2, \cdots)$ such that $\sum \epsilon_{i_k''}(a_{i_k''} - \delta b_{i_k''})$ converges and

$$\sum \epsilon_{i_k''}(a_{i_k''} - \delta b_{i_k''}) = (A - A_2) - \delta(B - B_2).$$

Putting $C' = C_2 + \quad \epsilon_{i_k''} c_{i_k''}$, we get $A - A' = \delta(B - B')$ and

$$|B - B'| \leq |B - B_1| + |B_1 - B_2| + |B_2 - B'| < \eta' + \sqrt{3}\,\eta' + \eta' = \eta'(2 + \sqrt{3})$$

whence

$$|C - C'| < \eta' \sqrt{1 + \delta^2}\,(2 + \sqrt{3}) < \eta.$$

The series $\sum \epsilon_i c_i$ is composed of a finite subseries $\sum_{i'=1}^{N_2} \epsilon_{i'} c_{i'}$ and two interwoven subseries $\sum \epsilon_{i_k''} c_{i_k''}$ and $\sum \epsilon_{i_k'''} c_{i_k'''}$ which are evidently convergent and in which the order of terms remains unchanged. Consequently $\sum \epsilon_i c_i = C'$.

3.2. In special cases every point of the convergence strip can be an attainable point of $\sum c_i$, but generally this is not true. A few examples are given showing the possibility that the attainable points do not cover the convergence strip and even are not dense on every straight line parallel to the axis of divergence:

a) For

$$c_n = \frac{1}{n} + \frac{1}{3^n}\, i,$$

on every line parallel to the axis of divergence (real axis) there is at most one attainable point.

b) When the convergence strip is connected, a similar example may serve, namely:

$$c_n = \frac{1}{n} + \frac{1}{2^n}\, i.$$

Here on every line parallel to the real axis there are at most two attainable points.

c) The case when the convergence strip covers the whole plane is more complicated. The following example may suit:

$$c_k = \frac{1}{n} + \frac{1}{10^{n^2}}\, i, \quad 1 + \sum_{j=0}^{n-1} 10^{j^2} \leq k \leq \sum_{j=0}^{n} 10^{j^2}, \; c_1 = 0.$$

No attainable point is, for example, on the line through $(0, i/9)$ parallel to the real axis. For let us suppose that $C^* = A^* + i/9$ is such a point; then

$$C^* = \sum t_n \left(\frac{1}{n} + \frac{1}{10^{n^2}}\, i \right), \text{ where } |t_n| \leq 10^{n^2}.$$

On the other hand, there exists N^* such that for

$$k_i > \sum_{j=0}^{N^*} 10^{j^2} \qquad (i = 1, 2)$$

we have $|c_{k_1} - c_{k_2}| < 1$. Consequently, for $n > N^*$, we have $|t_n| < n$. It follows that

$$\frac{1}{9} = \sum \frac{t_n}{10^{n^2}},$$

where $|t_n| < n$ for $n > N^*$, which clearly is impossible.

4. Plane of attainable points. We now turn to the special cases in which every point of the complex plane is an attainable point.

4.1. THEOREM. *Let $\sum c_i$ be a series of complex numbers which tend to zero, and let $\sum |c_i| = \infty$. If $\sum c_i$ has at least two axes of divergence, then every complex number C is an attainable point of $\sum c_i$.*

Proof. By an affine transformation the two axes of divergence may be identified with the real and imaginary axes respectively.

