# INVARIANT EXTENSION OF LINEAR FUNCTIONALS

V. L. KLEE, JR.

**1. Introduction.** The expression $[L, \mathfrak{J}, f, p]$ will mean (i) $L$ is a real linear space and $\mathfrak{J}$ a set of linear transformations of $L$ into $L$; (ii) $f$ is a linear functional on a linear subspace $D_f$ of $L$; (iii) $p$ is a positively homogeneous subadditive functional on $L$; (iv) $f \leqq p$; (v) $TD_f \subset D_f$ and $fT = f$ for each $T \in \mathfrak{J}$. If $[L, \mathfrak{J}, f, p]$, then $\langle L, \mathfrak{J}, f, p\rangle$ will denote the set of all $F$ such that $[L, \mathfrak{J}, F, p]$, $D_F = L$, and $F \mid D_f = f$. With $I$ denoting the identity transformation on $L$, the Hahn-Banach theorem [**2**, p. 28] asserts that if $[L, \{I\}, f, p]$, then $\langle L, \{I\}, f, p\rangle$ is nonempty. More general conditions have been obtained by Agnew and Morse [**1**] and Woodbury [**10**] under which $\langle L, \mathfrak{J}, f, p\rangle$ is nonempty, and by Dunford [**3**] and Yood [**11**] under which $\langle L, \mathfrak{J}, f, mp\rangle$ includes for some $m \geqq 1$ an $F$ which is not identically zero. Their results have applications to the extension and existence of measures [**1**; **3**; **10**; **11**], limits, and so on [**1**], and in proving the "normality" (as used in connection with the Banach-Tarski paradox) of certain sets [**6**]. We prove here a theorem whose corollaries include an extension of the results of Agnew and Morse and Woodbury, and also include the principal results of Dunford and Yood, although Yood's work with relaxed boundedness conditions is not covered here. In addition to the cases in which $\mathfrak{J}$ is a commutative semigroup or a finite or solvable group, we are able to handle the case in which $\mathfrak{J}$ is a compact group of bounded linear transformations.

**2. The theorem.** We shall use the following result.

(2.1) LEMMA. *Suppose $L$, $\mathfrak{J}$, $f$, and $p$ satisfy conditions* (i)-(iii), *and for each $x \in L$ let*

$$q(x) = \inf \left\{ p\left( x + \sum_{1}^{k} T_i y_i \right) \middle| k \text{ a positive integer, } T_i \in \mathfrak{J},\ y_i \in L \right\}.$$

*Then $f \leq q$ on $D_f$ if and only if there exists $F \in \langle L, \{I\}, f, p\rangle$ such that $FT \equiv 0$ on $L$ for each $T \in \mathfrak{J}$.*

Received January 16, 1953.
*Pacific J. Math.* 4 (1954), 37-46

*Proof.* If $F$ is as described, $x \in D_f$, $T_i \in \mathfrak{I}$, and $y_i \in L$, then

$$f(x) = F(x) = F\left(x + \sum_1^k T_i y_i\right) \leqq p\left(x + \sum_1^k T_i\, y_i\right),$$

and thus $f \leqq q$ on $D_f$. Now suppose conversely that $f \leqq q$ and note first that for arbitrary $x \in L$, $T_i \in \mathfrak{I}$, and $y_i \in L$,

$$0 = f(0) \leqq q(0) \leqq p\left(\sum_1^k T_i y_i\right) \leqq p(-x) + p\left(x + \sum_1^k T_i y_i\right),$$

whence $q(x) \geqq -p(-x)$ and $q$ is everywhere finite-valued. Clearly $q$ is positively homogeneous. Furthermore, if $x$ and $x'$ are points of $L$ and $\epsilon > 0$ there are transformations $T_i$, $T_i'$ in $\mathfrak{I}$ and points $y_i$, $y_i'$ in $L$ such that

$$p\left(x + \sum_1^k T_i\, y_i\right) \leqq q(x) + \epsilon,$$

and similarly for $x'$. But then

$$\begin{aligned} q(x + x') &\leqq p\left(x + x' + \sum_1^k T_i\, y_i + \sum_1^{k'} T_i{}' y_i{}'\right) \\ &\leqq p\left(x + \sum_1^k T_i\, y_i\right) + p\left(x' + \sum_1^{k'} T_i{}' y_i{}'\right) \\ &\leqq q(x) + q(x') + 2\epsilon. \end{aligned}$$

