ON TWO PROBLEMS OF KUREPA

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We prove [1]:

THEOREM 1. *There exists a denumerable ramified partially ordered set with the property that there is no chain meeting all maximal anti-chains and no anti-chain meeting all maximal chains.*

(Here a *chain* (*anti-chain*) is a set of elements every pair of which are comparable (incomparable). A *ramified* partially ordered set S is one in which for each x in S the set of elements $\leq x$ forms a chain.)

Proof. We denote by F the set of all finite sequences $(\alpha_1, \alpha_2, \cdots, \alpha_k)$ of integers. We use Greek letters α, β to denote elements of F, we denote by $l(\alpha)$ the length of α (that is, the number of terms in the sequence α) and by α_i (for $i = 1, \cdots, l(\alpha)$) the ith term in the sequence α; i, k are used throughout as variables for positive integers. If n is an integer we denote by (α, n) the sequence $(\alpha_1, \cdots, \alpha_{l(\alpha)}, n)$ obtained by adding the term n to the sequence α. We define $\alpha \leq \beta$ to hold when conditions

$$A : l(\alpha) \leq l(\beta),$$

$$B : \alpha_i = \beta_i \quad \text{for} \quad i = 1, \cdots, l(\alpha) - 1,$$

and

$$C : \alpha_{l(\alpha)} \leq \beta_{l(\alpha)},$$

are all satisfied. It is easily seen that this relation '$\leq$' is a ramified partial ordering of F.

Now let L_α denote the chain of elements $\leq \alpha$, let C_α denote the set of

[1] This answers two questions posed by Kurepa (Pacific J. Math. 2 (1952), 323-326). Answers to these questions were found independently by W. Gustin; see the reviews in Math. Rev. 14 (March, 1953), p. 255 by W. Gustin, and in Zentralblatt für Math., 64 (1953), p. 52, by J. C. Shepherdson.

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elements of the form (α, u), where u runs through all integer values, and let $L(C_\alpha)$ denote the set of elements less than all elements of C_α. Then we can easily prove

(i) C_α is a chain,

(ii) the elements of F which are comparable with all elements of C_α belong to $C_\alpha \cup L(C_\alpha)$,

(iii) $L(C_\alpha) = L_\alpha$,

and hence

(iv) $C_\alpha \cup L_\alpha$ is a maximal chain.

We now prove by *reductio ad absurdum* that no anti-chain meets all maximal chains. Suppose the anti-chain A meets all maximal chains. Clearly it has just one point in common with each maximal chain. It is easily seen that the set T_0 of all elements (u) of length one is a maximal chain. Hence there exists a unique integer a_1 such that $(a_1) \in A$. Take $n_1 = a_1 - 1$. Then the chain C_1 of elements $\leq (n_1)$ consists of all the elements (u) with $u < a_1$, and is therefore a subchain of T_0 not meeting A. We now define for each positive integer k by induction on k an integer n_k such that the chain C_k of elements $\leq (n_1, \cdots, n_k)$ does not meet A. We have just disposed of the case $k = 1$. Suppose then that $k > 1$ and that $n_1, \cdots, n_{k-1}$ are already defined so that the chain C_{k-1} of elements $\leq (n_1, \cdots, n_{k-1})$ does not meet A. By (iv) the set T_{k-1}[2] of all elements of the form $(n_1, \cdots, n_{k-1}, u)$ together with C_{k-1} forms a maximal chain. By hypothesis this meets A and C_{k-1} does not; hence there exists a unique integer a_k such that $(n_1, \cdots, n_{k-1}, a_k) \in A$. Take $n_k = a_k - 1$. Clearly C_k does not meet A. This completes the definition by induction of a sequence $n_1, n_2, \cdots$ of integers such that for all positive integers k the chain C_k of elements $\leq (n_1, \cdots, n_k)$ does not meet A. Now

$$(n_1, \cdots, n_k) < (n_1, \cdots, n_k, n_{k+1}),$$

so $C_k \subseteq C_{k+1}$. Hence the set

$$C = \sum_{k=1}^{\infty} C_k$$

[2] With the previous notation $T_{k-1} = C(n_1, \cdots, n_{k-1})$, $C_{k-1} = L(n_1, \cdots, n_{k-1})$.

is a chain. Now let α be an element of F comparable with all elements of C. Then

$$\alpha \not> (n_1, n_2, \cdots, n_{l(\alpha)+1}),$$

so

$$\alpha \leq (n_1, n_2, \cdots, n_{l(\alpha)+1});$$

that is, $\alpha \in C_{l(\alpha)+1}$, so $\alpha \in C$. Hence C is a maximal chain. But C cannot meet A since none of C_1, C_2, $\cdots$ meet A. Thus we have obtained a contradiction from the assumption that there exists an anti-chain meeting all maximal chains.

We now prove by *reductio ad absurdum* that no chain meets all maximal anti-chains. Suppose C is a chain meeting all maximal anti-chains. We note first that the lengths of the elements of C are unbounded. To prove this it is clearly enough to show that for each positive integer k there are maximal anti-chains all of whose elements are of length greater than k. It is easily seen that a set A_k with this property may be defined as follows: Denote by S_k the set of all elements of F of length k, and by N the set of elements $(\alpha_1, \cdots, \alpha_n)$ of F all of whose terms $\alpha_1, \cdots, \alpha_n$ are < 0. Let A_k be the set of all elements of the form $(\alpha, 0)$ for $\alpha \in S_k$ together with all elements of the form $(\alpha, \beta, 0)$ for $\alpha \in S_k$, $\beta \in N$. (Here $(\alpha, \beta, 0)$ stands for $(\alpha_1, \cdots, \alpha_{l(\alpha)}, \beta_1, \cdots, \beta_{l(\beta)}, 0)$.)

We note secondly that it follows easily from the definition of '$\leq$' that since C is a chain, all elements of C of length $> i$ have the same ith term.

In view of these two observations, it follows that we may define a unique sequence $n_1, n_2, n_3, \cdots$ of integers by putting n_i equal to the common ith term of the elements of C of length greater than i. Now let A be the set consisting of all sequences α such that $\alpha_i \leq n_i$ for $1 \leq i < l(\alpha)$ and $\alpha_{l(\alpha)} = n_{l(\alpha)} + 1$. This set A is easily seen to be a maximal anti-chain, so by hypothesis there exists an element α belonging to C and A. Let β be any element of C of greater length than α. Since $\alpha, \beta \in C$ they are comparable, so, since $l(\beta) > l(\alpha)$, we must have $\alpha < \beta$. From the definition of $n_1, n_2, \cdots$, we have $\alpha_i = n_i$ for $i < l(\alpha)$ and, since $\alpha < \beta$,

$$\alpha_{l(\alpha)} \leq \beta_{l(\alpha)} = n_{l(\alpha)}$$

(since $l(\beta) > l(\alpha)$). Hence

$$\alpha < (n_1, n_2, \cdots, n_{l(\alpha)-1}, n_{l(\alpha)} + 1);$$

but both these are elements of the anti-chain A and so are incomparable. So our hypothesis that there exists a chain meeting all maximal anti-chains leads to a contradiction; this completes the proof of Theorem 1.

By using the same sort of argument as Kurepa one can use the example of Theorem 1 to show, by means of the axiom of choice:

THEOREM 2. *A sufficient condition for a nonvoid set S to be finite is that in every ramified partial ordering of S there exists a chain meeting all maximal anti-chains* (*or*, '*··· there exists an anti-chain meeting all maximal chains*').

By Kurepa's result both these conditions are also necessary conditions for S to be finite.

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