CONGRUENT IMBEDDING IN F-METRIC SPACES

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1. Introduction. An F-metric space arises by associating with each pair x, y of elements ("points") of an abstract set S an element xy^2 (the "squared-distance") of a field F. It is required of the association merely that $xy^2 = yx^2$, $xx^2 = 0$, and if $x \neq y$ then $xz^2 \neq yz^2$ for at least one point z of S. In this note we establish some fundamental distance-geometric properties of the two F-metric spaces F_n, $F_n(a_1, \cdots, a_n)$ obtained by attaching to each two elements

$$x = (x_1, x_2, \cdots, x_n), \quad y = (y_1, y_2, \cdots, y_n)$$

of the set of ordered n-tuples of F the elements

$$xy^2 = \sum_{i=1}^{n} (x_i - y_i)^2 \quad \text{and} \quad xy^2 = \sum_{i=1}^{n} a_i (x_i - y_i)^2$$

Received April 24, 1953. This paper was left by the late Professor Abraham Wald as a manuscript (in German). It was translated and edited by Professor Leonard M. Blumenthal of the University of Missouri, whose friendship with Professor Wald began at Vienna during the period when the latter was making distinguished contributions to Distance Geometry.

Translator's note. This article was written while Professor Wald was at the University of Vienna, probably in 1934. He had previously proved similar metric characterization theorems for the space of all n-tuples of *complex* numbers with

$$xy^2 = \sum_{i=1}^{n} (x_i - y_i)^2$$

(*Ergebnisse eines mathematischen Kolloquiums* (Wien), Heft 5 (1933), pp. 32-42). It seems that the present paper was intended to follow one in that journal by Olga Taussky (Mrs. John Todd) in which the same problems were solved in the more abstract setting obtained upon replacing the complex number field by any field of characteristic zero *in which every element is a square.* (See footnote 1.) It was announced in Heft 6 of the *Ergebnisse* that Wald's paper (which complements Mrs. Todd's by treating the problems in *formally real fields*) would appear in Heft 7, but for some reason this intention was not carried out. Nor is it contained in Heft 8, the last number of the Vienna *Ergebnisse* that was published.

The remaining footnotes in this paper are comments by the translator.

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as squared-distances respectively, where, in the second instance, the coefficients $a_1\, a_2, \cdots, a_n$ belong to F. (Translators note: In the manuscript "distance" rather than "squared-distance" is used; for example, $\sum_{i=1}^{n} (x_i - y_i)^2$ is spoken of as the *distance* of the points $x = (x_1, x_2, \cdots, x_n)$, $y = (y_1, y_2, \cdots, y_n)$. In order that the developments of the paper should more exactly generalize the euclidean case (in which F is the real field) it seemed desirable to call $\sum_{i=1}^{n} (x_i - y_i)^2$ the *squared-distance* of x, y and to make the necessary minor changes in the manuscript. There is, of course, no implication that "distance" is meaningful. The reader is asked to interpret all such terms as "congruent", "congruence order", "metric basis", etc. in the sense of squared-distance. For definitions of these and other Distance Geometry concepts used in this paper see L.M. Blumenthal, *Theory and Applications of Distance Geometry*, The Clarendon Press, Oxford 1953.) It is assumed throughout that F has characteristic 0, while in § 3 it is further supposed that (1) each sum of squares of elements of F is a square of an element of F and (2) F does not contain $\sqrt{-1}$.

2. Congruence order of F_n. It is shown in this section that F_n has congruence order $n+3$ with respect to the class of F-metric spaces; that is, any F-metric space can be mapped into F_n with preservation of squared-distances whenever that is true for each $(n+3)$-tuple of the space. We prove first some lemmas.

LEMMA 2.1. *Each $(k+1)$-tuple $p_0, p_1, \cdots, p_k$ of F_n $(k = 1, 2, \cdots, n)$ for which the Cayley-Menger determinant*

$$D(p_0, p_1, \cdots, p_k) = \begin{vmatrix} 0 & 1 & 1 & \cdot & \cdot & 1 \\ 1 & 0 & p_0p_1^2 & \cdot & \cdot & p_0p_k^2 \\ 1 & p_1p_0^2 & 0 & \cdot & \cdot & p_1p_k^2 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 1 & p_kp_0^2 & p_kp_1^2 & \cdot & \cdot & 0 \end{vmatrix}$$

is not zero forms a metric basis for the k-dimensional subspace they determine.