Thus $q$ is subadditive, and since $f \leqq q$ the Hahn-Banach theorem quarantees the the existence of an $F \in \langle L, \{I\}, f, q\rangle \subset \langle L, \{I\}, f, p\rangle$. For $T \in \mathfrak{I}$ and $z \in L$ we have

$$F(Tz) \leqq p(Tz + T(-z)) = p(0) = 0.$$

Similarly, $F(T(-z)) \leqq 0$, so $F$ has the desired properties and the proof is complete.

(2.2) THEOREM. *If* $[L, \mathfrak{I}, f, p]$, *then the statements* $(\alpha) - (\gamma)$ *below are equivalent. If* $[L, \mathfrak{I}, f, p]$ *and* $\mathfrak{I}$ *is a semi-group, then* $(\alpha) - (\delta)$ *are equivalent:*

($\alpha$) *$\langle L, \mathfrak{J}, f, p\rangle$ is nonempty;*

($\beta$) *for each finite $\mathfrak{S} \subset \mathfrak{J}$, $\langle L, \mathfrak{S}, f, p\rangle$ is nonempty;*

($\gamma$) *if $x \in D_f$, $T_i \in \mathfrak{J}$, and $y_i \in L$, then $f(x) \leqq p(x + \sum_1^k (T_i - I)\, y_i)$;*

($\delta$) *there exists $g \in \langle L, \{I\}, f, p\rangle$ such that $gST = gTS$ and $gT \leqq p$ whenever $S, T \in \mathfrak{J}$.*

*Proof.* Since $F \in \langle L, \mathfrak{J}, f, p\rangle$ if and only if $F \in \langle L, \{I\}, f, p\rangle$ and $FR \equiv 0$ for each $R \in \mathfrak{J} - I$, from the lemma it follows at once that ($\alpha$) and ($\gamma$) are equivalent, and from this it follows that they are both equivalent to ($\beta$). Now if $F \in \langle L, \mathfrak{J}, f, p\rangle$ and $S, T \in \mathfrak{J}$, then $FST = F = FTS$ and $FT = F \leqq p$, so ($\alpha$) implies ($\delta$). We complete the proof by showing that if $[L, \mathfrak{J}, f, p]$ and $\mathfrak{J}$ is a semi-group, then ($\delta$) implies ($\gamma$).

Consider arbitrary $x \in D_f$, $T_i \in \mathfrak{J}$, $y_i \in L$, and let

$$\epsilon = p\left(x + \sum_1^k (T_i - I)\, y_i\right) - f(x).$$

We wish to prove $\epsilon \geqq 0$. Let $\Phi_n$ be the set of all functions on $\{1, \cdots, k\}$ to $\{0, \cdots, n\}$; and, for $1 \leqq i \leqq k$, let $\Phi_{n,i}$ be the set of all $\phi \in \Phi_n$ for which $\phi(i) = 0$. For $\phi \in \Phi_n$ let

$$S_\phi = T_1^{\phi(1)}\, T_2^{\phi(2)} \cdots T_k^{\phi(k)}.$$

Define

$$A = \sum_{i=1}^k \sum_{\phi \in \Phi_n} gS_\phi\, (T_i - I)\, y_i.$$

Now $g$ is linear, $g = f$ on $D_f$, $gS_\phi \leqq p$ on $L$, and $fS_\phi = f$ on $D_f$, so we have

$$A = \sum_{\phi \in \Phi_n} gS_\phi \left[x + \sum_1^k (T_i - I)\, y_i\right] - \sum_{\phi \in \Phi_n} fS_\phi x \leqq (n+1)^k\, \epsilon.$$

