ON LINEAR INDEPENDENCE OF SEQUENCES IN A BANACH SPACE

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1. A. Dvoretzky has raised the following problem:

Let $x_1, x_2, \cdots, x_n, \cdots$ be an infinite sequence of unit vectors in a Banach space which are linearly independent in the algebraic sense; that is,

$$\sum_{i=1}^{k} c_i x_{n_i} = 0 \implies c_i = 0 \qquad (i = 1, \cdots, k).$$

Does there exist an infinite subsequence $\{x_{n_i}\}$ which is linearly independent in a stronger sense?

We may consider three types of linear independence of a sequence of unit vectors in a normed linear space:

I. $$\sum_{n=1}^{\infty} c_n x_n = 0 \implies c_n = 0 \qquad (n = 1, 2, \cdots).$$

II. If $\phi(k) > 0$ is any function defined for $k = 1, 2, \cdots$, then

$$|c_n^{(k)}| < \phi(k) \qquad (n, k = 1, 2, \cdots)$$

and

$$\lim_{k \to \infty} \sum_{n=1}^{\infty} c_n^{(k)} x_n = 0$$

imply

$$\lim_{k \to \infty} c_n^{(k)} = 0 \qquad (n = 1, 2, \cdots).$$

Received October 29, 1952.
Pacific J. Math. 3 (1953), 689-694

III. $$\lim_{k\to\infty} \sum_{k=1}^{\infty} c_n^{(k)} x_n = 0 \implies \lim_{k\to\infty} c_n^{(k)} = 0 \qquad (n = 1, 2, \cdots).$$

It is obvious that III implies both II and I; and if

$$\liminf_{k\to\infty} \phi(k) > 0$$

then II implies I. It is easy to show that the converse implications do not hold.

In this note we give an affirmative answer to Dvoretzky's question if independence is defined in the sense I or even II for arbitrary $\phi(k)$. However the answer is in the negative if indepence is defined in the sense III.

2. **The negative part** is proved by the following example due to G. Szegö [**1**; I, p. 86]:

THEOREM. *If* $\{\lambda_n\}$ *is a sequence of positive number with* $\lambda_n \longrightarrow \infty$, *then the functions* $\{1/(x+\lambda_n)\}$ *are complete in every finite positive interval.*

Obviously every infinite subsequence of $\{1/(x+\lambda_n)\}$ satisfies the condition of the theorem and is therefore complete.

3. **For the affirmative part** of our result we prove the following:

THEOREM. *Let* $\{x_{n_i}\}$ *be an infinite sequence of algebraically linearly independent unit vectors in a Banach space and let* $\phi(k) > 0$ *be any function defined for* $k = 1, 2, \cdots$. *Then there exists an infinite subsequence* $\{x_{n_i}\}$ *such that* $|c_i^{(m)}| < \phi(i)$ $(i, m = 1, 2, \cdots)$ *and*

$$\lim_{m\to\infty} \sum_{i=1}^{\infty} c_i^{(m)} x_{n_i} = 0$$

imply

$$\lim_{m\to\infty} c_i^{(m)} = 0 \qquad (i = 1, 2, \cdots).$$

It was pointed out to us by the referee that it suffices to prove the theorem for a separable Hilbert space. The separability may be assumed since we may restrict our attention to the subspace spanned by $\{x_n\}$. Now every separable Banach space can be imbedded isometrically in the space $C(0, 1)$ of continuous functions over the interval $(0, 1)$; and $C(0, 1) \subset L_2(0, 1)$, where linear independence, in any of the above defined senses, in L_2 implies the same independence in C. Let $\{z_n\}$ be the orthonormal sequence obtained from $\{x_n\}$ by the Gram-Schmidt process; then

$$x_n = \sum_{m=1}^{n} a_{nm} z_m,$$

with $|a_{nm}| \leq 1$ and $a_{nn} \neq 0$.