Proof. Putting $p_0 = (0, 0, \cdots, 0)$, we note that each point p of the subspace can be written

$$p = \lambda_1 p_1 + \lambda_2 p_2 + \cdots + \lambda_k p_k.$$

Then a *necessary* condition that such a point p have assigned squared-distances from $p_0, p_1, \cdots, p_k$ is that $\lambda_1, \lambda_2, \cdots, \lambda_k$ satisfy the system of linear equations

$$(p, p_i) = (1/2)(p_0p^2 + p_0p_i^2 - pp_i^2)$$

$$= \lambda_1(p_1, p_i) + \lambda_2(p_2, p_i) + \cdots + \lambda_k(p_k, p_i)$$

$(i = 1, 2, \cdots, k)$, where $(\cdot, \cdot)$ denotes a scalar product. The coefficient determinant $|(p_i, p_j)|$ $(i, j = 1, 2, \cdots, k)$ is the Gram determinant $G(v_1, v_2, \cdots, v_k)$ of the *vectors* $v_i = [p_0, p_i]$ $(i = 1, 2, \cdots, k)$, and from the relation

$$G(v_1, v_2, \cdots, v_k) = [(-1)^{k+1}/2^k] \cdot D(p_0, p_1, \cdots, p_k)$$

it does not vanish. Hence $\lambda_1, \lambda_2, \cdots, \lambda_k$ are uniquely determined.

LEMMA 2.2. *Let k be any one of the first n integers, and let $p_0, p_1, \cdots, p_k$ be a $(k+1)$-tuple of F_n with $D(p_0, p_1, \cdots, p_k) \neq 0$. If $p_0', p_1', \cdots, p_k'$ is a $(k+1)$-tuple of F_n with $p_i p_j^2 = p_i' p_j'^2$ $(i, j = 0, 1, \cdots, k)$ (symbolized by writing $p_0, p_1, \cdots, p_k \approx_s p_0', p_1', \cdots, p_k'$), then a nonsingular linear transformation that maps p_i on p_i' $(i = 0, 1, \cdots, k)$ also maps the k-dimensional subspace S_k determined by $p_0, p_1, \cdots, p_k$ congruently (that is, with preservation of squared-distances) onto the k-dimensional subspace S_k' determined by $p_0', p_1', \cdots, p_k'$.*

Proof. Putting $p_0 = p_0' = (0, 0, \cdots, 0)$, we note that any such transformation clearly maps S_k onto S_k' and associates with each point

$$p = \lambda_1 p_1 + \lambda_2 p_2 + \cdots + \lambda_k p_k$$

of S_k the point

$$p' = \lambda_1 p_1' + \lambda_2 p_2' + \cdots + \lambda_k p_k'$$

of S_k'. If

$$q = \mu_1 p_1 + \mu_2 p_2 + \cdots + \mu_k p_k$$

and

$$q' = \mu_1 p_1' + \mu_2 p_2' + \cdots + \mu_k p_k'$$

are corresponding points of S_k and S_k', respectively, then it is seen that

$$pq^2 = \sum_{i=1}^{k} \sum_{j=1}^{k} (\lambda_i - \mu_i)(\lambda_j - \mu_j)(p_i, p_j)$$

$$= \sum_{i=1}^{k} \sum_{j=1}^{k} (\lambda_i - \mu_i)(\lambda_j - \mu_j)(p_i', p_j') = p'q'^2,$$

since

$$(p_i, p_j) = (1/2)(p_0 p_i^2 + p_0 p_j^2 - p_i p_j^2)$$

$$= (1/2)(p_0' p_i'^2 + p_0' p_j'^2 - p_i' p_j'^2) = (p_i', p_j') \quad (i, j = 1, 2, \cdots, k).$$

LEMMA 2.3. *Let $p_0, p_1, \cdots, p_k, p, q$ be $k+3$ points of $F_n (0 < k \le n)$ for which $D(p_0, p_1, \cdots, p_k, p, q)$ vanishes, along with each of its bordered principal minors of order $k+3$. If $D(p_0, p_1, \cdots, p_k) \neq 0$ then the k-dimensional subspace $S_k(p_0, p_1, \cdots, p_k)$ determined by $p_0, p_1, \cdots, p_k$ contains points $\bar{p}, \bar{q}$ such that*

$$p_0, p_1, \cdots, p_k, p, q \approx_s p_0, p_1, \cdots, p_k, \bar{p}, \bar{q}.$$