But $\mathfrak{J}$ is a semi-group and $gST = gTS$ for $S, T \in \mathfrak{J}$, so the terms $S_\phi T_i y_i$ for $\phi(i) \neq n$ and $S_\phi(-I)\, y_i$ for $\phi(i) \neq 0$ all cancel out, and we have

$$A = \sum_{i=1}^k \sum_{\phi \in \Phi_{n,i}} gS_\phi\, (T_i^{n+1} - I)\, y_i.$$

Now since $gT \leqq p$ for all $T \in \mathfrak{J}$,

$$gS_\phi(T_i^{n+1} - I)y_i = -[\, gS_\phi T_i^{n+1}(-y_i) + gS_\phi y_i \,] \geqq -[p(-y_i) + p(y_i)].$$

Hence

$$A \geqq -\sum_{i=1}^{k}{}' \sum_{\phi \in \Phi_{n,i}} [p(-y_i) + p(y_i)] = (n+1)^{k-1} B,$$

with

$$B = -\sum_{1}^{k} [p(-y_i) + p(y_i)].$$

Thus $B \leqq (n+1)\,\epsilon$ for each $n$, and since $B$ is finite and independent of $n$ this implies that $\epsilon \geqq 0$, completing the proof.

The expression $[L, \mathfrak{J}, f, p]_b$ will mean that $[L, \mathfrak{J}, f, p]$, $b \geqq 1$, and $\mathfrak{J}$ is p-bounded with bound $b$, that is, $pT \leqq bp$ for each $T \in \mathfrak{J}$.

By using the lemma and the equivalence of ($\alpha$) and ($\delta$) in (2.2), one can prove that if $[L, \mathfrak{J}, f, p]_1$ and $\mathfrak{J}$ is a semi-group, then ($\alpha$) – ($\delta$) are equivalent to

($\epsilon$) if $x \in D_f$, $S_i \in \mathfrak{J}$, $T_i \in \mathfrak{J}$, and $y_i \in L$, then

$$f(x) \leqq p(x + \sum_{1}^{k} (S_i T_i - T_i S_i)\, y_i).$$

**3. The corollaries.** The closure and convex hull of a set $X$ will be denoted by $\mathrm{Cl}\, X$ and $\mathrm{conv}\, X$ respectively.

(3.1) COROLLARY. *Suppose $[L, \mathfrak{J}, f, p]_b$ and either $b = 1$ or $p \geqq 0$. Then each of the following implies that $\langle L, \mathfrak{J}, f, bp\rangle$ is nonempty:*

(a) *$\mathfrak{J}$ is a commutative semi-group;*

(b) *there is a linear transformation $R$ of $L$ into $L$ such that $R\mathfrak{J} = R$, $fR = f$, and $pR(z) \in \mathrm{Cl}\, p(\mathrm{conv}\, \mathfrak{J}(z))$ for each $z \in L$;*

(c) *every finite subset of $\mathfrak{J}$ is contained in a finite subgroup of $\mathfrak{J}$;*

(d) *p is a norm for $L$ and $\mathfrak{J}$ is a group which is compact in the uniform topology of operators.*

*Proof.* For $b = 1$ we have $p = bp$; and for $p \geqq 0$, $p \leqq bp$ (since, as always, $b \geqq 1$). Thus $[L, \mathfrak{J}, f, bp]$, and (2.2) is applicable.

(a) The Hahn-Banach theorem guarantees the existence of a

$$g \in \langle L, \{I\}, f, p\rangle \subset \langle L, \{I\}, f, bp\rangle .$$

Since $\mathfrak{J}$ is commutative, $gST = gTS$ for $S, T \in \mathfrak{J}$. Furthermore, $gT \leqq pT \leqq bp$. Thus ($\delta$) of (2.2) is satisfied for $[L, \mathfrak{J}, f, bp]$, and the desired conclusion follows from (2.2).

