CONCERNING TOTAL DIFFERENTIABILITY OF FUNCTIONS OF CLASS P

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1. Introduction. Since 1900 some eight or more writers have formulated conditions under which a function of two real variables shall be said to be of bounded variation. In two papers Adams and Clarkson [1, 4] examined and compared most of these definitions. In particular, they established the relations

$$\overline{T} \cap M \supset P \supset A, \qquad P \cap M = P,$$

where $\overline{T}$, P, and A represent respectively the classes of functions which are of bounded variation in an extended Tonelli sense [1], in the sense of Pierpont, and in the sense of Arzelà [4], and M stands for the class of plane measurable functions. An explicit definition of the class P will be given presently. For other definitions see Clarkson and Adams [4].

Burkill and Haslam-Jones [2] have shown that each function in A is totally differentiable almost everywhere. Adams and Clarkson [1] have proved that each function in $\overline{T} \cap M$ is approximately totally differentiable almost everywhere, although not necessarily totally differentiable anywhere. The question of whether each function in P is totally differentiable almost everywhere has been left open; the object of the present paper is to settle this question.

Saks [6] has shown that in a certain subset $E \subset \overline{T} \cap M$, suitably metrized, the functions which are nowhere totally differentiable form a residual set. One might naturally raise the question as to the category of the set $P \cap E$, for if this set were of second category in E, our question would be answered at once; but it turns out that $P \cap E$ is of first category in E.

In this paper (see §§ 3, 4, and 5) we show by exhibiting an example constructed along lines suggested by A. P. Morse that there exist functions which are in P and which are nowhere totally differentiable. We then show (see §§ 6-10 and § 11) that functions which are nowhere totally differentiable form residual sets (complements of sets of first category) in the classes $P \cap C$ and $P \cap E$, where C represents the class of functions continuous on the unit

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square. These two results may be compared in the following manner. The first shows that nowhere totally differentiable functions are relatively numerous in a large set of functions. The second shows that nowhere totally differentiable functions are also relatively numerous in a highly restricted subset of the larger set. Since different metrics are used, the first result does not imply the second. Finally it can be shown that $P \cap E$ is of first category in E. All category proofs depend upon the theorem of Baire which states that a complete metric space is of second category in itself.

2. Notations and preliminary definitions. Let R_1 and R_2 denote Euclidean one-space and two-space, respectively. In R_2 we shall use rectangular cartesian coordinates. We shall usually employ the letter p, sometimes with superscripts or subscripts, to denote a point in R_2, and x and y with corresponding superscripts or subscripts to denote the coordinates of the point. By $\rho(p, p')$ we mean the Euclidean distance $((x - x')^2 + (y - y')^2)^{1/2}$. If p is a point and E is a set, by $\rho(p, E)$ we mean the *inf* of numbers of the form $\rho(p, p')$, where $p' \in E$. Unless otherwise stated, the letters n, m, k, h, i, j, K, N, M, ν, λ will denote nonnegative integers. By the unit square in R_2, we mean

$$[(x, y) \mid (0 \leq x \leq 1)(0 \leq y \leq 1)].$$

We shall denote this set by I, and throughout the paper shall be concerned only with points of this set.

2.1. DEFINITION. Let I be divided into λ^2 congruent squares. We call this partition *the* λ*-net*. Each subdivision is called a λ-cell and is considered as a closed region. The cells of the λ-net can be ordered; we designate them by the index ν $(\nu = 1, 2, \cdots, \lambda^2)$.

2.2. DEFINITION. Let f be a real-valued function on the unit square. Let $\omega_\nu^\lambda(f)$ be the oscillation of f on the νth cell of the λ-net. Let

$$P_\lambda(f) = \sum_{\nu=1}^{\lambda^2} \frac{\omega_\nu^\lambda(f)}{\lambda} \; ;$$

let

$$P(f) = \sup_\lambda P_\lambda(f) \, .$$

If there exists a finite number A such that $P(f) < A$, then f is said to be of *bounded variation in the sense of Pierpont,* and we write $f \in P$.

This is Hahn's version of Pierpont's definition [**5**, p. 539], stated for the unit square. Clarkson and Adams [**4**] have shown that the Hahn version is equivalent to the definition as given by Pierpont.

2.3. DEFINITION. Let E be an open set in R_2, f a function on E, and p a point of E. If there exist functions R on R_2 and S on R_1, and finite numbers A and B such that

$$S(0+) = S(0) = 0;$$

$$|R(x'-x, y'-y)| \leq S(\rho(p, p')); \qquad \text{for each } p' \in R_2;$$

$$f(p') - f(p) = A(x'-x) + B(y'-y) + \rho(p, p') \cdot R(x'-x, y'-y),$$

$$\text{for each } p' \in E,$$

then f is *totally differentiable* at p [**3**, p. 644].

We note that if f is totally differentiable at p, then the numbers A and B above are the ordinary partial derivatives of f at p. Also the directional derivative of f must exist in each direction.

3. The function F. In this section we shall define the function mentioned in § 1. We first define a sequence of integers and state some of its properties.

3.1. DEFINITION.

$$Q(n) = 2^{2^n} \qquad (n = 1, 2, 3, \cdots).$$

3.2. LEMMA. $Q(n)^2 = Q(n+1)$.

The proof is evident from the definition.

3.3. LEMMA.

$$\sum_{n=1}^{N} Q(n) < 2Q(N).$$

This can be proved by induction.

3.4. LEMMA.

$$\sum_{n=N}^{\infty} Q(n)^{-1} < (Q(N)-1)^{-1}.$$

Proof.

$$\sum_{n=N}^{\infty} Q(n)^{-1} = Q(N)^{-1} + Q(N+1)^{-1} + Q(N+2)^{-1} + \cdots$$

$$= Q(N)^{-1} + Q(N)^{-2} + Q(N)^{-4} + \cdots$$

$$= \sum_{n=0}^{\infty} Q(N)^{-2^n} < \sum_{n=1}^{\infty} Q(N)^{-n} = (Q(N)-1)^{-1}.$$

3.5. DEFINITION. For each positive integer n, let

$$X_i^n = [x \mid i/Q(n) < x < i/Q(n) + 1/Q(n+1)] \qquad (i = 0, 1, 2, \cdots, Q(n)-1);$$

$$Y_j^n = [y \mid j/Q(n) < y < j/Q(n) + 1/Q(n+1)] \qquad (j = 0, 1, 2, \cdots, Q(n)-1);$$

$$E_{ij}^n = [(x, y) \mid x \in X_i^n,\ y \in Y_j^n];$$

$$E_n = \bigcup_{i,j=0}^{Q(n-1)} E_{ij}^n ;$$

$$I = [(x, y) \mid 0 \leq x \leq 1,\ 0 \leq y \leq 1].$$

If $p \in E_n$, let

$$F_n(p) = 2\rho(p, I - E_n)\, Q(n)^{-1}.$$

If $p \in I - E_n$, let

$$F_n(p) = 0.$$

For each point $p \in I$, let

$$F(p) = \sum_{n=1}^{\infty} F_n(p).$$

Let us consider this definition from a geometric viewpoint. One sees at once that the graph of each function F_n is a surface consisting of $Q(n)^2$ right square pyramids of altitude $1/Q(n)$ erected on those cells of the $Q(n+1)$-net whose lower left corners coincide with the lower left corners of the cells of the $Q(n)$-net, while the surface is of height zero at all other points. We also notice that the slope of each lateral edge is $\sqrt{2}\,Q(n)$, and that the slope of a line on a lateral face and perpendicular to a base edge is $2Q(n)$. The function F is the sum of a uniformly convergent series of nonnegative continuous functions, and therefore we can deduce the following result.

3.6. THEOREM. *F is continuous and nonnegative.*

We shall devote the next two sections to showing that F is of bounded variation in Pierpont's sense and also is nowhere totally differentiable.

