APPLICATION OF A THEOREM OF PÓLYA TO THE SOLUTION OF AN INFINITE MATRIX EQUATION

J. COPPING

1. Introduction. G. Pólya [2] has given various sufficient conditions on the infinite matrix A to ensure that the infinite system of linear equations $Au = b$, where b and u are column vectors, has a solution in u. It is remarkable that there are no conditions on the given column vector b.

R. G. Cooke [1, pp. 34-35] established the existence of reciprocals of a matrix A satisfying Pólya's conditions, given in the following theorem.

THEOREM 1 (Pólya). *In the infinite system of linear equations*

$$\sum_{j=1}^{\infty} a_{i,j}\, u_j = b_i \qquad (i = 1, 2, 3, \cdots), \tag{1.1}$$

where $\{b_i\}$ *is an arbitrary sequence, let* $(a_{i,j})$ *satisfy the conditions*

(i) *the first row* $a_{1,j}$ *contains an infinity of nonzero elements, and*

(ii) $\liminf_{j\to\infty} \dfrac{|a_{1j}| + |a_{2j}| + \cdots + |a_{i-1,j}|}{|a_{i,j}|} = 0$ *for every fixed* $i \geq 2$.

Then there exists an infinite sequence $\{u_j\}$ *satisfying* (1.1), *such that all the left sides are absolutely convergent.*

It follows [1, pp. 34-35] that if a matrix $A = (a_{i,j})$ satisfies (i) and (ii), then A has an infinity of linearly independent right-hand reciprocals, and that if A', the transpose of A, satisfies (i) and (ii), then A has an infinity of linearly independent left-hand reciprocals.

In this paper it is shown that Pólya's theorem can be applied to establish the existence of solutions of the infinite matrix equation

Received December 8, 1952, and in revised form June 15, 1953. The author wishes to express his thanks to Dr. R. G. Cooke and Dr. P. Vermes, who have read the manuscript and made some useful comments. He also thanks the referee for his comments on Theoems 2 and 4, which were of assistance in clarifying the proofs.

Pacific J. Math. 4 (1954), 21-28

$$AX - XB = C,$$

where B and C are arbitrary given matrices, and A is a given matrix satisfying (i) and (ii) of Theorem 1. The principal tool is given in Theorem 2, where the doubly infinite set $(x_{r,s})$ of the matrix elements of X is found as a solution of a simple infinite set of equations. Theorem 3 then gives the main result. In §§3 and 4, other solutions are obtained in the general case, and in the special case

$$AX - XB = 0;$$

and the nature of the solutions is discussed.

2. **We shall establish** the following result.

THEOREM 2. *Suppose A satisfies conditions* (i) *and* (ii) *of Theorem* 1, *and B and C are arbitrary given infinite matrices. Let the set of linear equations*

$$\sum_{r,s} (\delta_{k,s}\, a_{n,r} - \delta_{n,r}\, b_{s,k})\, x_{r,s} = c_{n,k} \tag{2.1}$$

be written in the order $(n = 1,\ k = 1)$, $(n = 2,\ k = 1)$, $(n = 1,\ k = 2), \cdots$, *where* $\sum_{r,s}$ *denotes the "Cauchy sum",* $(r = 1,\ s = 1)$, $(r = 2,\ s = 1)$, $(r = 1,\ s = 2)$, $\cdots$. *Then the matrix of the system of equations* (2.1) *satisfies the conditions* (i) *and* (ii) *of Theorem* 1.

Let M be the matrix of the system of equations (2.1), so that $\delta_{k,s}\, a_{n,r} - \delta_{n,r}\, b_{s,k}$ is the element of M in the row defined by the pair n, k, and in the column defined by the pair r, s. The elements of the first row of M (that is, the row $n = k = 1$), for which $s = 1$, $r = 2, 3, \cdots$, are $a_{1,2}, a_{1,3}, \cdots$; hence the first row of M contains infinitely many nonzero elements, so that condition (i) of Theorem 1 is satisfied by M.

To show that M satisfies (ii) of Theorem 1, observe that if $\{\mu_j\}$ is any subsequence of the positive integers, then the condition

$$\liminf_{j \to \infty} \frac{|a_{1,\mu_j}| + |a_{2,\mu_j}| + \cdots + |a_{i-1,\mu_j}|}{|a_{i,\mu_j}|} = 0 \tag{2.11}$$

for a particular value of i implies condition (ii) of Theorem 1 for this value of i. Hence it is sufficient to show that, corresponding to each fixed integer $\lambda > 1$, there is a semi-infinite submatrix M_λ of M, of order $\lambda \times \infty$, consisting of the first λ elements of a certain infinite subset of the columns of M, such that M_λ satisfies condition (ii) of Theorem 1 for the particular value λ of i. We shall find such submatrices M_λ $(\lambda = 2, 3, \cdots)$, in each of which all the elements are either zero or elements of the matrix A, and are so arranged that (2.11) is satisfied for $i = \lambda$.