Proof. Put $p_0 = (0, 0, \cdots, 0)$ and denote by F^* the closed algebraic extension of F. Now every element of F^* is a square, and according to a theorem of O. Taussky F_n^* contains a $(k+3)$-tuple $p_0, p_1^*, \cdots, p_k^*, p^*, q^*$ with

$$p_0, p_1, \cdots, p_k, p, q \approx_s p_0, p_1^*, \cdots, p_k^*, p^*, q^*,$$

and

$$p^*, q^* \in S_k^*(p_0, p_1^*, p_2^*, \cdots, p_k^*).$$ [1]

Let T denote a linear transformation of F_n^* with $p_i = T(p_i^*)$ $(i = 1, 2, \cdots, k)$, $p_0 = T(p_0)$, and let $\bar{p} = T(p^*)$, $\bar{q} = T(q^*)$. By Lemma 2.2,

$$p_0, p_1^*, \cdots, p_k^*, p^*, q^* \approx_s p_0, p_1, p_2, \cdots, p_k, \bar{p}, \bar{q},$$

[1] O. Taussky (Mrs. John Todd), *Abstracte Körper und Metrik. Erste Mitteilung: Endliche Mengen und Körperpotenzen,* Ergebnisse eines mathematischen Kolloquiums (Wien) Heft 6 (1935), pp. 20-23. The reference is to Theorem II (p. 23) in which it is assumed that every element of the base field is a square.

$$(*) \qquad p_0, p_1, \cdots, p_k, \bar{p}, \bar{q} \approx_s p_0, p_1, \cdots, p_k, p, q.$$

Since T carries $S_k^*(p_0, p_1^*, \cdots, p_k^*)$ into $S_k^*(p_0, p_1, \cdots, p_k)$, the k-dimensional subspace of F_n^* determined by $p_0, p_1, \cdots, p_k$, then

$$\bar{p}, \bar{q} \in S_k^*(p_0, p_1, \cdots, p_k),$$

and so

$$\bar{p} = \lambda_1 p_1 + \lambda_2 p_2 + \cdots + \lambda_k p_k, \quad \bar{q} = \mu_1 p_1 + \mu_2 p_2 + \cdots + \mu_k p_k,$$

where $\lambda_i, \mu_i \in F^*$ $(i = 1, 2, \cdots, k)$. To complete the proof, we show that λ_i, μ_i are also elements of $F (i = 1, 2, \cdots, k)$.

Now

$$(\bar{p}, p_i) = \lambda_1 (p_1, p_i) + \lambda_2 (p_2, p_i) + \cdots + \lambda_k (p_k, p_i) \quad (i = 1, 2, \cdots, k),$$

$$(\bar{q}, p_i) = \mu_1 (p_1, p_i) + \mu_2 (p_2, p_i) + \cdots + \mu_k (p_k, p_i) \quad (i = 1, 2, \cdots, k),$$

and since

$$p_0, p_1, \cdots, p_k, p, q \in F_n$$

it follows from $(*)$ that all of the coefficients in these two systems of equations (as well as the left members) belong to F. The relation between the determinant $G(v_1, v_2, \cdots, v_k)$ and $D(p_0, p_1, \cdots, p_k)$ exhibited in Lemma 2.1, together with the nonvanishing (by hypothesis) of the latter determinant, imply that $\lambda_1, \lambda_2, \cdots, \lambda_k$ and $\mu_1, \mu_2, \cdots, \mu_k$ are uniquely determined and belong to F.

THEOREM 2.1. *The space F_n has congruence order $n + 3$ with respect to the class of F-metric spaces, where F is any field of characteristic zero.*

Proof. Let S be any F-metric space with the property that each $(n + 3)$-tuple of S may be mapped in a squared-distance preserving manner into F_n. We show that S itself may be so mapped into F_n.