4. F is of bounded variation in Pierpont's sense.

4.1. DEFINITION. For each positive integer k and for each point $p \in I$ let

$$S_k(p) = \sum_{n=1}^{k} F_n(p),$$

$$R_k(p) = \sum_{n=k+1}^{\infty} F_n(p).$$

Clearly $F(p) = S_k(p) + R_k(p)$ and, recalling 2.2. we have the following lemmas.

4.2. LEMMA. *For each positive integer λ and each positive integer k,*

$$P_\lambda(F) \leq P_\lambda(S_k) + P_\lambda(R_k).$$

Proof. Let $I(\nu)$ denote the νth cell of the λ-net. For each ν we then have

$$\omega_\nu^\lambda(F) = \sup_{I(\nu)} |F(p) - F(p')| = \sup_{I(\nu)} |S_k(p) - S_k(p') + R_k(p) - R_k(p')|$$

$$\leq \sup_{I(\nu)} |S_k(p) - S_k(p')| + \sup_{I(\nu)} |R_k(p) - R_k(p')| = \omega_\nu^\lambda(S_k) + \omega_\nu^\lambda(R_k).$$

The remainder of the proof is evident.

4.3. LEMMA. *For each* λ,

$$\limsup_{k\to\infty} P_\lambda(S_k) \geq P_\lambda(F).$$

Proof. Since each function F_n is nonnegative and since the convergence is uniform, for each positive ϵ there exists K such that for each $k \geq K$ and for each point p,

$$0 \leq R_k(p) < \frac{\epsilon}{\lambda}.$$

It follows that for each $k > K$, we have $P_\lambda(R_k) < \epsilon$, and in view of 4.2 we have

$$P_\lambda(S_k) \geq P_\lambda(F) - P_\lambda(R_k) > P_\lambda(F) - \epsilon,$$

and the lemma follows.

4.4. LEMMA. *For some finite number* A *and for each* λ,

$$\sum_{n=1}^{\infty} P_\lambda(F_n) < A.$$

Proof. Let λ be a fixed positive integer. Let the λ-net be constructed and its cells ordered. If $\lambda \geq 4$, then for some integer $N > 0$ we must have

$$Q(N) \leq \lambda < Q(N+1).$$

If $n \geq N+1$, then $1/Q(n) \leq 1/Q(N+1) < 1/\lambda$ and therefore each cell of the λ-net must contain at least one point at which F_n attains its maximum, $1/Q(n)$, and at least one point at which F_n attains its minimum, zero. Consequently, we conclude that for each ν,

$$\omega_\nu^\lambda(F_n) = \frac{1}{Q(n)},$$

and hence that

4.4.1 $$P_\lambda(F_n) = \sum_{\nu=1}^{\lambda^2} \omega_\nu^\lambda(F_n)/\lambda = \lambda^2(1/Q(n))/\lambda = \lambda/Q(n) < Q(N+1)/Q(n).$$

Since $Q(N) \leq \lambda < Q(N+1)$, the oscillation of F_N will be zero except on at most $4Q(N)^2$ λ-cells. On these cells we must have

$$\omega_\nu^\lambda(F_N) \leq 1/Q(N).$$

It follows that

4.4.2 $$P_\lambda(F_N) \leq 4Q(N)^2/(Q(N)\lambda) = 4Q(N)/\lambda \leq 4.$$

If $1 \leq n < N$, then $1/\lambda \leq 1/Q(N) < 1/Q(n)$ and the oscillation of F_n will be zero except on at most $Q(n)^2 \cdot ([\lambda/Q(n+1)]+2)^2$ cells. (The symbol "$[\lambda/Q(n+1)]$" means "largest integer not exceeding $\lambda/Q(n+1)$".) On these cells,

$$\omega_\nu^\lambda(F_n) \leq \lambda^{-1} Q(n)^{-1} \cdot 2Q(n+1) = 2Q(n)/\lambda,$$

and hence

4.4.3 $$\begin{aligned} P_\lambda(F_n) &\leq Q(n)^2 ([\lambda/Q(n+1)]+2)^2 (2Q(n)/\lambda)/\lambda \\ &\leq 2Q(n)(Q(n)^2/\lambda^2)\{(\lambda/Q(n+1))+2\}^2 \\ &= \{2/Q(n)\}\{1+(2Q(n+1)/\lambda)\}^2 \leq \{2/Q(n)\}(1+2)^2, \end{aligned}$$

since $n+1 \leq N$, and hence $Q(n+1) \leq Q(N) \leq \lambda$.

If $N > 1$, we have, in view of 4.4.1, 4.4.2, and 4.4.3,

$$\begin{aligned} \sum_{n=1}^{\infty} P_\lambda(F_n) &= \sum_{n=1}^{N-1} P_\lambda(F_n) + P_\lambda(F_N) + \sum_{n=N+1}^{\infty} P_\lambda(F_n) \\ &\leq \sum_{n=1}^{N-1} 18/Q(n) + 4 + \sum_{n=N+1}^{\infty} Q(N+1)/Q(n) \\ &< 18 \sum_{n=1}^{\infty} Q(n)^{-1} + 4 + Q(N+1) \sum_{n=N+1}^{\infty} Q(n)^{-1}. \end{aligned}$$

In view of 3.4, this yields

4.4.4 $$\sum_{n=1}^{\infty} P_\lambda(F_n) < 18(Q(1)-1)^{-1} + 4 + Q(N+1)(Q(N+1)-1)^{-1}$$

$$= 6 + 4 + (1 - Q(N+1)^{-1})^{-1}$$

$$\leq 10 + (1 - 1/Q(1))^{-1} = 10 + 4/3 < 12.$$

If $N = 1$, we have (recalling 4.4.1 and 4.4.2)

$$\sum_{n=1}^{\infty} P_\lambda(F_n) = P_\lambda(F_1) + \sum_{n=2}^{\infty} P_\lambda(F_n) \leq 4 + \sum_{n=2}^{\infty} \frac{Q(2)}{Q(n)}$$

$$= 4 + Q(2) \sum_{n=2}^{\infty} Q(n)^{-1}.$$

As before, we use 3.4 to obtain

4.4.5 $$\sum_{n=1}^{\infty} P_\lambda(F_n) \leq 4 + Q(2)(Q(2) - 1)^{-1} = 4 + 16/15 < 12.$$

If $1 \leq \lambda \leq 3$, then

4.4.6 $$\sum_{n=1}^{\infty} P_\lambda(F_n) = \sum_{n=1}^{\infty} \sum_{\nu=1}^{\lambda^2} \omega_\nu^\lambda(F_n)/\lambda \leq \sum_{n=1}^{\infty} \lambda/Q(n)$$

$$= \lambda \sum_{n=1}^{\infty} Q(n)^{-1} < \lambda/(Q(1) - 1) = \lambda/3 \leq 1.$$

Comparing 4.4.4, 4.4.5, and 4.4.6, we see that for each λ

$$\sum_{n=1}^{\infty} P_\lambda(F_n) < 12$$

and our proof is complete.

4.5. THEOREM. $F \in P$.

Proof. In view of 4.3, the proof of 4.2, and 4.4, we have for each λ,

$$P_\lambda(F) \leq \limsup_{k\to\infty} P_\lambda(S_k) \leq \limsup_{k\to\infty} \sum_{n=1}^{k} P_\lambda(F_n) = \sum_{n=1}^{\infty} P_\lambda(F_n) < 12.$$

5. F is nowhere totally differentiable. We introduce the following notation:

$$P_n = [p \mid F_n(p) > 0], \qquad Z_n = [p \mid F_n(p) = 0].$$

If A is a set, then $\overline{A}$ is the closure of the set A, and $A - B$ is the set consisting of those points which are in A and are not in B.