(b) Let $R$ be as described. Then for $x \in D_f$, $T_i \in \mathfrak{J}$, and $y_i \in L$, we have

$$f(x) = fR(x) \leqq pR(x) = p\left(Rx + \sum_1^k (RT_i - R)\, y_i\right) = pR(z),$$

with

$$z = x + \sum_1^k (T_i - I)\, y_i .$$

But $pR(z) \in \mathrm{Cl}\; p(\mathrm{conv}\; \mathfrak{J}(z))$, so for each $\epsilon > 0$ there are $S_1, \cdots, S_n$ in $\mathfrak{J}$ and $t_1, \cdots, t_n \geqq 0$ with sum 1 such that $pR(z)$ differs by less than $\epsilon$ from $pS(z)$, with

$$S = \sum_1^n t_i\, S_i .$$

Now

$$pS(z) \leqq \sum_1^n t_i\, pS_i(z) \leqq \sum_1^n t_i\, bp(z) = bp(z).$$

Thus $f(x) \leqq bp(z)$, so ($\gamma$) of (2.2) is satisfied and the desired conclusion follows from (2.2).

(c) In order to show that $\langle L, \mathfrak{J}, f, bp\rangle$ is nonempty it will suffice, in view of ($\alpha$) and ($\beta$) in (2.2), to show that (b) must hold if $\mathfrak{J}$ is a finite group. Let $T_1, \cdots, T_n$ be the members of $\mathfrak{J}$ and $R = (1/n)(T_1 + \cdots + T_n)$. Then the last two conditions of (b) are clearly satisfied, and $\mathfrak{J}\, T_j$ is merely a permutation of $\mathfrak{J}$, so $RT_j = R$ for each $j$.

The proof of (3.1) will be completed by showing that (d) implies (b), but we defer this until § 4.

(3.2) COROLLARY. *Suppose* $[L, \mathfrak{I}, f, p]_1$, $\mathfrak{I}$ *is a semi-group, and for some* $x \in D_f$, $f(x) = p(x)$ *and* p *has a unique supporting functional at* $x$ *(that is, there is a unique linear functional* $F \leqq p$ *on* $L$ *such that* $F(x) = p(x)$*). Then* $\langle L, \mathfrak{I}, f, p\rangle$ *is nonempty.*

*Proof.* Consider an arbitrary $S \in \mathfrak{I}$; $S$ generates in $\mathfrak{I}$ a commutative semi-group $S^*$, so by (3.1) (a) there must exist $F_S \in \langle L, S^*, f, p\rangle$. Now $F_S \leqq p$ and $F_S(x) = f(x) = p(x)$, so $F_S = F$. Thus we have

$$F \in \bigcap_{S \in \mathfrak{I}} \langle L, S^*, f, p\rangle = \langle L, \mathfrak{I}, f, p\rangle,$$

completing the proof.

(3.3) COROLLARY. *Suppose* $[L, \mathfrak{I}, f, p]_b$, *either* $b = 1$ *or* $p \geqq 0$, *and* $j$ *is a positive integer. Then each of the following implies that* $\langle L, \mathfrak{I}, f, b^j p\rangle$ *is nonempty*:

(e) $\mathfrak{I}$ *is a group and* $\langle L, \mathfrak{I}^{(k)}, f, b^{j-k}p\rangle$ *is nonempty for some integer* $k$ *with* $0 \leqq k \leqq j$, *where* $\mathfrak{I}^{(k)}$ *is the* $k^{th}$ *derived group of* $\mathfrak{I}$;

(f) $\mathfrak{I}$ *is a solvable group, with* $\mathfrak{I}^{(j)} = \{I\}$.