The definition of axes of divergence implies the existence of two disjoint subseries $\sum c_{i_k'}$ and $\sum c_{i_k''}$ of $\sum c_i$ such that:

$$\frac{b_{i_k'}}{a_{i_k'}} \longrightarrow 0, \sum |a_{i_k'}| = \infty \text{ and } a_{i_k'} \neq 0 \qquad (k = 1, 2, \cdots),$$

$$\frac{a_{i_k''}}{b_{i_k''}} \longrightarrow 0, \sum |b_{i_k''}| = \infty \text{ and } b_{i_k''} \neq 0 \qquad (k = 1, 2, \cdots).$$

We shall now fix finite subseries

$$\sum_{l=1}^{k_n'} c_{i_l'(n)} \quad \text{and} \quad \sum_{l=1}^{k_n''} c_{i_l''(n)} \qquad (n = 1, 2, \cdots)$$

of $\sum c_{i_k'}$ and $\sum c_{i_k''}$, respectively, and N_n $(n = 1, 2, \cdots)$ as follows:

a) for every $i > N_n$, $|c_i| < 2^{-n}$;

b) for every $i_k' > N_n$, $|b_{i_k'}/a_{i_k'}| < 2^{-n}$, and

for every $i_k'' > N_n$, $|a_{i_k''}/b_{i_k''}| < 2^{-n}$;

c) $i'^{(n-1)}_{k'_{n-1}} \le N_n < i_1'^{(n)}$ and $i''^{(n-1)}_{k''_{n-1}} \le N_n < i_1''^{(n)}$;

d) $1 \le \sum_{l=1}^{k_n'} |a_{i_l'(n)}| < 1 + 2^{-n}$ and $1 \le \sum_{l=1}^{k_n''} |b_{i_l''(n)}| < 1 + 2^{-n}$.

From b) and d) we obtain

$$\sum_{l=1}^{k_n'} |b_{i_l'(n)}| < 2^{-n+1} \quad \text{and} \quad \sum_{l=1}^{k_n''} |a_{i_l''(n)}| < 2^{-n+1}.$$

We denote by $\sum c_{i_k'''}$ what remains of the series $\sum c_i$ after the subseries

$$\sum_n \sum_{l=1}^{k_n'} c_{i_l'(n)} \quad \text{and} \quad \sum_n \sum_{l=1}^{k_n''} c_{i_l''(n)}$$

are removed.

According to 1.2, there exists a sequence $\epsilon_{i_k'''}$ $(k = 1, 2, \cdots)$ such that $\sum \epsilon_{i_k'''} c_{i_k'''}$ converges. We put $C_1 = \sum \epsilon_{i_k'''} c_{i_k'''}$. We construct by induction a sequence of points C_n $(n = 1, 2, \cdots)$. Suppose that we have already fixed $C_1, C_2, \cdots, C_n$; we proceed to construct C_{n+1}. We fix signs $\epsilon_{i_l'(n)}$ $(l = 1, 2, \cdots k_n')$ so that, by addition of $-\epsilon_{i_l'(n)}\, a_{i_l'(n)}$ to

$$A - \left(A_n + \sum_{q=1}^{l-1} \epsilon_{i_q'(n)}\, a_{i_q'(n)} \right),$$

this expression either diminishes in absolute value or changes sign*. We put then

$$C_n' = C_n + \sum_{l=1}^{k_n'} \epsilon_{i_l'(n)}\, c_{i_l'(n)}.$$

Similarly we put

$$C_{n+1} = C_n' + \sum_{l=1}^{k_n''} \epsilon_{i_l''(n)}\, c_{i_l''(n)},$$

where $\epsilon_{i_l''(n)}$ $(l = 1, 2, \cdots, k_n'')$ are fixed so that, by adding $-\epsilon_{i_l''(n)}\, b_{i_l''(n)}$ to

$$B - \left(B_n' + \sum_{q=1}^{l-1} \epsilon_{i_q''(n)}\, b_{i_q''(n)}\right),$$

this expression either diminishes in absolute value or changes sign*. The series $\sum \epsilon_i c_i$ is composed of three interwoven subseries

$$\sum_n \sum_{l=1}^{k_n'} \epsilon_{i_l'(n)}\, c_{i_l'(n)}, \quad \sum_n \sum_{l=1}^{k_n''} \epsilon_{i_l''(n)}\, c_{i_l''(n)}, \quad \text{and} \quad \sum_k \epsilon_{i_k'''}\, c_{i_k'''},$$

which evidently are convergent and in which the order of terms remains unchanged. Consequently $\sum \epsilon_i c_i$ converges; and, as $C_n \longrightarrow C$, also $\sum \epsilon_i c_i = C$.