Since the Cayley-Menger determinants of all $(n + 2)$-tuples and $(n + 3)$-tuples of F_n vanish, the same is true of such subsets of S.[2] Let $k (k \leq n)$

[2] The vanishing of the Cayley-Menger determinant D of each m-tuple of F_n for $m > n + 1$ may be proved in the same way in which this result is established for euclidean n-space. See, for example, Theorem 40.1 (page 99) of the book referred to in § 1.

denote the largest natural number for which $k+1$ points of S exist with non-vanishing determinant D, and let $p_0, p_1, \cdots, p_k$ be such a $(k+1)$-tuple. The F_n contains, by hypothesis, $k+1$ points $p_0' p_1', \cdots, p_k'$ with

$$p_0, p_1, \cdots, p_k \approx_s p_0', p_1', \cdots, p_k'.$$

If $p, q \in S(p \neq q)$, elements $p_0'', p_1'', \cdots, p_k'', p'', q''$ of F_n exist such that

$$p_0, p_1, \cdots, p_k, p, q \approx_s p_0'', p_1'', \cdots, p_k'', p'', q'',$$

and by Lemma 2.3 we may suppose that p'', q'' belong to $S_k(p_0'', p_1'', \cdots, p_k'')$.

A linear transformation that carries $p_0'', p_1'', \cdots, p_k''$ into $p_0', p_1', \cdots, p_k'$, respectively, carries $S_k(p_0'', p_1'', \cdots, p_k'')$ into $S_k(p_0', p_1', \cdots, p_k')$ and consequently sends the points p'', q'' into points, say p', q', of the latter k-dimensional hyperplane. According to Lemma 2.2, this linear transformation preserves squared-distances, and so

$$p_0, p_1, \cdots, p_k, p, q \approx_s p_0'', p_1'', \cdots, p_k'', p'', q'' \approx_s p_0', p_1', \cdots, p_k', p', q'.$$

Use of Lemma 2.1 shows that the point p' corresponding to p by the procedure described is unique, and so S is mapped in a squared-distance preserving manner into a k-dimensional subspace of F_n.

THEOREM 2.2. *Let $F_n(a_1, a_2, \cdots, a_n)$ denote the space obtained by associating with each two n-tuples*

$$x = (x_1, x_2, \cdots, x_n), \quad y = (y_1, y_2, \cdots, y_n)$$

of a field F, with characteristic zero, the element

$$xy^2 = \sum_{i=1}^{n} a_i(x_i - y_i)^2$$

as squared-distance, where $a_1, a_2, \cdots, a_n$ are n selected elements of F. The space $F_n(a_1, a_2, \cdots, a_n)$ has congruence order $n+3$ with respect to the class of all F-metric spaces.

Proof. The closed algebraic extension F^* of F contains elements $\sqrt{a_i}$ $(i = 1, 2, \cdots, n)$. If to each point $(x_1, x_2, \cdots, x_n)$ of $F_n(a_1, a_2, \cdots, a_n)$

there is associated the point

$$(\sqrt{a_1}\cdot x_1,\ \sqrt{a_2}\cdot x_2,\cdots,\sqrt{a_n}\cdot x_n)$$

of F_n^* it is seen that $F_n(a_1, a_2, \cdots, a_n)$ is congruently contained in F_n^* which has, by Theorem 2.1, congruence order $n+3$ with respect to F^*-metric spaces and hence also with respect to F-metric spaces. Thus, if S is any F-metric space with each $(n+3)$-tuple imbeddable congruently in $F_n(a_1, a_2, \cdots, a_n)$ then S is congruent with a subset of F_n^*.

Let k be the greatest natural number such that a $(k+1)$-tuple $p_0, p_1, \cdots, p_k$ of S exists with

$$D(p_0, p_1, \cdots, p_k) \neq 0.$$

Then $k \leq n$, and $p_0, p_1, \cdots, p_k$ are congruent with a $(k+1)$-tuple of $F_n(a_1, a_2, \cdots, a_n)$ which, in turn, is congruent with a $(k+1)$-tuple $(p_0', p_1', \cdots, p_k')$ of F_n^*. We have, putting $p_0' = (0, 0, \cdots, 0)$,

$$p_j' = (\sqrt{a_1}\cdot x_{j1},\ \sqrt{a_2}\cdot x_{j2},\cdots,\sqrt{a_n}\cdot x_{jn}) \qquad (j=1, 2, \cdots, k),$$

where $x_{jm} \in F$ $(j=1, 2, \cdots, k;\ m=1, 2, \cdots, n)$. We see that S is congruent with a subset S' of the k-dimensional hyperplane of F_n^* determined by p_0', $p_1', \cdots, p_k'$.