5.1. LEMMA.

$$I = \bigcap_{k=1}^{\infty} \bigcup_{n=k}^{\infty} \overline{(P_n - P_{n+1})} \cup \bigcup_{k=1}^{\infty} \bigcap_{n=k}^{\infty} \overline{P}_n \cup \bigcup_{k=1}^{\infty} \bigcap_{n=k}^{\infty} (Z_{n+1} - \overline{P}_{n+1}).$$

Proof. If $p \in I$, then one of the three following cases must occur:

(1) If k is sufficiently large, then for each $n > k$ there exists an open neighborhood of p such that for each point p' in that neighborhood, $F_n(p') = 0$.

(2) If k is sufficiently large, then for each $n > k$ there exists a sequence of points converging to p and such that for each point p' of this sequence, $F_n(p') > 0$.

(3) For each k, there exists $n > k$ and a sequence of points converging to p such that for each point p' of that sequence, $F_n(p') > 0$ and $F_{n+1}(p') = 0$.

Since each function F_n is continuous, (1) implies

$$p \in \bigcup_{k=1}^{\infty} \bigcap_{n=k}^{\infty} (Z_{n+1} - \overline{P}_{n+1}),$$

and (2) implies

$$p \in \bigcup_{k=1}^{\infty} \bigcap_{n=k}^{\infty} \overline{P}_n.$$

Existence of the sequences mentioned in case (3) and continuity imply that for some n exceeding each k, p must be either an inner point or a boundary point of a region where F_n is positive and F_{n+1} vanishes. Hence in this case,

$$p \in \bigcap_{k=1}^{\infty} \bigcup_{n=k}^{\infty} \overline{(P_n - P_{n+1})}.$$

5.2. LEMMA. *If* $p \in \overline{(P_n - P_{n+1})}$ *then there exists at least one point* p_n *with* *the following properties:*

5.2.1 $$\rho(p, p_n) = \sqrt{2}/Q(n+2);$$

5.2.2 $$|F_{n-k}(p) - F_{n-k}(p_n)| \leq 2Q(n-k)/Q(n+2) \qquad (k = 1, 2, \cdots, n-1);$$

5.2.3 $$|F_n(p) - F_n(p_n)| = 2Q(n)/Q(n+2);$$

5.2.4 $$|F_{n+k}(p) - F_{n+k}(p_n)| = 0 \qquad (k = 1, 2, 3, \cdots).$$

Proof. Let the coordinates of p be denoted by x and y. Consider the four points whose coordinates are

$$(x + Q(n+2)^{-1},\ y + Q(n+2)^{-1}),\ (x - Q(n+2)^{-1},\ y + Q(n+2)^{-1}),$$

$$(x - Q(n+2)^{-1},\ y - Q(n+2)^{-1}),\ (x + Q(n+2)^{-1},\ y - Q(n+2)^{-1}).$$

We require that p_n shall be included in $(\overline{P_n - P_{n+1}})$, and also that the line segment joining p to p_n shall either have no point other than p in common with any diagonal of that cell of the $Q(n)$-net to which the points belong, or shall coincide with one diagonal of that cell and shall have no point other than p in common with the other diagonal of the cell. When the latter situation obtains, we say that the segment in question does not cross any diagonal of the cell. In view of the structure of the set and the fact that

$$Q(n+1) \geq Q(2) = 16,$$

we see that at least one of the four points must satisfy these conditions, and we select any one of these as p_n.

Evidently, 5.2.1 is satisfied. The line segment joining p and p_n is parallel to some diagonal of each cell of each net, and in the sense of the preceding paragraph "does not cross any diagonal", nor does it cross the boundary of any cell of the $Q(n-k)$-net for $0 \leq k \leq n-1$. If the line segment joining p to p_n "does not cross any diagonal" nor cross the boundary of any cell of the $Q(n-k)$-net, then it is the projection of a line on the surface representing F_{n-k-1} which lies either entirely on a horizontal plane or entirely on one side of a pyramid. In the latter event, one can easily calculate $|F_{n-k}(p) - F_{n-k}(p_n)|$, recalling the remarks following 3.5. It follows that for $1 \leq k \leq n-1$ we must have either

$$|F_{n-k}(p) - F_{n-k}(p_n)| = 0,$$

or

$$|F_{n-k}(p) - F_{n-k}(p_n)| = \{\sqrt{2}/Q(n+2)\}\{2Q(n-k+1)/\sqrt{2}Q(n-k)\}$$

$$= 2Q(n-k)/Q(n+2).$$

Therefore 5.2.2 is verified. Also since p and p_n are in $\overline{P_n - P_{n+1}}$, we have, in particular,

$$|F_n(p) - F_n(p_n)| = 2Q(n)/Q(n+2);$$

that is 5.2.3 holds. Furthermore, for the same reason we must have

$$F_{n+1}(p) = F_{n+1}(p_n) = 0.$$

Since the points p and p_n are similarly placed in their respective cells of the $Q(n+k)$-net for each $k \geq 2$, we have also

$$F_{n+k}(p) = F_{n+k}(p_n) \qquad (k = 2, 3, 4, \cdots).$$

and these last two relations imply 5.2.4. This completes the proof.

5.3. THEOREM. *If*

$$p \in \bigcap_{k=1}^{\infty} \bigcup_{n=k}^{\infty} \overline{P_n - P_{n+1}},$$

then F is not totally differentiable at p.

Proof. There exists a sequence of integers $\{n_\lambda\}$ such that for each λ, $p \in \overline{P_{n_\lambda} - P_{n_\lambda+1}}$. We consider the points $\{p_{n_\lambda}\}$ satisfying the conditions of 5.2 and whose existence is guaranteed by 5.2. We have, recalling 3.3 and 5.2,

$$|F(p) - F(p_{n_\lambda})| = \left| \sum_{n=1}^{n_\lambda - 1} \{F_n(p) - F_n(p_{n_\lambda})\} + F_{n_\lambda}(p) - F_{n_\lambda}(p_{n_\lambda}) \right.$$

$$\left. + \sum_{n=n_\lambda+1}^{\infty} \{F_n(p) - F_n(p_{n_\lambda})\} \right|$$

$$= \left| \sum_{n=1}^{n_\lambda - 1} \{F_n(p) - F_n(p_{n_\lambda})\} + F_{n_\lambda}(p) - F_{n_\lambda}(p_{n_\lambda}) \right|$$

$$\geq |F_{n_\lambda}(p) - F_{n_\lambda}(p_{n_\lambda})| - \sum_{n=1}^{n_\lambda - 1} |F_n(p) - F_n(p_{n_\lambda})|$$

$$\geq 2Q(n_\lambda)/Q(n_\lambda + 2) - \sum_{n=1}^{n_\lambda - 1} 2Q(n)/Q(n_\lambda + 2)$$

$$= 2\{Q(n_\lambda) - \sum_{n=1}^{n_\lambda - 1} Q(n)\}/Q(n_\lambda + 2)$$

$$\geq 2\{Q(n_\lambda) - 2Q(n_\lambda - 1)\}/Q(n_\lambda + 2)$$

$$= 2\{Q(n_\lambda - 1) - 2\}Q(n_\lambda - 1)/Q(n_\lambda + 2).$$

We then have

$$|F(p) - F(p_{n_\lambda})| \quad \rho(p, p_{n_\lambda})^{-1} \geq \sqrt{2}\{Q(n_\lambda - 1) - 2\}Q(n_\lambda - 1),$$

and hence

$$\limsup_{\lambda \to \infty} |F(p) - F(p_{n_\lambda})| \quad \rho(p, p_{n_\lambda})^{-1} \geq \lim_{\lambda \to \infty} Q(n_\lambda) = \infty .$$

As $\lambda \longrightarrow \infty$, $p_{n_\lambda} \longrightarrow p$, and hence in view of 2.3 the proof is complete.