4.2. THEOREM. *Let $\sum c_i$ be a series of complex numbers which tend to zero, having exactly one axis of divergence. If $\sum c_i$ can be separated into two subseries $\sum c_{i_k}^-$ and $\sum c_{i_k}^=$, such that the convergence strip of $\sum c_{i_k}^-$ is the whole plane, and the attainable points of $\sum c_{i_k}^=$ cover a segment not parallel to the axis of divergence, then every complex number C is an attainable point of $\sum c_i$.*

This theorem is a direct outcome of Theorem 3.1.

4.3. THEOREM. *Let $\sum c_i$ be a series of complex numbers which tend to zero, having exactly one axis of divergence, and let γ' be a direction perpendicular to this axis. If $\sum c_i$ can be separated into two subseries $\sum c_{i_k'}$ and*

* Whenever this expression equals zero we put the next ϵ equal to $+1$.

$\sum c_{i_k''}$ such that the convergence strip of $\sum c_{i_k'}$ is the whole plane, $\sum |c_{i_k''}|$ converges, and

$$(1) \quad 0 < |(c_{i_k''}, \gamma')| \leq \sum_{l=k+1}^{\infty} |(c_{i_l''}, \gamma')| \qquad (k = 1, 2, \cdots),$$

then every complex number C is an attainable point of $\sum c_i$.

Proof. As usual, we assume that the axis of divergence is the real axis. Let $\sum c_{\bar{i}_k''(1)}$ be a tail of the series $\sum c_{i_k''}$ such that

$$\sum |a_{\bar{i}_k''(1)}| < \frac{1}{2},$$

and let η be any real number satisfying

$$(2) \qquad 0 < \eta < |b_{\bar{i}_1''(1)}|.$$

We form finite subseries

$$\sum_{l=1}^{k_n'} c_{i_l'(n)} \qquad (n = 1, 2, \cdots)$$

of $\sum c_{i_k'}$ so that the following conditions are satisfied:

$$i_{k_{n-1}'}^{\prime(n-1)} < i_1^{\prime(n)};$$

for every term $c_{i_l'(n)}$ $(l = 1, 2, \cdots, k_n'; \ n = 1, 2, \cdots)$, we have

$$(3) \qquad |a_{i_l'(n)}| < 2^{-n-3}$$

and

$$0 < \left| \frac{b_{i_l'(n)}}{a_{i_l'(n)}} \right| < \eta \cdot 2^{-n-2};$$

$$1 \leq \sum_{l=1}^{k_n'} |a_{i_l'(n)}| < 1 + 2^{-n-3}. \tag{4}$$

Consequently we have also

$$\sum_{l=1}^{k_n'} |b_{i_l'(n)}| < \eta \cdot 2^{-n-1}. \tag{5}$$

We denote by $\sum c_{i_k'''}$ what remains from the series $\sum c_i$ after the series

$$\sum_n \sum_{l=1}^{k_n'} c_{i_l'(n)} \text{ and } \sum c_{\bar{i}_k''(1)}$$

are removed. In consideration of

$$\sum |b_{i_k'}| = \infty \text{ and } \sum_n \sum_{l=1}^{k_n'} |b_{i_l'(n)}| < \frac{1}{2}\eta,$$

we get

$$\sum |b_{i_k'''}| = \infty,$$

and consequently also

$$\sum |a_{i_k'''}| = \infty.$$

The convergence strip of $\sum c_{i_k'''}$ is therefore the whole plane. By Theorem 3.1, there exists a sequence $\epsilon_{i_k'''}$ $(k = 1, 2, \cdots)$ such that $\sum \epsilon_{i_k'''} c_{i_k'''} = C_1$, with