If $p' \in S'$ then $p' = \lambda_1 p_1' + \lambda_2 p_2' + \cdots + \lambda_k p_k'$ and $\lambda_1, \lambda_2, \cdots, \lambda_k$ satisfy the system of equations.

$$(p', p_i') = \lambda_1(p_1', p_i') + \lambda_2(p_2', p_i') + \cdots + \lambda_k(p_k', p_i') \quad (i=1, 2, \cdots, k),$$

with all coefficients in F and with nonvanishing determinant. Hence $\lambda_1, \lambda_2, \cdots,$ $\lambda_k \in F$, and so each point p' of S' has coordinates

$$\sqrt{a_1}\cdot x_1,\ \sqrt{a_2}\cdot x_2,\cdots,\sqrt{a_n}\cdot x_n) \text{ with } x_1, x_2, \cdots, x_n \in F.$$

If, now, we make correspond to each such point p' of S' the point $(x_1, x_2, \cdots, x_n)$ of $F_n(a_1, a_2, \cdots, a_n)$, the correspondence is clearly a congruence, and S' is mapped congruently onto a subset of $F_n(a_1, a_2, \cdots, a_n)$. It follows that S is congruent with a subset of $F_n(a_1, a_2, \cdots, a_n)$, and the theorem is proved.

3. F-metric spaces with F formally real. Let F be a field with characteristic

0 such that every sum of squares of elements of F is the square of an element of F. In case F contains $\sqrt{-1}$, Taussky has shown that every $(n+1)$-tuple of an F-metric space is congruently imbeddable in F_n, and an $(n+3)$-tuple p_0, $p_1, \cdots, p_{n+2}$ has this property if and only if $D(p_0, p_1, \cdots, p_{n+2})$ vanishes along with each of its bordered principal minors of order $n+3$.[3] In this section we suppose that $\sqrt{-1}$ is *not* an element of F; that is, F is a *formally real field.*

Let $p_0, p_1, \cdots, p_k$ be an F-metric space of $k+1$ elements. As in Lemma 2.1, we call the ordered pointpairs $[p_0, p_i]$ *vectors* v_i, and define the scalar product (v_i, v_j) of vectors v_i, v_j by

$$(v_i, v_j) = (1/2)(p_0 p_i^2 + p_0 p_j^2 - p_i p_j^2) \qquad (i, j = 1, 2, \cdots, k).$$

The Gram determinant $|(v_i, v_j)|$ $(i, j = 1, 2, \cdots, k)$ is denoted as before, by $G(v_1, v_2, \cdots, v_k)$.

THEOREM 3.1. *A necessary and sufficient condition that an F-metric $(k+1)$-tuple $p_0, p_1, \cdots, p_k$ be congruently imbeddable in the F_n is that for $j > n$ every j of the vectors $v_1, v_2, \cdots, v_k$ have a vanishing Gram determinant, while for $j \leq n$, the Gram determinant $G(v_{i_1}, v_{i_2}, \cdots, v_{i_j})$ of each j of the vectors $v_i, v_2, \cdots, v_k$ be the square of an element of F.*

Proof. Let $p_0', p_1', \cdots, p_k'$ be a $(k+1)$-tuple of F_n.

Then $D(p_0', p_{i_1}', p_{i_2}', \cdots, p_{i_j}')$ vanishes for every j of the $k+1$ points when $j > n$, and consequently (by the relation in Lemma 2.1) $G(v_{i_1}', v_{i_2}', \cdots, v_{i_j}') = 0$ for every j of the vectors $v_1', v_2', \cdots, v_k'$ when $j > n$. Since, moreover, $G(v_{i_1}', v_{i_2}', \cdots, v_{i_j}')$ is easily shown to be the square of the determinant (the "volume-determinant") formed by annexing a column of 1's to the $j \times (j+1)$ rectangular array of the j coordinates $(j \leq m)$ of the points $p_0', p_{i_1}', p_{i_2}', \cdots, p_{i_j}'$ with respect to a j-dimensional subspace F_j of F_n containing them (we put $p_0' = (0, 0, \cdots, 0)$), the necessity is established.

To prove the sufficiency, let $p_0, p_1, \cdots, p_k$ form an F-metric space satisfying the conditions of the theorem, and suppose $v_{i_1}, v_{i_2}, \cdots, v_{i_j}$ are j of the vectors with nonvanishing Gram determinant. We show that the corresponding $(j+1)$-tuple $p_0, p_{i_1}, \cdots, p_{i_j}$ is imbeddable in F_j. Assume this the case for all positive integers less than j—an assumption that is obviously valid for $j = 1$.