5.4. LEMMA. *If*

$$p \in \bigcap_{n=h}^{\infty} \overline{P}_n$$

then there exists an integer k *and for each integer* λ *there exists a point* p_λ *satisfying the following conditions:*

5.4.1 $$\rho(p, p_\lambda) = \sqrt{2}/2Q(k + \lambda),$$

5.4.2 $$F_{k+\lambda-1}(p_\lambda) = 1/Q(k + \lambda - 1),$$

5.4.3 $\quad F_{k+\lambda+m}(p_\lambda) = 0, \qquad (m = 0, 1, 2, \cdots),$

5.4.4 $\quad |F_m(p_\lambda) - F_m(p)| \leq Q(m)/Q(k+\lambda), \qquad (m = 1, 2, 3, \cdots, k+\lambda-2).$

Proof. Let k be the least integer which permits representation of the coordinates of p in the form $(a/Q(k), b/Q(k))$, where a and b are integers. In view of the structure of the set $\bigcap_{n=h}^{\infty} \overline{P}_n$ such an integer must exist. We then let p_λ be the point whose coordinates are

$$(\{a/Q(k)\} + \{1/2Q(k+\lambda)\}, \{b/Q(k)\} + \{1/2Q(k+\lambda)\}).$$

Clearly 5.4.1 is satisfied, and since p is a corner of some cell of the $Q(m)$-net for $m \geq k$, 5.4.2 can be verified by direct computation. Similarly, since p_λ is a corner of some cell of the $Q(m)$-net for $m > k + \lambda$, 5.4.3 is evident. One sees from the construction of p_λ that the line segment joining p and p_λ "does not cross any diagonal" (see 5.2) of any cell of any $Q(m)$-net for $m \leq k + \lambda - 1$. The segment joining p to p_λ either underlies a flat portion of the surface representing F_m or is parallel to the projection of a portion of a lateral edge of a pyramid. It follows that for such m, either

$$|F_m(p_\lambda) - F_m(p)| = Q(m)/Q(k+\lambda),$$

or

$$F_m(p_\lambda) = F_m(p) = 0.$$

Consequently 5.4.4 is verified and the proof is complete.

5.5. THEOREM. *If*

$$p \in \bigcup_{h=1}^{\infty} \bigcap_{n=h}^{\infty} \overline{P}_n$$

then F is not totally differentiable at p.

Proof. Since

$$p \in \bigcup_{h=1}^{\infty} \bigcap_{n=h}^{\infty} \overline{P}_n,$$

for some h we must have

$$p \in \bigcap_{n=h}^{\infty} \overline{P}_n.$$

Then in view of 5.4 there exist points p_λ ($\lambda = 1, 2, 3, \cdots$) and k such that the conditions of 5.4 are fulfilled for each λ. We have then

$$|F(p_\lambda) - F(p)| = \left| \sum_{n=1}^{\infty} \{F_n(p_\lambda) - F_n(p)\} \right|$$

$$= \left| \sum_{n=1}^{k+\lambda-2} \{F_n(p_\lambda) - F_n(p)\} + F_{k+\lambda-1}(p_\lambda) - F_{k+\lambda-1}(p) \right.$$

$$\left. + \sum_{n=k+\lambda}^{\infty} \{F_n(p_\lambda) - F_n(p)\} \right|.$$

Since p is a corner of the $Q(m)$-net for $m \geq k$, $F_m(p) = 0$ for such m, and, in view of 5.4, we have

$$|F(p_\lambda) - F(p)| \geq |F_{k+\lambda-1}(p_\lambda)| - \sum_{n=1}^{k+\lambda-2} |F_n(p_\lambda) - F_n(p)|$$

$$\geq 1/Q(k+\lambda-1) - \sum_{n=1}^{k+\lambda-2} Q(n)/Q(k+\lambda)$$

$$= \{Q(k+\lambda-1) - \sum_{n=1}^{k+\lambda-2} Q(n)\}/Q(k+\lambda)$$

$$\geq \{Q(k+\lambda-1) - 2Q(k+\lambda-2)\}/Q(k+\lambda)$$

$$= Q(k+\lambda-2)\{Q(k+\lambda-2) - 2\}/Q(k+\lambda).$$

It follows that

$$\limsup_{\lambda\to\infty} |F(p_\lambda) - F(p)| \quad \rho(p, p_\lambda)^{-1} \geq \limsup_{\lambda\to\infty} \sqrt{2}Q(k+\lambda-2)\{Q(k+\lambda-2)-2\}$$

$$\geq \limsup_{n\to\infty} Q(n) = \infty.$$

Since $\lim_{\lambda \to \infty} (p, p_\lambda) = 0$, the proof is complete.

5.6. LEMMA. *If*

$$p \in \bigcap_{n=k}^{\infty} (Z_{n+1} - \overline{P}_{n+1})$$

then on each half-line having p as its initial point, there exists a sequence of points $\{p_\lambda\}$ *such that for each* λ,

5.6.1 $$\rho(p, p_\lambda) \leq \sqrt{2}/Q(k+\lambda),$$

5.6.2 $$F_{k+m}(p_\lambda) = 0 \qquad (m = 1, 2, 3, \cdots).$$

Proof. Let $C(\lambda)$ denote that cell of the $Q(k+\lambda)$-net which contains p and also contains points of the half-line in question other than p. Since $p \in Z_{k+\lambda} - \overline{P}_{k+\lambda}$ for each λ, p must be either an interior point of some cell or a boundary point of two cells, hence there exists one and only one cell satisfying these requirements unless the half-line considered is the boundary, in which case either cell may be taken. We let p_λ denote the point other than p in which the half-line under consideration intersects the boundary of $C(\lambda)$. If the half-line is a boundary, we take either of the adjacent corners of $C(\lambda)$ as p_λ. Condition 5.6.1 is evident. If $m \geq \lambda$, then p_λ is a boundary point of some cell of the $Q(k+m)$-net and hence for such m

$$F_{k+m}(p_\lambda) = 0.$$

If $1 \leq m \leq \lambda$ then p and p_λ belong to the same cell of the $Q(k+m)$-net since $C(\lambda)$ can have inner points in common with at most one cell of each lower order net. Since $p \in Z_{k+m} - \overline{P}_{k+m}$ for such m, we conclude that

$$F_{k+m}(p_\lambda) = F_{k+m}(p) = 0,$$

for if $F_{k+m}(p_\lambda) > 0$ then we must conclude that $p \in \overline{P}_{k+m}$, which is contrary to our hypothesis.

5.7. LEMMA. *If*

$$p \in \bigcap_{n=k}^{\infty} (Z_{n+1} - \overline{P}_{n+1})$$

then for each λ *there exists a point* p_λ *such that*

5.7.1 $$\rho(p, p_\lambda) \le \sqrt{2}/Q(k+\lambda),$$

5.7.2 $$F_{k+m}(p_\lambda) = 0, \ \ 0 < m < \lambda,$$

5.7.3 $$F_{k+\lambda}(p_\lambda) = 1/Q(k+\lambda),$$

5.7.4 $$F_{k+m}(p_\lambda) = 0, \ m > \lambda.$$

Proof. Let $C(\lambda)$ denote any cell of the $Q(k+\lambda)$-net which contains p. Let the coordinates of the lower left corner of this cell be denoted by x_λ and y_λ. Let p_λ be the point whose coordinates are

$$(x_\lambda + \{1/2Q(k+\lambda+1)\}, \ y_\lambda + \{1/2Q(k+\lambda+1)\}).$$

Evidently 5.7.1 is satisfied. Since p and p_λ are contained in the same cell of the $Q(k+m)$-net for $m < \lambda$ and since

$$p \in \bigcap_{n=k}^{\infty} (Z_{n+1} - \overline{P}_{n+1}),$$

5.7.2 is also satisfied. Moreover 5.7.3 follows at once from 3.5; and since p_λ is a boundary point of some cell of the $Q(k+m)$-net for $m > \lambda$, 5.7.4 also follows from 3.5.