*Proof* (e). It suffices to show that if $1 \leqq k \leqq j$ and there exists $g \in \langle L, \mathfrak{I}^{(k)}, f, b^{j-k}p\rangle$, then $\langle L, \mathfrak{I}^{(k-1)}, f, b^{j-k+1}p\rangle$ is nonempty. Now for $S$, $T \in \mathfrak{I}^{(k-1)}$ we have $TST^{-1}S^{-1} \in \mathfrak{I}^{(k)}$, so

$$gST = g(TST^{-1}S^{-1})ST = gTS.$$

Furthermore,

$$gT \leqq b^{j-k}pT \leqq b^{j-k}bp = b^{j-k+1}p.$$

Thus it follows from (2.2) that $\langle L, \mathfrak{I}^{(k-1)}, f, b^{j-k+1}p\rangle$ is nonempty, and this completes the proof.

(f) If $\mathfrak{I}^{(j)} = \{I\}$, then it follows from the Hahn-Banach theorem that (e) holds with $k = j$.

(3.4) COROLLARY. *Let* $h$ *be the linear functional with domain* $D_h = \{0\}$. *Suppose* $[L, \mathfrak{I}, h, p]$, $\mathfrak{I}$ *is* p*-bounded,* $p > 0$ *on* $L - \{0\}$, *and there is an* $x \in L - \{0\}$ *such that* $Tx = x$ *for each* $T \in \mathfrak{I}$. *Then if* (a), (c), (d), *or* (f) *is satisfied,* *there exists an* $m < \infty$ *and a not-identically-zero* $F \in \langle L, \mathfrak{T}, h, mp\rangle$.

*Proof.* For each real $t$, let $f(tx) = tp(x)$. Then $[L, \mathfrak{T}, f, p]$, so from (3.1) and (3.3) it follows that $\langle L, \mathfrak{T}, f, mp\rangle$ is nonempty for sufficiently large $m < \infty$. Since $f$ is not identically zero, this completes the proof.

For the case in which $p$ is $\mathfrak{T}$-invariant ($pT = p$ for each $T \in \mathfrak{T}$), (3.1) (a) was given by Woodbury [**10**], and (3.3) was proved by Agnew and Morse [**1**]. (3.4) (a), (c), and (f) were proved by Yood [**11**], who obtained his results under boundedness assumptions weaker than those employed here.

By an argument analogous to that of Agnew and Morse [**1**, p. 24-25] the following can be proved.

(3.5) HYPOTHESES: $X$ is a set; $\mathfrak{M}$ is a set of subsets of $X$ such that if $A$, $B \in \mathfrak{M}$, then $A \cup B \in \mathfrak{M}$ and $A - B \in \mathfrak{M}$; $m$ is a finitely additive real-valued measure on $\mathfrak{M}$; $\mathfrak{T}$ is a group of biunique transformations of $X$ onto $X$ which is either solvable or such that every finite subset is contained in a finite subgroup; $k_T \mid T \in \mathfrak{T}$ is a positive-valued function such that $mT = k_T m$ for each $T \in \mathfrak{T}$; $\mathfrak{S}$ is the set of all sets which are contained in some member of $\mathfrak{M}$.

CONCLUSION: *There is a finitely additive real-valued measure* $\mu$ *on* $\mathfrak{S}$ *such that* $\mu \mid \mathfrak{M} = m$ *and* $\mu T = k_T$ *for each* $T \in \mathfrak{T}$.

In particular, there is a finitely additive extension of Lebesgue measure in the plane which is defined for every set of finite outer measure and multiplies properly under every similarity transformation.

**4. Compact groups.** In this section we complete the proof of (3.1), but we must first develop some tools. (4.1) collects some well-known facts. (4.2) follows from a theorem of Šmulian [8], but is proved here for the sake of completeness.