Since $\{a_{nm}\}$ is bounded for fixed m, we can select a subsequence $\{x_{n_i}\}$ such that

$$\lim_{i \to \infty} a_{n_i m} = b_m$$

exists for every m.

If we prove the theorem for $\psi(k) \geq \phi(k)$, then it is proved *a fortiori* for $\phi(k)$. Hence we may set

$$\psi(n) = \max\{1, \phi(1), \cdots, \phi(n)\},$$

so that $\psi(n) \geq 1$ and $\psi(n)$ is nondecreasing.

If the theorem we false then for every infinite subsequence $\{y_k\}$ of $\{x_{n_i}\}$ there would exist a sequence of sequences $\{c_k^{(m)}\}$ with

$$|c_k^{(m)}| < \psi(k) \qquad (k, m = 1, 2, \cdots)$$

and

$$\lim_{m \to \infty} \sum_{k=1}^{\infty} c_k^{(m)} y_k = Q$$

while

$$\limsup_{m \to \infty} |c_{k_0}^{(m)}| \neq 0 \quad \textit{for some fixed } k_0.$$

We can then select a subsequence of sequences $\{c_k^{(m_i)}\}$ such that

$$\lim_{i \to \infty} c_k^{(m_i)} = c_k$$

exists for every k, and $c_{k_0} \neq 0$. For convenience of notation we assume

$$\lim_{n \to \infty} c_k^{(m)} = c_k.$$

Since $c_{k_0} \neq 0$, there would exist a least $k_1 \geq k_0$ such that

$$|c_k| < 2^{k-k_0} \psi(k) |c_{k_0}| \qquad \text{for all } k > k_1.$$

This implies

$$(1) \qquad |c_k^{(m)}| \leq 2^{k-k_1} \psi(k) \; |c_{k_1}^{(m)}| \qquad \text{for all } k \geq k_1; \; m > m_0.$$

Case A: $b_{n_{i_j}} = 0$ *for* $j = 1, 2, \cdots$.

In order to simplify notation we assume $b_{n_i} = 0$ for all $i = 1, 2, \cdots$ by omitting all terms with $n_i \neq n_{i_j}$ from our subsequence. We select the subsequence $\{y_k\}$ as follows:

$$y_1 = x_{n_1}, \; y_{k+1} = x_{n_{i_{k+1}}}$$

where

$$|a_{n_{i_{k+1}}, n_j}| < \frac{|a_{n_j, n_j}|}{4^{k+1} \psi(k+1)} \qquad \text{for } j = 1, 2, \cdots, i_k.$$

We write $y_k = x_{l_k}$.

If the theorem were false then there would exist a sequence of sequences $\{c_k^{(m)}\}$ with the above properties such that

$$\left\| \sum_{k=1}^{\infty} c_k^{(m)} y_k \right\| = \epsilon_m \longrightarrow 0 \quad \text{as} \quad m \longrightarrow \infty.$$

If we take the k_1 defined in (1), then

$$(2) \qquad \left| \sum_{k=k_1}^{\infty} c_k^{(m)} a_{l_k, l_{k_1}} \right| \leq \epsilon_m;$$

but for all $m > m_0$ we have

$$|c_k^{(m)}| \le 2^{k-k_1} \psi(k) |c_{k_1}| \qquad (k = k_1, k_1+1, \cdots),$$

and hence

$$\left| \sum_{k=k_1+1}^{\infty} c_k^{(m)} a_{l_k, l_{k_1}} \right| < \sum_{k=k_1+1}^{\infty} \frac{2^{k-k_1} \psi(k) |c_{k_1}| |a_{l_{k_1}, l_{k_1}}|}{4^k \psi(k)} = 2^{-2k_1} |c_{k_1}| |a_{l_{k_1}, l_{k_1}}|.$$

We can now choose m so large that

$$|c_{k_1}^{(m)} - c_{k_1}| < 2^{-4k_1} |c_{k_1}| |a_{l_{k_1}, l_{k_1}}| \text{ and } \epsilon_m < 2^{-4k_1} |c_{k_1}| |a_{l_{k_1}, l_{k_1}}|.$$