5.8. LEMMA. *If*

$$p \in \bigcap_{n=k}^{\infty} (Z_{n+1} - \overline{P}_{n+1}),$$

then R_k *is not totally differentiable at* p.

Proof. Lemma 5.6 implies that $\partial R_k/\partial x$ and $\partial R_k/\partial y$ either must fail to exist or must be equal to zero. Hence if R_k is totally differentiable at p we must have

$$\limsup_{p' \to p} \frac{|R_k(p') - R_k(p)|}{\rho(p', p)} = 0.$$

However, 5.7 implies

$$\limsup_{p' \to p} \frac{|R_k(p') - R_k(p)|}{\rho(p', p)} \geq 1/\sqrt{2} > 0,$$

and we conclude that R_k is not totally differentiable at p.

5.9. THEOREM. *If*

$$p \in \bigcap_{n=k}^{\infty} (Z_{n+1} - \bar{P}_{n+1}),$$

then F is not totally differentiable at p.

Proof. We recall that $F(p) = S_k(p) + R_k(p)$. If S_k is totally differentiable at p, 5.8 implies that F is not. We therefore need only to consider a point p at which S_k is not totally differentiable. In view of 2.3, we see that there must exist at least one direction in which the surface representing S_k has no tangent line. On the other hand, it is clear that this surface has half-tangents in every direction. Now if F is totally differentiable at p, then its graph must possess a tangent line in each direction. Also 5.6 implies that if F possesses a half-tangent, then it must coincide with the half-tangent of S_k. Since the half-tangents of S_k are distinct in at least one direction, it follows that F can have no tangent line in this direction and is not totally differentiable there. This completes the proof.

In view of 5.1, 5.3, 5.5, and 5.9 we have:

5.10. THEOREM. *If $p \in I$, then F is not totally differentiable at p.*

Thus F is a function in the class $P \cap C$ which is nowhere totally differentiable.

6. The space $P \cap C$. Having already established the existence of at least one continuous function of class P which is nowhere totally differentiable, we now show that in certain spaces the subset of all such functions is a residual set. Our proofs will be based upon the well-known theorem of Baire. We retain the notation of the preceeding sections (cf. §§ 1, 2). Let C represent the space of all continuous functions defined on I. As is well known we can norm the space by letting

$$||f||_C = \sup_{p \in I} |f(p)|.$$

Under this norm the space C is complete. If $f \in P \cap C$, we let

$$||f|| = ||f||_C + P(f)$$

(cf. 2.2). Clearly $||f|| = 0$ if and only if $f(p) = 0$ for each $p \in I$. One can easily show that this is a proper norm for the space $P \cap C$. If f and g are two functions of $P \cap C$, we define a distance $\rho(f, g)$ in the usual manner, letting

$$\rho(f, g) = ||f - g||.$$

7. Completeness of $P \cap C$. In this section we show that the space $P \cap C$ is a complete space when metrized as is indicated above.

7.1. LEMMA. *If*

$$\lim_{m, n \to \infty} P(f_n - f_m) = 0,$$

then

$$\lim_{n \to \infty} P(f_n)$$

exists.

Proof. In view of 2.1, we see that

$$\omega_\nu^\lambda(f_m - f_n) = \sup_{I(\nu)} |f_m(p) - f_n(p) - f_m(p') + f_n(p')|$$

$$\geq |\sup_{I(\nu)} |f_m(p) - f_m(p')| - \sup_{I(\nu)} |f_n(p) - f_n(p')||$$

$$= |\omega_\nu^\lambda(f_m) - \omega_\nu^\lambda(f_n)|.$$

It follows that

$$P_\lambda(f_m - f_n) \geq |P_\lambda(f_m) - P_\lambda(f_n)|.$$

We have then

$$P(f_m - f_n) = \sup_\lambda P_\lambda(f_m - f_n) \geq |\sup_\lambda P_\lambda(f_m) - \sup_\lambda P_\lambda(f_n)| = |P(f_m) - P(f_n)|,$$

that is

$$|P(f_m) - P(f_n)| \leq P(f_m - f_n),$$

and our lemma clearly follows.

7.2. THEOREM. *$P \cap C$ is complete.*

Proof. If

$$\lim_{m,n\to\infty} ||f_m - f_n|| = 0,$$

then

$$\lim_{m,n\to\infty} ||f_m - f_n||_C = 0.$$

This implies the existence of a continuous function f such that

$$\lim_{n\to\infty} ||f_n - f||_C = 0.$$

We shall show that $f \in P \cap C$ and that

$$\lim_{n\to\infty} ||f_n - f|| = 0.$$

We note that for each positive ϵ and each integer λ there exist integers M and N having the properties:

7.2.1 $M > N$;

7.2.2 for each $n > N$ and each ν,

$$\omega_\nu^\lambda(f) < \omega_\nu^\lambda(f_n) + (\epsilon/\lambda);$$

7.2.3 $P(f_M) < \liminf_{n\to\infty} P(f_n) + \epsilon.$

Property 7.2.2 implies that for each $n > N$,

$$P_\lambda(F) < P_\lambda(f_n) + \epsilon < P(f_n) + \epsilon.$$

In particular,

$$P_\lambda(f) < P_\lambda(f_M) + \epsilon < P(f_M) + \epsilon,$$

and consequently 7.2.3 implies

$$P_\lambda(f) < \liminf_{n\to\infty} P(f_n) + 2\epsilon .$$

Since this must hold for each ϵ and each λ, we conclude that

$$P(f) \le \liminf_{n\to\infty} P(f_n).$$

In view of 7.1, this implies $f \in P$. Also, since

$$\lim_{m\to\infty} ||f_m - f_n||_C = ||f - f_n||_C$$

for each n, we conclude that

$$P(f - f_n) \le \liminf_{m\to\infty} P(f_m - f_n)$$

and hence that

$$\lim_{n\to\infty} P(f - f_n) \le \lim_{m,n\to\infty} P(f_m - f_n) = 0.$$

We have then

$$\lim_{n\to\infty} ||f - f_n|| = 0,$$

completing the proof.

8. Preliminary lemmas. If a function is totally differentiable at a point, then its graph is a surface which has a non-vertical tangent plane at that point. It follows that the surface must have a non-vertical tangent line in each direction. We let:

8.1 $U = P \cap C \cap [f \mid \text{for each } p \in I, \limsup_{p'\to p} \{|f(p) - f(p')|/\rho(p,p')\} = \infty]$.

Clearly, if $f \in U$, then f is nowhere totally differentiable. For each positive integer K, let A_K be defined as follows:

8.2. DEFINITION. If $f \in P \cap C$ and if for some $p \in I$ and each p' such that $\rho(p, p') < 1/K$, we have

$$\frac{|f(p) - f(p')|}{\rho(p, p')} \leq K,$$

then $f \in A_K$.

8.3. LEMMA.

$$P \cap C - U = \bigcup_{K=1}^{\infty} A_K.$$

Proof. If $f \in A_K$ for some K, then $f \in (P \cap C - U)$.

Conversely, if $f \in (P \cap C - U)$, then for some point $p \in I$,

$$\limsup_{p' \to p} \frac{|f(p) - f(p')|}{\rho(p, p')} < \infty,$$

and hence for some integer N

$$\limsup_{p' \to p} \frac{|f(p) - f(p')|}{\rho(p, p')} < N - 1.$$

It follows that for some positive δ, $\rho(p, p') < \delta$ implies

$$\frac{|f(p) - f(p')|}{\rho(p, p')} < N.$$

If K is any integer exceeding both N and $1/\delta$, we must have $f \in A_K$, and our proof is complete.