(4.1) *If* $E$ *is a Banach space and* $F \in E^{**}$, *there is at most one point* $y_F \in E$ *such that* $F(f) = f(y_F)$ *for each* $f \in E^*$. *If* $E$ *is finite-dimensional, the point* $y_F$ *exists for each* $F \in E^{**}$. *If* $T$ *is a bounded linear transformation of* $E$ *into* $E$, $F \in E^{**}$, $y_F$ *exists, and* $H \in E^{**}$ *is defined by* $H(f) = F(fT)$ $(f \in E^*)$, *then* $y_H$ *exists and* $y_H = Ty_F$.

(4.2) *Suppose* $E$ *is a Banach space,* $F \in E^{**}$, *and there is a compact set* $X \subset E$ *such that* $F(f) \leqq \sup_X f$ *for each* $f \in E^*$. *Then* $y_F$ *exists and* $y_F \in$ Cl conv $X$.

*Proof.* Since $f(y_F) = F(f) \leqq \sup_X f$, $y_F$ is included in every closed half-space containing $X$, and hence $y_F \in \mathrm{Cl\ conv}\ X$. We still must show that $y_F$ exists and for finite-dimensional $E$ this follows from (4.1). Let $\mathcal{S}$ be the set of all finite-dimensional linear subspaces of $E^*$, and for each $S \in \mathcal{S}$ let $K_S$ be the set of all points $x \in \mathrm{Cl\ conv}\ X$ such that $F(f) = f(x)$ for each $f \in S$. We shall show that $K_S$ is non-empty. Since, by a theorem of Mazur [5], Cl conv $X$ is compact, $\{K_S \mid S \in \mathcal{S}\}$ must then be a family of compact sets which has the finite intersection property. But then there exists a point $p \in \bigcap_{S \in \mathcal{S}} K_S$ and it is clear that $p = y_F$.

We now complete the proof by showing that $K_S$ is nonempty for each $S \in \mathcal{S}$. Let

$$E_2 = \{x \mid f(x) = 0 \text{ for each } f \in S\},$$

and let $E_1$ be a subspace of $E$ which is complementary to $E_2$. Then $E_1$ is finite-dimensional, and each point $x$ of $E$ has a unique expression in the form $x = x_1 + x_2$ with $x_i \in E_i$. The map $x_1 \mid x \in E$ is continuous. For each $f \in S$ let $f_1 = f \mid E_1$. Then the map $f_1 \mid f \in S$ is an isomorphism of $S$ onto $E_1^*$. For each $f \in S$ let $G(f_1) = F(f)$. Then $G \in E_1^{**}$, and for each $f_1 \in E_1^*$ we have

$$G(f_1) = F(f) \leqq \sup_X f = \sup_{X_1} f_1,$$

where $X_1 = \{x_1 \mid x \in X\}$. Thus by the finite-dimensional case of (4.2) (already established) there is a point $q_s \in \mathrm{Cl\ conv} \mid X_1$ such that $G(f_1) = f_1(q_s)$ for each $f_1 \in E_1^*$. Since Cl conv $X$ is compact and $(\mathrm{conv}\ X)_1 = \mathrm{conv}\ X_1$, we have $\mathrm{Cl\ conv}\ X_1 = (\mathrm{Cl\ conv}\ X)_1$. Thus $q_s = p_{s_1}$ for some $p_s \in \mathrm{Cl\ conv}\ X$, and we have

$$F(f) = G(f_1) = f_1(q_s) = f(q_s) = f(p_s)$$

for each $f \in S$. Hence $p_s \in K_s$, and the proof of (4.2) is complete.

(4.3) *Suppose $A$ is a Banach algebra and $G$ a compact multiplicative subgroup of $A$. Then there are points $u$, $v \in$* Cl conv *$G$ such that $ug = u$ and $gv = v$ for each $g \in G$.*

*Proof.* Let $\mu$ be the right-invariant Haar measure on $G$ with $\mu G = 1$. For each $f \in E^*$ let $F(f) = \int_G f d\mu$. Then $F$ satisfies the conditions of (3.2), so $y_F$ exists, with $y_F \in \mathrm{Cl\ conv}\ G$. Consider an arbitrary $g \in G$. Let $Tx = xg$ for each $x \in E$, and let $H$ be defined as in (4.1). Then by (4.1), $y_H = Ty_F$. But

$$H(f) = F(fT) = \int_G f(xg) \mid x \, d\mu = \int_G f d\mu = F(f),$$

so $H = F$. Hence $Ty_F = y_F$, and since $Ty_F = y_F g$ we see that $y_F$ is the desired point $u$. Using left-invariant measure, we obtain the desired $v$.