$$A_1 = A, \; |B - B_1| < \eta. \tag{6}$$

We denote by $\sum_{l=1}^{k_n''} c_{i_l''(n)}$ some head and by $\sum_k c_{\bar{i}_k''(n+1)}$ the corresponding tail of $\sum c_{\bar{i}_k''(n)}$, and we construct by induction a sequence of points C_p, a increasing sequence of integers n_p and a sequence of integers k_p'' $(p = 1, 2, \cdots)$ having the following properties:

$$(7) \qquad |A - A_p| < \sum_{l=1}^{k''_{p-1}} |a_{i_l''(p-1)}| + 2^{-n_{p-2}-1} \qquad (n_{-1} = n_0 = 0),$$

$$(8) \qquad |B - B_p| < \frac{1}{2} \sum |b_{i_k''(p)}^{-}|,$$

$$(9) \qquad \eta \cdot 2^{-n_{p-1}} < \frac{1}{2} \sum |b_{i_k''(p)}^{-}|.$$

It can easily be verified with the use of (6), (2), and (1) that (7), (8), and (9) hold for $p = 1$.

Let us now suppose that we have already

$$n_q \text{ and } \sum_{l=1}^{k''_q} c_{i_l''(q)} \quad (q = 1, 2, \cdots, p-1) \text{ and } C_q \quad (q = 1, 2, \cdots, p),$$

and we proceed to construct

$$n_p, \sum_{l=1}^{k''_p} c_{i_l''(p)}, \text{ and } C_{p+1}.$$

We fix $\epsilon_{i_l'(n_{p-1}+1)}$ $(l = 1, 2, \cdots, k'_{n_{p-1}+1} - 1)$ so that by addition of

$$-\epsilon_{i_l'(n_{p-1}+1)}\, a_{i_l'(n_{p-1}+1)} \text{ to } \left[A - \left(A_p + \sum_{q=1}^{l-1} \epsilon_{i_q'(n_{p-1}+1)}\, a_{i_q'(n_{p-1}+1)}\right)\right],$$

this expression either diminishes in absolute value or changes sign. Now $\epsilon_{i_Q'(n_{p-1}+1)}$, where $Q = k'_{n_{p-1}+1}$, is fixed so that

$$B - \left(B_p + \sum_{l=1}^{a} \epsilon_{i_l'(n_{p-1}+1)}\, b_{i_l'(n_{p-1}+1)}\right) \neq 0.$$

We put then

$$C'_p = C_p + \sum_{l=1}^{a} \epsilon_{i_l'(n_{p-1}+1)}\, c_{i_l'(n_{p-1}+1)},$$

and fix $n_p > n_{p-1}$ so that

$$\eta \cdot 2^{-n_p} < \frac{1}{6} \, |B - B_p'| . \tag{10}$$

We proceed as before and fix $\bar{\epsilon}_{i_l'(q)}$ ($l = 1, 2, \cdots, k_q'$; $q = n_{p-1}+2, n_{p-1}+3, \cdots, n_p$) so that by addition of

$$-\bar{\epsilon}_{i_l'(q)} \, a_{i_l'(q)} \text{ to } \left[A - \left(A_p' + \sum_{r=n_{p-1}+2}^{q-1} \sum_{s=1}^{k_r'} \bar{\epsilon}_{i_s'(r)} \, a_{i_s'(r)} + \sum_{s=1}^{l-1} \bar{\epsilon}_{i_s'(q)} \, a_{i_s'(q)} \right) \right]$$

this expression either diminishes in absolute value or changes sign. If

$$\left| B - \left(B_p' + \sum_{q=n_{p-1}+2}^{n_p} \sum_{l=1}^{k_q'} \bar{\epsilon}_{i_l'(q)} \, b_{i_l'(q)} \right) \right| \geq |B - B_p'| ,$$

we leave $\epsilon_{i_l'(q)} = \bar{\epsilon}_{i_l'(q)}$; otherwise we put $\epsilon_{i_l'(q)} = -\bar{\epsilon}_{i_l'(q)}$. In either case, we denote

$$C_p'' = C_p' + \sum_{q=n_{p-1}+2}^{n_p} \sum_{l=1}^{k_q'} \epsilon_{i_l'(q)} \, c_{i_l'(q)} .$$