8.4. LEMMA. *For each* K, $A_K = \overline{A}_K$.

Proof. We suppose $\{f_n\}$ to be a sequence of functions and K a fixed integer with $f_n \in A_K$ for each n and

$$\lim_{m,n \to \infty} ||f_n - f_m|| = 0.$$

Since $P \cap C$ is complete (see 7.2) there exists f such that $f \in P \cap C$ and

$$\lim_{n\to\infty} ||f_n - f|| = 0.$$

We must show that $f \in A_K$. Since $f_n \in A_K$ for each n, there exists a sequence of points p_n ($n = 1, 2, 3, \cdots$) with the properties

8.4.1 $p_n \in I$ for each n;

and

8.4.2 if $\rho(p_n, q) < 1/K$ with $q \in I$, then

$$|f_n(p_n) - f_n(q)| \leq K\rho(p_n, q).$$

This sequence must have at least one limit point p and a subsequence $\{p_{n_\nu}\}$ converging to it. Let p' be an interior point of I with $\rho(p, p') < 1/K$, and for each ν let p'_{n_ν} be a point having the property that $\rho(p_{n_\nu}, p'_{n_\nu}) = \rho(p, p')$ and so placed that the directed line segment from p_{n_ν} to p'_{n_ν} shall have the same direction as that from p to p'. Then clearly the sequence $\{p'_{n_\nu}\}$ converges to p', and, since p' is an interior point of I, there exists N such that $\nu > N$ implies $p'_{n_\nu} \in I$. We note also that for each ν, $\rho(p, p_{n_\nu}) = \rho(p', p'_{n_\nu})$. Since our sequence of functions converges uniformly, for each positive ϵ there exists an integer N_1 such that for each $\nu > N_1$ and each $q \in I$,

8.4.3 $$|f_{n_\nu}(q) - f(q)| < \epsilon.$$

Moreover since f is uniformly continuous, there exists N_2 such that for $\nu > \max(N, N_2)$,

8.4.4 $$|f(p) - f(p_{n_\nu})| < \epsilon, \quad |f(p') - f(p'_{n_\nu})| < \epsilon.$$

For $\nu > \max(N, N_1, N_2)$, we have $p'_{n_\nu} \in I$, and relations 8.4.2, 8.4.3, and 8.4.4 hold. It follows that

$$|f(p) - f(p')| \leq |f(p) - f(p_{n_\nu})| + |f(p_{n_\nu}) - f_{n_\nu}(p_{n_\nu})|$$

$$+ |f_{n_\nu}(p_{n_\nu}) - f_{n_\nu}(p'_{n_\nu})| + |f_{n_\nu}(p'_{n_\nu}) - f(p'_{n_\nu})| + |f(p'_{n_\nu}) - f(p')|$$

$$< \epsilon + \epsilon + K\rho(p_{n_\nu}, p'_{n_\nu}) + \epsilon + \epsilon = K\rho(p, p') + 4\epsilon.$$

Our lemma clearly follows.

9. The sets A_K are nowhere dense. The whole of this section is devoted to a method for constructing functions which we shall need in our proof that the sets A_K are nowhere dense.

Let n be a fixed positive integer. Construct the $Q(n)$-net (cf. 3.1, 2.1) and order its cells in any manner whatever. In particular, this ordering may be entirely independent of position. We let $I(k)$ $(k = 1, 2, \cdots, Q(n)^2)$ represent the kth cell. In each of these cells we construct a subnet as follows: On the kth cell we superimpose those cells of the $Q(n+k)$-net which have inner points in common with that cell. Since $Q(n+k)$ is divisible by $Q(n)$, there are exactly $Q(n+k)^2/Q(n)^2$ cells in the kth subnet. Evidently we may order the cells of each subnet. We do so, again in any manner, and agree to let $I(k, m)$ represent the mth cell of the kth subnet, where

$$m = 1, 2, 3, \cdots, Q(n, + k)^2/Q(n)^2; \quad k = 1, 2, 3, \cdots, Q(n)^2.$$

9.1. CONSTRUCTION. Let N be a sequence of positive integers and let e be a sequence of positive numbers. Let $R(k, m, j)$ $(j = 1, 2, 3, \cdots, N(n))$ be a set of $N(n)$ open regions, disjoint or not, and each having points in common with $I(k, m)$. Now if $e(n)$ is a sufficiently small positive number, then for each triple (k, m, j) such that

$$(1 \leq k \leq Q(n), \ 1 \leq m \leq Q(n+k)^2/Q(n)^2, \ 1 \leq j \leq N(n))$$

we can determine a point $p(k, m, j)$ and an open circular region $C(k, m, j)$ with center at $p(k, m, j)$ and diameter $e(n)$ and such that

9.1.1 $$C(k, m, j) \subset R(k, m, j) \cap I(k, m);$$

9.1.2 if $j_1 \neq j_2$ then $C(k, m, j_1)$ and $C(k, m, j_2)$ are disjoint;

9.1.3. no straight line has points in common with more than two of the regions $C(k, m, j)$.

Since the number of regions $R(k, m, j)$ is finite, it is clear that the indicated construction can be carried out. We shall call any system of circular regions constructed in the manner just described a system of type B_n.

9.2. DEFINITION. Let there be given a system of type B_n; let $p \in I$. If for some triple (k, m, j) we have $\rho(p, p(k, m, j)) \leq e(n)/2$ then

$$b_n(p) = \frac{Q(n)}{Q(n+k)}\left\{1 - \frac{2\rho(p, p(k,m,j))}{e(n)}\right\};$$

otherwise let

$$b_n(p) = 0.$$

The function b_n just defined is called a function of type B_n.

Evidently the graph of this function is a surface consisting of a finite number of right circular cones, each cone being erected over one of the circular regions of the system, and of a plane of height zero elsewhere. The height of each cone on $I(k)$ is $Q(n)/Q(n+k)$, $(1 \le k \le Q(n)^2)$. We note that with each system of type B_n there is associated one and only one function of type B_n. We also note that the only properties of the sequence Q which have been employed in this section are that $Q(n)$ is an integer and that $Q(n+k)^2/Q(n)^2$ is an integer for $1 \le k \le Q(n)$, and $n = 1, 2, 3, \cdots$.

9.3. THEOREM. *If f is a function of type B_n, then $f \in P \cap C$.*

Proof. To show that $f \in P$ we must prove that the sequence $P_\lambda(f)$ $(\lambda = 1, 2, 3, \cdots)$ is bounded. We notice that in view of 9.1.1 we have

$$e(n) \le 1/Q(n + Q(n)^2).$$

We consider four cases.

Case I. $1 \le \lambda \le Q(n)$.