If the identity map $m$ of $G$ onto $G$ is regarded as a function from the measure-space $(G, \mu)$ to the Banach space $A$, then $u$ is merely the Pettis integral [7] of $m$.

Now returning to the proof of (3.1), suppose (d) holds. Then applying (4.3) to the Banach algebra of bounded linear transformations of $E$ into $E$, where $E = (L, p)$, we obtain a bounded linear transformation $R$ of $E$ into $E$ such that $R \in \mathrm{Cl}\ \mathrm{conv}\ \mathfrak{I}$ and $RT = R$ for each $T \in \mathfrak{I}$. From $R \in \mathrm{Cl}\ \mathrm{conv}\ \mathfrak{I}$ it follows easily that $fR = f$ and $pR(z) \in \mathrm{Cl}\ p(\mathrm{conv}\ \mathfrak{I}(z))$ for each $z \in L$, so (b) holds and the proof of (3.1) is complete.

We conclude with:

(4.4) *Suppose $K$ is a compact convex subset of the Banach space $E$ and $\mathfrak{I}$ is a compact group (in the uniform topology of operators) of bounded linear transformations of $E$ onto $E$, each mapping $K$ into $K$. Then there is a point of $K$ which is invariant under every transformation in $\mathfrak{I}$.*

*Proof.* By (4.3) there is an $R \in \mathrm{Cl}\ \mathrm{conv}\ \mathfrak{I}$ such that $TR = R$ for each $T \in \mathfrak{I}$. Since $K$ is convex, if follows that $RK \subset K$. Then by the fixed-point theorem of Tychonoff [9], $R$ admits a fixed-point $x \in K$. For each $T \in \mathfrak{I}$ we have $Tx = TR(x) = R(x) = x$, so the proof is complete.

For related results see Kakutani [4].

## REFERENCES


1. R. P. Agnew and A. P. Morse, *Extensions of linear functionals, with applications to limits, integrals, measures, and densities*, Ann. of Math. 39 (1938), 20-30.

2. Stefan Banach, *Théorie des opérations linéaires*, Warsaw, 1932.

3. N. Dunford, *Direct decompositions of Banach spaces*, Bol. Soc. Mat. Mexicana 3 (1946), 1-12.

4. Shizuo Kakutani, *Two fixed-point theorems concerning bicompact convex sets*, Proc. Imp. Acad. Tokyo 14 (1938), 242-245.

5. S. Mazur, *Über die kleinste konvexe Mengen, die eine gegebene kompakte Menge enthält*, Studia Math. 2 (1930), 7-9.

6. Anthony P. Morse, *Squares are normal*, Fund. Math. 36 (1949), 35-39.

7. B. J. Pettis, *On integration in vector spaces*, Trans. Amer. Math. Soc. 44 (1938), 277-304.

8. V. Šmulian, *Sur les espaces lineaires topologiques*, II. Rec. Math. [Mat. Sbornik] N.S. 9[51] (1941), 727-730.

9. A. Tychonoff, *Ein Fixpunktsatz*, Math. Ann. 111 (1935), 767-776.

10. M. A. Woodbury, *Invariant functionals and measures*, Bull. Amer. Math. Soc. 56 (1950), 172, Abstract 168t.

11. Bertram Yood, *On fixed points for semi-groups of linear operations*, Proc. Amer. Math. Soc. 2 (1951), 225-233.

UNIVERSITY OF WASHINGTON