By (8), (5), and (9), we have

$$|B - B_p''| \leq |B - B_p| + |B_p - B_p''|$$

$$< \frac{1}{2} \sum |b_{\bar{i}_k''(p)}| + \eta \cdot 2^{-n_{p-1}-1} < \frac{3}{4} \sum |b_{\bar{i}_k''(p)}| .$$

On the other hand we have, by (10), $|B - B_p''| \geq |B - B_p'| > 6\eta \cdot 2^{-n_p}$. Consequently,

$$6\eta \cdot 2^{-n_p} < |B - B_p''| < \frac{3}{4} \sum |b_{\bar{i}_k''(p)}| . \tag{11}$$

We now fix $\epsilon_{i_1''(p)}$ so that

$$|B - (B_p'' + \epsilon_{i_1''(p)}\, b_{i_1''(p)})| < |B - B_p''|,$$

or

$$[B - (B_p'' + \epsilon_{i_1''(p)}\, b_{i_1''(p)})] \cdot (B - B_p'') < 0,$$

and $\epsilon_{i_l''(p)}$ $(l = 2, 3, \cdots, k_p'')$ so that by addition of

$$-\epsilon_{i_l''(p)}\, b_{i_l''(p)} \text{ to } \left[B - \left(B_p'' + \sum_{q=1}^{l-1} \epsilon_{i_q''(p)}\, b_{i_q''(p)}\right)\right],$$

this expression diminishes in absolute value without changing sign, $b_{i_{k_p''}''(p)}$ being the last term of $\sum b_{\bar{i}_l''(p)}$ for which such operation is possible.

Such $b_{i_{k_p''}''(p)}$ exists in view of (11) and (1).

We put

$$C_{p+1} = C_p'' + \sum_{l=1}^{k_p''} \epsilon_{i_l''(p)}\, c_{i_l''(p)}.$$

The construction of n_p, $\sum_{l=1}^{k_p''} c_{i_l''(p)}$, and C_{p+1} is thus completed. It remains to show that conditions (7)-(9) are fulfilled for these indices.

We have

$$|A - A_{p+1}| \leq |A - A_p'| + |A_p' - A_p''| + |A_p'' - A_{p+1}|;$$

but in view of (7), (4), and (3),

$$|A - A_p'| \leq 2 \max_{l=1,2,\cdots,k'_{n_{p-1}+1}} |a_{i_l'(n_{p-1}+1)}| < 2^{-n_{p-1}-3},$$

$$|A_p' - A_p''| \leq |A - A_p'| + \max_{l=1,2,\cdots,k'_{n_p}} |a_{i_l'(n_p)}| < 3.2^{-n_{p-1}-4},$$

and

$$|A_p'' - A_{p+1}| \leq \sum_{l=1}^{k_p''} |a_{i_l''(p)}|.$$

Consequently,

$$|A - A_{p+1}| < \sum_{l=1}^{k_p''} |a_{i_l''(p)}| + 2^{-n_{p-1}-1},$$

so that (7) holds for this index.