Then for the left side of (2) we obtain

$$\left| \sum_{k=k_1}^{\infty} c_k^{(m)} a_{l_k, l_{k_1}} \right| \ge |c_{k_1}^{(m)}| |a_{l_{k_1}, l_{k_1}}| - \left| \sum_{k=k_1+1}^{\infty} c_k^{(m)} a_{l_k, l_{k_1}} \right|$$

$$\ge |c_{k_1}| |a_{l_{k_1}, l_{k_1}}| - 2^{-4k_1} |c_{k_1}| |a_{l_{k_1}, l_{k_1}}| - 2^{-2k_1} |c_{k_1}| |a_{l_{k_1}, l_{k_1}}|$$

$$> 2^{-4k_1} |c_{k_1}| |a_{l_{k_1}, l_{k_1}}|,$$

while for the right side of (2) we have

$$\epsilon_m < 2^{-4k_1} |c_{k_1}| |a_{l_{k_1}, l_{k_1}}|,$$

a contradiction.

Case B: $b_{n_i} \neq 0$ except for a finite number of i.

Without loss of generality we may assume $b_{n_i} \neq 0$ for all i by omitting a finite number of elements from $\{x_{n_i}\}$. We select the subsequence $\{y_k\}$ as follows:

$$y_1 = x_{n_1}, \quad y_{k+1} = x_{n_{i_{k+1}}},$$

where

$$|a_{n_{i_{k+1}}, n_j} - b_{n_j}| < \frac{|b_{n_{i_{k+1}}}|}{4^{k+1}\,\psi(k+1)} \qquad \text{for } j = 1, 2, \cdots, i_k .$$

For simplicity we again write $y_k = x_{l_k}$.

If the theorem were false then there would exist sequences $\{c_k^{(m)}\}$ with the foregoing properties such that

$$\|\sum c_k^{(m)} y_k\| = \epsilon_m \longrightarrow 0 \quad \text{as} \quad m \longrightarrow \infty.$$

If we let k_1 be defined as in (1), then on the one hand we have

$$Q = \left| \frac{1}{b_{l_{k_1}}} \sum_{k=k_1}^{\infty} c_k^{(m)} a_{l_k, l_{k_1}} - \frac{1}{b_{l_{k_1+1}}} \sum_{k=k_1+1}^{\infty} c_k^{(m)} a_{l_k, l_{k_1+1}} \right|$$

$$\geq |c_{k_1}^{(m)}| - \frac{|c_{k_1}^{(m)}|}{4^{k_1}} - \sum_{k=k_1+1}^{\infty} \frac{2|c_k^{(m)}|}{4^k\psi(k)}$$

$$\geq |c_{k_1}^{(m)}| \left(1 - \frac{1}{4^{k_1}} - \sum_{k=k_1+1}^{\infty} \frac{2\cdot 2^{k-k_1}\psi(k)}{4^k\,\psi(k)}\right) > \frac{1}{2}\,|c_{k_1}^{(m)}| > \frac{1}{4}\,|c_{k_1}| > 0$$

for all $m > m_0$; on the other hand, we have

$$Q \leq \frac{1}{b_{l_{k_1}}} \left\| \sum_{k=1}^{\infty} c_k^{(m)} y_k \right\| + \frac{1}{b_{l_{k_1+1}}} \left\| \sum_{k=1}^{\infty} c_k^{(m)} y_k \right\|$$

$$\leq \left(\frac{1}{b_{l_{k_1}}} + \frac{1}{b_{l_{k_2}}} \right) \epsilon_m < \frac{1}{4}\,|c_{k_1}|$$

for all sufficiently large m, a contradiction.

REFERENCES

1. R. Courant, D. Hilbert, *Mothoden der mathematischen Physik*, Berlin, 1931.

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