We have here $1/Q(n) \le 1/\lambda$, and it follows that each cell of the $Q(n)$-net can have points in common with at most four cells of the λ-net. Since the oscillation of f on a cell $I(k)$ cannot exceed $Q(n)/Q(n+k)$, we have

9.3.1
$$P_\lambda(f) \le \sum_{k=1}^{Q(n)^2} \frac{1}{\lambda}\cdot\frac{4Q(n)}{Q(n+k)} \le 4\sum_{k=1}^{Q(n)^2}\frac{Q(n)}{Q(n+k)}$$

$$= 4\sum_{k=1}^{Q(n)^2} Q(n)^{1-2^k} < 4\sum_{k=1}^{Q(n)^2} Q(n)^{-k}$$

$$< \frac{4}{Q(n)}\cdot\frac{1}{1-1/Q(n)} = \frac{4}{Q(n)-1}.$$

Case II. $Q(n) \le Q(n+k_0) \le \lambda \le Q(n+k_0+1) \le Q(n+Q(n)^2) \le 1/e(n)$. Since equality cannot hold simultaneously between the second and third and the third and fourth members, we conclude that $k_0 < Q(n)^2$. Now if $k > k_0$, we have $1/Q(n+k) \le 1/\lambda$. The cell $I(k)$ has points in common with at most $([\lambda/Q(n)]+2)^2$ cells of the λ-net, and on these cells the oscillation of f does not exceed $Q(n)/Q(n+k)$. It follows that the contribution of the cells $I(k)$, $k = k_0+1, \cdots, Q(n)^2$, to the sum $P_\lambda(f)$ is at most

$$\sum_{k=k_0+1}^{Q(n)^2} \lambda^{-1}([\lambda/Q(n)]+2)^2\, Q(n)/Q(n+k).$$

Using the inequalities

$$2^{k_0+1} - 2^k - 1 \le -k + k_0,$$

$$-2^k < -k + k_0,$$

$$1 - 2^{k_0} - 2^k < -k + k_0,$$

which are valid if $k_0 \ge 0$ and $k \ge k_0 + 1$, we have

$$\sum_{k=k_0+1}^{Q(n)^2} \frac{1}{\lambda}\left(\left[\frac{\lambda}{Q(n)}\right]+2\right)^2 \cdot \frac{Q(n)}{Q(n+k)}$$

$$\le \sum_{k=k_0+1}^{Q(n)^2} \frac{1}{\lambda}\left(\frac{\lambda^2}{Q(n)^2} + \frac{4\lambda}{Q(n)} + 4\right)\frac{Q(n)}{Q(n+k)}$$

$$= \sum_{k=k_0+1}^{Q(n)^2}\left(\frac{\lambda}{Q(n)Q(n+k)} + \frac{4}{Q(n+k)} + \frac{4Q(n)}{\lambda Q(n+k)}\right)$$

$$\le \sum_{k=k_0+1}^{Q(n)^2}\left(\frac{Q(n+k_0+1)}{Q(n)Q(n+k)} + \frac{4}{Q(n+k)} + \frac{4Q(n)}{Q(n+k_0)Q(n+k)}\right)$$

$$= \sum_{k=k_0+1}^{Q(n)^2}\left(Q(n)^{2^{k_0+1}-2^k-1} + 4Q(n)^{-2^k} + 4Q(n)^{1-2^{k_0}-2^k}\right)$$

$$\leq 9 \sum_{k=k_0+1}^{Q(n)^2} Q(n)^{-k+k_0} = 9 \sum_{k=1}^{Q(n)^2-k_0} Q(n)^{-k}$$

$$\leq 9 \sum_{k=1}^{\infty} Q(n)^{-k} = \frac{9}{Q(n)-1} .$$

We now consider the cells $I(k)$ for $k = 1, 2, \cdots, k_0$. For such k we have $1/\lambda \leq 1/Q(n+k)$. Since $e(n) \leq 1/\lambda$, each circle $C(k, m, j)$ can have points in common with at most four cells of the λ-net. Since each $I(k)$ contains exactly $N(n)\ Q(n+k)^2/Q(n)^2$ such circles, we conclude that for each cell there are at most $4N(n) \cdot Q(n+k)^2/Q(n)^2$ cells of the λ-net on which the oscillation of f is not zero. Furthermore on these cells the oscillation cannot exceed $Q(n)/Q(n+k)$, and we conclude that the contribution of these cells to $P_\lambda(f)$ is certainly not greater than

$$\sum_{k=1}^{k_0} \frac{1}{\lambda} \cdot 4 \cdot \frac{N(n)Q(n+k)^2}{Q(n)^2} \cdot \frac{Q(n)}{Q(n+k)} .$$

Since $1/\lambda \leq 1/Q(n+k_0)$, this sum is in turn dominated by the sum

$$\sum_{k=1}^{k_0} \frac{4N(n)Q(n+k)}{Q(n+k_0)Q(n)} .$$

Recalling 3.3, we may simplify this as follows:

$$\sum_{k=1}^{k_0} \frac{4N(n)Q(n+k)}{Q(n+k_0)Q(n)} = \frac{4N(n)}{Q(n)Q(n+k_0)} \sum_{k=1}^{k_0} Q(n+k)$$

$$\leq \frac{4N(n)}{Q(n)Q(n+k_0)} \sum_{k=1+n}^{n+k_0} Q(k)$$

$$\leq \frac{4N(n)}{Q(n)Q(n+k_0)} \cdot 2Q(n+k_0) = \frac{8N(n)}{Q(n)} .$$

Combining the two estimates obtained above, we have

9.3.2 $$P_\lambda(f) \le \frac{9}{Q(n)-1} + \frac{8N(n)}{Q(n)} \le \frac{9+8N(n)}{Q(n)-1} \le \frac{17N(n)}{Q(n)-1}.$$

Case III. $Q(n+Q(n)^2) \le Q(n+k) \le \lambda \le 1/e(n)$.

Here the argument used in obtaining the second estimate in Case II applies directly yielding an estimate for the entire sum $P_\lambda(f)$. There are minor changes in computation which we give here:

9.3.3 $$P_\lambda(f) \le \frac{4N(n)}{Q(n)Q(n+k_0)} \sum_{k=n+1}^{n+Q(n)^2} Q(k)$$

$$\le \frac{4N(n)}{Q(n)Q(n+k_0)} \cdot 2Q(n+Q(n)^2) \le \frac{8N(n)}{Q(n)}.$$

Case IV. $1/e(n) < \lambda$.

Consider one circular region $C(k,m,j)$. Its diameter is $e(n)$, so it can have points in common with at most $([\lambda e(n)]+2)^2$ cells of the λ-net. Over this region the graph of f is a right circular cone, the slope of whose element is

$$\frac{2Q(n)}{Q(n+k)e(n)}.$$

Since the longest line segment in each λ-cell is of length $\sqrt{2}/\lambda$, the oscillation on each λ-cell is at most

$$\frac{2\sqrt{2}Q(n)}{Q(n+k)e(n)}.$$

It follows that the contribution to $P_\lambda(f)$ of any one circle is at most

$$\frac{1}{\lambda} \cdot ([\lambda e(n)]+2)^2 \cdot \frac{2\sqrt{2}Q(n)}{Q(n+k)e(n)}.$$

Since the kth cell of the $Q(n)$-net contains $Q(n+k)^2/Q(n)^2$ sub-cells, and since each sub-cell contains $N(n)$ of the circular regions $C(k,m,j)$, we have

9.3.4 $$P_\lambda(f) \le \sum_{k=1}^{Q(n)^2} N(n)\frac{Q(n+k)^2}{Q(n)^2} \cdot \frac{1}{\lambda}([\lambda e(n)]+2)^2 \frac{2\sqrt{2}Q(n)}{Q(n+k)e(n)}$$

$$\leq \frac{2\sqrt{2}N(n)}{Q(n)} \sum_{k=1}^{Q(n)^2} \frac{Q(n+k)}{e(n)} \left(e(n) + \frac{2}{\lambda}\right)^2$$

$$\leq \frac{2\sqrt{2}N(n)}{Q(n)} \sum_{k=1}^{Q(n)^2} \frac{Q(n+k)}{e(n)} (3e(n))^2$$

$$= \frac{18\sqrt{2}N(n)e(n)}{Q(n)} \sum_{k=1}^{Q(n)^2} Q(n+k)$$

$$= \frac{18\sqrt{2}N(n)e(n)}{Q(n)} \sum_{k=n+1}^{n+Q(n)^2} Q(k)$$

$$< \frac{18\sqrt{2}N(n)}{Q(n)} \cdot \frac{1}{Q(n+Q(n)^2)} \cdot 2Q(n+Q(n)^2) = \frac{36\sqrt{2}N(n)}{Q(n)} .$$

Upon examining the relations 9.3.1, 9.3.2, 9.3.3, and 9.3.4, we conclude that, for each λ,

$$P_\lambda(f) < \frac{36\sqrt{2}N(n)}{Q(n)-1} .$$

This clearly implies $f \in P$; and since f is evidently continuous, our theorem follows.