For (8), we note that clearly

$$|B - B_{p+1}| < |b_{\bar{i}_1''(p+1)}|,$$

and therefore, in view of (1),

$$|B - B_{p+1}| < \frac{1}{2} \sum |b_{\bar{i}_k''(p+1)}|.$$

Finally, if

$$[B - (B_p'' + \epsilon_{i_1''(p)}\, b_{i_1''(p)})] \cdot (B - B_p'') \geq 0,$$

then in view of (11) we have

$$\sum |b_{\bar{i}_k''(p+1)}| = \sum |b_{\bar{i}_k''(p)}| - \sum_{l=1}^{k_p''} |b_{i_l''(p)}| \geq \sum |b_{\bar{i}_k''(p)}| - |B - B_p''|$$

$$> \frac{1}{4} \sum |b_{\bar{i}_k''(p)}| > 2\eta \cdot 2^{-n_p}.$$

If, on the other hand

$$[B - (B_p'' + \epsilon_{i_1''(p)}\, b_{i_1''(p)})] \cdot (B - B_p'') < 0,$$

then by (1) and (11) we have

$$\sum |b_{\bar{i}_k''(p+1)}| \geq |B - B_p''| > 6\eta \cdot 2^{-n_p}.$$

Thus (9) holds in either case.

In order to prove that $\sum \epsilon_i c_i$ converges it is sufficient to point out that this series is composed of three interwoven subseries

$$\sum \epsilon_{i_k'''} c_{i_k'''}\,, \quad \sum_n \sum_{l=1}^{k_n'} \epsilon_{i_l'(n)} c_{i_l'(n)}\,, \quad \text{and} \quad \sum \epsilon_{\bar{i}_k''(1)} c_{\bar{i}_k''(1)}$$

which are evidently convergent and in which the order of the terms remains unchanged.

As, according to (7) and (8) above, we have $C_p \longrightarrow C$, it follows that $\sum \epsilon_i c_i = C$.

4.4. The following examples illustrate the way in which the above result may be applied:

a) Let

$$\sum_n \left(\frac{1}{\sqrt{n}} + \frac{1}{n} i \right)$$

be the series in question.

If we put

$$\sum_k c_{n_k''} = \sum_k \left(\frac{1}{\sqrt{2^k}} + \frac{1}{2^k} i \right),$$

that is, the subseries of those terms for which n is a power of 2, and $\sum_l c_{n_l'}$, the remaining subseries, then the assumptions of Theorem 4.3 are fulfilled, and therefore every complex number C is an attainable point of our series.

b) The terms of the subseries $\sum_k c_{i_k''}$ may be composed of two or more terms of the series $\sum c_i$, as the following example shows:

$$\sum c_n = \sum_n \left(a_n + \frac{1}{n} i \right),$$

where

$$0 < a_{n+1} \leq a_n \quad (n = 1, 2, \cdots),\ a_n \longrightarrow 0, \ \text{and} \ na_n \longrightarrow \infty.$$

If we put

$$\sum_k c_{n_k''} = \sum_{k=2}^{\infty} \left[(a_{3k} - a_{3k+1}) + \left(\frac{1}{3k} - \frac{1}{3k+1} \right) i \right],$$

and $\sum_l c_{n_l}'$ the remaining subseries, the assumptions of Theorem 4.3 are fulfilled, and in this case too every complex number C is an attainable point of c_i.

5. Further considerations. We make the following observations.

5.1. For an absolutely convergent series $\sum c_i$ of complex numbers, the attainable points form a perfect set. The proof does not vary from the proof of a well-known similar theorem for series of real numbers (see 1.1).

5.2. Instead of $\epsilon = \pm 1$, more general convergence- and sum-factors have been introduced by E. Calabi and A. Dvoretzky [1]. They call a set Z of complex numbers a sum-factor set if, given any series $\sum c_i$ $(\sum |c_i| = \infty,\ c_i \longrightarrow 0)$, and any number C, there exists a sequence $\zeta_n \in Z$ $(n = 1, 2, \cdots)$ for which $\sum_n \zeta_n c_n = C$. It was shown by them that a bounded set Z is a sum-factor set if and only if 0 is an interior point of its convex hull.

5.3. All the theorems proved in this paper may reasonably be extended to results concerning vectors in n-dimensional Euclidean spaces.

References

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2. A. Dvoretzky and H. Hanani, *Sur les changements des signes des termes d'une série à termes complexes*, C. R. Acad. Sci. Paris 225 (1947), 516-518.

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