There exists a *regular arrangement*, say $v_{r_1}, v_{r_2}, \cdots, v_{r_j}$, of the vectors $v_{i_1}, v_{i_2}, \cdots, v_{i_j}$ such that in the sequence

[3] See footnote 1.

$$G(v_{r_1}),\ G(v_{r_1}, v_{r_2}),\cdots,\ G(v_{r_1}, v_{r_2}, \cdots, v_{r_j})$$

no two neighboring elements are zero. We consider two cases.

Case I. $G(v_{r_1}, v_{r_2}, \cdots, v_{r_{j-1}}) \neq 0$. According to the inductive hypothesis, $p_0, p_{r_1}, p_{r_2}, \cdots, p_{r_{j-1}}$ are imbeddable in F_{j-1}. If $p_0', p_{r_1}', \cdots, p_{r_{j-1}}'$ are the image points in F_{j-1}, we put

$$p_0' = (0, 0, \cdots, 0),\ p_{r_i}' = (\alpha_{i1}, \alpha_{i2}, \cdots, \alpha_{i,j-1}) \quad (i = 1, 2, \cdots, j-1),$$

where the coordinates are all elements of F. Now by Taussky's theorem the points $p_0, p_{r_1}, p_{r_2}, \cdots, p_{r_j}$ are imbeddable in F_j^*, where F^* is the closed algebraic extension of F and, indeed, in such a manner that the images of $p_0, p_{r_1}, \cdots, p_{r_{j-1}}$ are $p_0', p_{r_1}', \cdots, p_{r_{j-1}}'$, respectively. Let

$$p_{r_j}' = (\alpha_1, \alpha_2, \cdots, \alpha_j)$$

be the image of p_{r_j}, where $\alpha_1, \alpha_2, \cdots, \alpha_j$ belong to F^*. We have

$$(v_{r_i}, v_{r_j}) = \alpha_1\alpha_{i1} + \alpha_2\alpha_{i2} + \cdots + \alpha_{j-1}\alpha_{i,j-1} \qquad (i = 1, \cdots, j-1),$$

which uniquely determine the elements $\alpha_1, \alpha_2, \cdots, \alpha_{j-2}$, since the square of the determinant $|\alpha_{im}|$ $(i, m = 1, 2, \cdots, j-1)$ of the coefficients is

$$G(v_{r_1}', v_{r_2}', \cdots, v_{r_{j-1}}') \neq 0.$$

It follows that $\alpha_1, \alpha_2, \cdots, \alpha_{j-1} \in F$.

In F_j^* we have

$$G(v_{r_1}, v_{r_2}, \cdots, v_{r_{j-1}}) = |\alpha_{im}|^2 \qquad (i, m = 1, 2, \cdots, j-1),$$

$$G(v_{r_1}, v_{r_2}, \cdots, v_{r_{j-1}}, v_{r_j}) = |\beta_{im}|^2 \qquad (i, m = 1, 2, \cdots, j),$$

with

$$\beta_{im} = \alpha_{im} \quad (i, m = 1, 2, \cdots, j-1) \quad \beta_{ij} = 0 \quad (i = 1, 2, \cdots, j-1),$$

$$\beta_{jm} = \alpha_m (m = 1, 2, \cdots, j),$$

and hence

$$G(v_{r_1}, v_{r_2}, \cdots, v_{r_j}) = \alpha_j^2 \, G(v_{r_1}, v_{r_2}, \cdots, v_{r_{j-1}}).$$

But the two Gram determinants in this relation are different from zero, and are squares of elements of F. Consequently $\alpha_j \in F$ and the theorem is proved in this case.

Case II. $G(v_{r_1}, v_{r_2}, \cdots, v_{r_{j-1}}) = 0$. Then $G(v_{r_1}, v_{r_2}, \cdots, v_{r_{j-2}}) \neq 0$, and, as in Case I, the points $p_0, p_{r_1}, \cdots, p_{r_j}$ are imbeddable in F_j^* with the respective image points

$$p_0' = (0, 0, \cdots, 0),$$

$$p_{r_i}' = (\alpha_{i1}, \cdots, \alpha_{i,j-2}, 0, 0) \qquad (i = 1, 2, \cdots, j-2),$$