Let the λth row of M correspond to the suffixes $n = p$, $k = q$, p and q being fixed positive integers, not both equal to 1. Consider the column of M for which $r \equiv r(t) = p + q + t$ and $s = q$, where t is any fixed positive integer. The first λ elements in this column correspond to pairs of suffixes n, k, such that their sum is nondecreasing (that is, 1, 1; 2, 1; 1, 2; 3, 1; 2, 2; $\cdots$), so that both n and k are less than $p + q$. Hence, for these pairs of values of n and k, $\delta_{n,r} = 0$, so that no elements of the matrix B occur among the first λ elements of this column.

The only nonzero elements among the first λ elements of this column are therefore those for which $\delta_{k,q} = 1$, that is, $k = q$, and $n = 1, 2, \cdots, p-1, p$; and these elements are

$$a_{1,p+q+t},\ a_{2,p+q+t}, \cdots,\ a_{p-1,p+q+t},\ a_{p,p+q+t}\ .$$

We now select the columns of M_λ by keeping $s = q$ fixed, and letting $r \equiv r(t)$ assume in succession all the values of $p + q + t$, where p, q are fixed, and $t = 1, 2, 3, \cdots$, in succession. Hence, to show that M_λ satisfies condition (ii) of Theorem 1 for the particular value $i = \lambda$, we must show that

$$\liminf_{t \to \infty} \frac{\sum_{n=1}^{p-1} |a_{n,p+q+t}|}{|a_{p,p+q+t}|} = 0,$$

which is clearly true, since, by hypothesis, the matrix A satisfies condition (ii) of Theorem 1. Thus Theorem 2 is now proved.

THEOREM 3. *Let A satisfy conditions* (i) *and* (ii) *of Theorem* 1. *Then the equation*

$$AX - XB = C, \tag{2.2}$$

where B and C are arbitrary given infinite matrices, has an infinity of solutions.

For, by Theorem 2, the equations (2.1) have an infinity of sets of solutions, and, for each set, all the series on the left of (2.1) are absolutely convergent. Take any such set and rearrange each of the series in (2.1) as a "sum by rows". This gives

$$\sum_{r=1}^{\infty} a_{n,r}\, x_{r,k} - \sum_{s=1}^{\infty} x_{n,s}\, b_{s,k} = c_{n,k}\,.$$

Thus $X = (x_{r,s})$ is a solution of (2.2).

3. Certain conditions may be imposed on the solutions obtained by the method of Theorem 3.

THEOREM 4. *If A satisfies conditions* (i) *and* (ii) *of Theorem* 1, *then there is an infinity of solutions of the equation*

$$AX - XB = C,$$

each of which is a lower semimatrix [**1**, p. 6] *whose principal diagonal elements are zero.*

Returning to Theorem 1, observe that if the given conditions of that theorem are satisfied when the column-suffix j is restricted to a subsequence S of the positive integers, where S is independent of the row-suffix i, then solutions $\{u_j\}$ exist such that $u_j = 0$ whenever j is not in S.

For, let H be the matrix obtained from A by selecting the columns of A whose suffixes are in S, so that

$$h_{i,p} = a_{i,k_p} \; (p = 1, 2, 3, \cdots,\; i = 1, 2, 3, \cdots),$$

where $\{k_p\}$ is the subsequence S of the positive integers. Then H satisfies the conditions of Theorem 1, so that, given any column-vector b, there are vectors v such that $Hv = b$; that is,

$$\sum_{p=1}^{\infty} a_{i,k_p}\, v_p \equiv \sum_{p=1}^{\infty} h_{i,p}\, v_p = b_i \qquad (i = 1, 2, 3, \cdots),$$

where each of the series is absolutely convergent.

If we now write $u_j = v_p$ when $j = k_p$ $(p = 1, 2, 3, \cdots)$, and $u_j = 0$ otherwise, we have

$$\sum_{j=1}^{\infty} a_{i,j}\, u_j = b_i \qquad (i = 1, 2, 3, \cdots),$$

where each of the series is absolutely convergent.

This result clearly can be applied to the matrix of (2.1). For, in the proof of Theorem 2, in which it was shown that M satisfies conditions (i) and (ii) of Theorem 1 we considered only the columns of M for which $r > s$. Hence there are solutions $\{x_{r,s}\}$ of the equations (2.1) for which $x_{r,s} = 0$ when $r \leq s$, and the property of absolute convergence of all the double series involved holds as before, so that Theorem 3 again follows, with solutions X for which $x_{r,s} = 0$ when $r \leq s$. This completes the proof of Theorem 4.

THEOREM 5. *Suppose A satisfies conditions* (i) *and* (ii) *of Theorem* 1, *and let $P = (p_{n,k})$ be any given matrix such that AP and PB exist. Then there is an infinity of solutions of the equation*

$$AX - XB = C,$$

for each of which $x_{n,k} = p_{n,k}$ for all $k \geq n$.