$$p_{r_{j-1}}' = (\alpha_1, \cdots, \alpha_{j-2}, \alpha_{j-1}, \alpha_j),$$

$$p_{r_j}' = (\beta_1, \cdots, \beta_j),$$

where $\alpha_{im} \in F$ $(i, m = 1, 2, \cdots, j-2)$, and $\alpha_1, \cdots, \alpha_j$, $\beta_1, \cdots, \beta_j$ are elements of F^*. The argument used in Case I may be applied here to show that $\alpha_1, \cdots, \alpha_{j-2}$ and $\beta_1, \cdots, \beta_{j-2}$ are elements of F. Hence $(\alpha_1, \alpha_2, \cdots, \alpha_{j-2}) \in F_{j-2}$ along with p_0' and p_{r_i}' $(i = 1, 2, \cdots, j-2)$, and it is easy to show that

$$\alpha_{j-1}^2 + \alpha_j^2 = 0$$

follows from the vanishing of $G(v_{r_1}, v_{r_2}, \cdots, v_{r_{j-1}})$. We may suppose that

$$G(v_{r_1}, v_{r_2}, \cdots, v_{r_{j-2}}, v_{r_j}) = 0,$$

for in the contrary case the argument of Case I may be applied with v_{r_j} taking the place of $v_{r_{j-1}}$. Hence we have

$$\beta_{j-1}^2 + \beta_j^2 = 0.$$

Putting $\alpha_{j-1} = \lambda$, we take $\alpha_j = \lambda\sqrt{-1}$. Then from $\beta_{j-1} = \mu$ it follows that $\beta_j = -\mu\sqrt{-1}$, since $G(v_{r_1}, v_{r_2}, \cdots, v_{r_j})$ vanishes (contrary to hypothesis) if $\beta_j = \mu\sqrt{-1}$.

The scalar product $(v_{r_{j-1}}, v_{r_j}) \in F$; that is,

$$\alpha_1 \beta_1 + \alpha_2 \beta_2 + \cdots + \alpha_{j-2} \beta_{j-2} + \lambda\mu + \lambda\mu \in F,$$

and so $\lambda\mu \in F$ since $\alpha_1, \cdots, \alpha_{j-2}$ and $\beta_1, \cdots, \beta_{j-2}$ belong to F. Further

$$G(v_{r_1}, v_{r_2}, \cdots, v_{r_j}) = -4\lambda^2 \mu^2 \cdot G(v_{r_1}, v_{r_2}, \cdots, v_{r_{j-2}}),$$

and since these nonvanishing Gramians are squares of elements of F then $-\lambda^2 \mu^2$ is the square of an element of F. It follows that $\sqrt{-1}$ is the square of an element of F, contrary to the assumed character of the field F. *This contradiction shows that* Case II *is impossible.*

Now let j be an integer such that $G(v_1, v_2, \cdots, v_j) \neq 0$ (with appropriate labelling of the vectors), while the Gram determinant of every $j+1$ of the vectors vanishes. Let

$$p_0' = (0, 0, \cdots, 0), \; p_i' = (\alpha_{i1}, \alpha_{i2}, \cdots, \alpha_{ij}) \qquad (i = 1, 2, \cdots, k)$$

be points of F_j^* congruent (in the "squared-distance" sense) to $p_0, p_1, \cdots, p_k$, with p_i and p_i' corresponding $(i = 0, 1, \cdots, k)$. By the previous part of the proof, $p_0, p_1, \cdots, p_j$ are imbeddable in F_j, and the imbedding in F_j^* can be done so that α_{im} $(i, m = 1, 2, \cdots, j)$ are elements of F. The proof is completed by showing that also for $t > j$, $\alpha_{t1}, \alpha_{t2}, \cdots, \alpha_{tj} \in F$. The system of linear equations

$$\alpha_{i1}\alpha_{t1} + \alpha_{i2}\alpha_{t2} + \cdots + \alpha_{ij}\alpha_{tj} = (v_i, v_t) \qquad (i = 1, 2, \cdots, j)$$

has nonvanishing determinant $|\alpha_{im}|$ $(i, m = 1, 2, \cdots, j)$ (the square of this determinant is $G(v_1, v_2, \cdots, v_j)$) and all coefficients, together with (v_i, v_t) $(i = 1, 2, \cdots, j)$ are in F. It follows that $\alpha_{t1}, \alpha_{t2}, \cdots, \alpha_{tj} \in F$, and the theorem is proved.