Consider the equation

$$AX - XB = C - AP + PB. \tag{3.1}$$

By Theorem 4, this equation has an infinity of solutions, each of which is a lower semimatrix whose principal diagonal elements are zero. Let Y be such a solution. Then $Y + P$ is a solution, of the type required of the equation

$$AX - XB = C.$$

The theorem is thus proved.

Theorem 5 may be applied to obtain transformations of the form

$$Y^{-1} \cdot AY = B,$$

where A satisfies conditions (i) and (ii) of Theorem 1 and B is an arbitrary matrix, and then $Y^{-1} \cdot AY$ is associative for multiplication. For, in Theorem 5, put $C = 0$, $P = I$, the unit matrix. Thus solutions of the equation

$$AX - XB = 0 \tag{3.2}$$

exist, which are lower semimatrices with no zero elements in the principal diagonal. Let Y be such a solution. Then

$$AY - YB = 0,$$

and Y has a two-sided reciprocal Y^{-1} which is a lower semimatrix [1, p. 19, 22].

Since Y is row-finite, YB exists, and hence AY exists; that is, $\sum_{j=k}^{\infty} a_{i,j}\, y_{j,k}$ exists for each i and k. Hence

$$\{Y^{-1}(AY)\}_{n,k} = \sum_{i=1}^{n} \sum_{j=k}^{\infty} y_{n,i}^{-1}\, a_{i,j}\, y_{j,k}$$

$$= \sum_{j=k}^{\infty} \sum_{i=1}^{n} y_{n,i}^{-1}\, a_{i,j}\, y_{j,k}$$

$$= \{(Y^{-1}A)Y\}_{n,k}\,.$$

Also

$$\{Y^{-1}(YB)\}_{n,k} = \{(Y^{-1}Y)B\}_{n,k} = b_{n,k},$$

since the double series concerned are finite, so that

$$Y^{-1}AY = B,$$

where $Y^{-1}AY$ is associative.

4. In considering the nature of the solutions obtained by the methods of Theorems 3, 4, and 5, it would be desirable to know whether solutions exist which belong to a given "associative field" [1, pp. 9, 26]. For example, the equation

$$AX - XD = 0, \tag{4.1}$$

where D is a given diagonal matrix, is of fundamental importance in quantum mechanics, and in the theory of consistency of Toeplitz transformations of divergent sequences. For such applications, solutions Y of (4.1) are required such that

$$YDY^{-1} = A, \quad Y^{-1}AY = D,$$

respectively [1, pp. 41, 101], and such that Y, Y^{-1} and A belong to the same "associative field."

The method of § 3 above fails to give solutions Y of (4.1) such that

$$YDY^{-1} = A\,.$$

For the solutions obtained by this method are lower semimatrices, so that

$$YDY^{-1} = A$$

would imply that A is a lower semimatrix, which is impossible under the given conditions (i) and (ii) of Theorem 1, Moreover, although the method of § 3 gives solutions Y of (4.1) such that

$$Y^{-1} \cdot AY = D,$$

these solutions cannot satisfy the condition that Y, Y^{-1} and A should belong to the same "associative field," for this would again imply

$$YDY^{-1} = A,$$

which is impossible.

Another case in which the existence of a solution belonging to a given "associative field" can be shown to be impossible is provided by the following theorem.

THEOREM 6. *If A belongs to a "field with an associative bound,"* [**1**, p. 27][1], *then no solution of the equation*

$$AX - XA = I$$

belongs to the same field.

For suppose if possible that Y is a solution belonging to the same "field" as A. Then

$$AY - YA = I,$$

and from the associative property it follows by induction that for each positive integer n,

$$AY^n - Y^nA = nY^{n-1}. \tag{4.2}$$

Denoting the bound of Y by $|Y|$, and applying its properties [**1**, pp. 26, 27] to equation (4.2), we obtain

$$n|Y^{n-1}| = |nY^{n-1}| \leq |AY^n| + |Y^nA| \leq 2|A||Y||Y^{n-1}|.$$

[1] For example, K, K_r, K_c, and Hilbert matrices [**1**, pp. 63, 25, 29, 243].

Observe that $Y^n \neq 0$ for any positive integer n, for if p is the smallest positive integer such that $Y^p = 0$, then equation (4.2) is contradicted when $n = p$.

Hence $|Y^{n-1}| \neq 0$, so that

$$|Y| \geq \frac{n}{2|A|}\cdot$$

for each positive integer n, contradicting the hypothesis that $|Y|$ exists.

References

1. R. G. Cooke, *Infinite matrices and sequence spaces*, Macmillan, New York, 1950.

2. G. Pólya, *Eine einfache, mit funktiontheoretischen Aufgaben verknüpfte, hinreichende Bedingung für die Auflösbarkeit eines Systems unendlich vieler linearer Gleichungen*, Comment. Math. Helv. **11** (1938-9), 234-252.

Birkbeck College,
University of London