As an immediate consequence of the proof of the preceding theorem we have:

9.4. COROLLARY. *If $f_n \in B_n$ for each n and if*

$$\lim_{n\to\infty} \frac{N(n)}{Q(n)} = 0,$$

then

$$\lim_{n\to\infty} P(f_n) = 0.$$

9.5. REMARK. If in 9.2, the expression $Q(n)/Q(n+k)$ is replaced by $h(n)Q(n)/Q(n+k)$, where $h(n)$ is any positive number, 9.3 still holds as does 9.4 if we replace $N(n)/Q(n)$ by $h(n)N(n)/Q(n)$.

10. U is residual in $P \cap C$. We are now in position to prove:

10.1. THEOREM. *For each positive integer K, A_K is nowhere dense in $P \cap C$.*

Proof. Suppose $f \in P \cap C$; let n be any fixed positive integer. We shall now construct a function of type B_n (see § 9). We construct the $Q(n)$-net and the subnets as specified in § 9 and retain the notation there employed. Since f is continuous on I, there exists within or on the boundary of each cell of each subnet at least one point at which f assumes a maximum for that cell. More precisely, for each pair (k, m) $1 \leq k \leq Q(n)$, $1 \leq m \leq Q(n+k)^2/Q(n)^2$,

there exists a point $q(k, m)$ such that

$$q(k, m) \in I(k, m);$$

$$\text{if } p \in I(k, m), \text{ then } f(q(k, m)) - f(p) \geq 0.$$

Furthermore since f is uniformly continuous, there exists a positive number $r(n)$ such that if $\rho(p, q) \leq r(n)$, then, for $k \leq Q(n)^2$,

$$|f(p) - f(q)| \leq Q(n)/3Q(n + Q(n)^2) \leq Q(n)/3Q(n+k).$$

For each pair (k, m) let $\alpha(k, m)$ be a circular region of radius $r(n)/2$ and with center at $q(k, m)$. Let $N(n) = 1$, and let the region $R(k, m, 1)$ be the interior of the set $I(k, m) \cap \alpha(k, m)$. We now make any permissible choice of the points $p(k, m, 1)$ and the number $e(n)$ and define a function of type B_n which we shall call g_n. In view of 9.2 and earlier remarks in § 9, it is clear that that infinitely many permissible choices must exist. We note in passing that our choice of the regions $R(k, m, 1)$ has assured satisfaction of the inequality

$$e(n) \leq r(n).$$

We can evidently repeat this process for each n (letting $N(n) = 1$ for each n) and obtain a sequence of functions $\{g_n\}$. For each n and each $p \in I$, let

$$f_n(p) = f(p) + g_n(p).$$

The sequence $\{f_n\}$ is evidently uniformly convergent to f in the ordinary sense, and in view of 9.4 we see that

$$\lim_{n\to\infty} \|f_n - f\| = \lim_{n\to\infty} \|g_n\| = 0.$$

We shall now consider a set A_K and shall show that for n sufficiently large no function f_n can belong to A_K. Let $p \in I$; then for some pair (k, m), we have $p \in I(k, m)$. For simplicity, let $v = p(k, m, 1)$, also let $q = q(k, m)$. Now if $p \in I(k, m) - C(k, m, 1)$, we have

$$\begin{aligned} f_n(v) - f_n(p) &= g_n(v) + f(v) - g_n(p) - f(p) = g_n(v) + f(v) - f(p) \\ &= g_n(v) + f(v) - f(q) + f(q) - f(p) \\ &\geq g_n(v) + f(v) - f(q). \end{aligned}$$

Since $\rho(v, q) \leq r(n)$,

$$f(q) - f(v) < Q(n)/3Q(n+k).$$

In addition, $g_n(v) = Q(n)/Q(n+k)$; hence the above yields

$$f_n(v) - f_n(p) \geq 2Q(n)/3Q(n+k).$$

Since $\rho(p, v) < \sqrt{2}/Q(n+k)$, it follows that

10.1.1 $$\frac{f_n(v) - f_n(p)}{\rho(v, p)} \geq \frac{2Q(n)}{3\sqrt{2}}.$$

On the other hand, suppose $p \in C(k, m, 1)$. We now let e represent the point in which the directed line segment from v to p extended intersects the boundary of the region $C(k, m, 1)$. We have then

10.1.2 $$\rho(e, p) + \rho(p, v) = e(n)/2,$$

and also $g_n(e) = 0$. Recalling that $e(n) \leq r(n)$, we have, in view of 9.2,

10.1.3 $$\begin{aligned} |f_n(v) - f_n(p)| &= |g_n(v) + f(v) - g_n(p) - f(p)| \\ &\geq |g_n(v) - g_n(p)| - |f(v) - f(p)| \\ &\geq \rho(p, v)\frac{2Q(n)}{e(n)Q(n+k)} - \frac{Q(n)}{3Q(n+k)} \end{aligned}$$

Since $\rho(p, v) = (e(n)/2) - \rho(p, e)$, we see by direct computation that

$$g_n(p) = \rho(p, e)\,\frac{2Q(n)}{e(n)Q(n+k)},$$

and hence we have

10.1.4 $$|f_n(p) - f_n(e)| = |g_n(p) + f(p) - g_n(e) - f(e)|$$
$$\geq |g_n(p)| - |f(p) - f(e)|$$
$$\geq \rho(p, e)\,\frac{2Q(n)}{e(n)Q(n+k)} - \frac{Q(n)}{3Q(n+k)}\ .$$

Since v, p, and e are collinear points on a radius of a circular region lying entirely within $I(k, m)$, we see that

$$\rho(p, v) \leq 1/2Q(n+k) \text{ and } \rho(p, e) \leq 1/2Q(n+k).$$

Consequently, if $\rho(p, v) \geq e(n)/4$, 10.1.3 yields

10.1.5 $$\frac{|f_n(v) - f_n(p)|}{\rho(p, v)} \geq \frac{Q(n)}{6Q(n+k)\rho(p, v)} \geq \frac{Q(n)}{3}.$$

Also, if $\rho(p, e) \geq e(n)/4$, 10.1.4 implies

10.1.6 $$\frac{|f_n(p) - f_n(e)|}{\rho(p, e)} \geq \frac{Q(n)}{6Q(n+k)\rho(p, e)} \geq \frac{Q(n)}{3}\ .$$

In view of 10.1.2, either 10.1.5 or 10.1.6 (or both) must hold. Combining this with 10.1.1, we may now assert that for each $p \in I$ there exists p' with

$$\rho(p, p') < 1/Q(n)$$

and

$$\frac{|f_n(p) - f_n(p')|}{\rho(p, p')} > \frac{Q(n)}{3}.$$

It is easily established that if

$$n > \log(\log 3K/\log 2)/\log 2,$$

then f_n does not belong to A_K, and hence A_K is nowhere dense since each element of the space $P \cap C$ is a limit point of the complement of A_K.

Recalling now that

$$P \cap C - U = \bigcup_{k=1}^{\infty} A_k$$

and the theorem of Baire, we have:

10.2. THEOREM. *U is a residual set in $P \cap C$.*

Otherwise stated, the set of functions in $P \cap C$ which are nowhere totally differentiable is a residual set.

11. Category of nondifferentiable functions in $P \cap E$. We now turn our attention to the space $P \cap E$, where E is the set of functions mentioned in § 1 and introduced by Saks [**6**]. Let $\|f\|_E$ be the least upper bound of the total variations of f along all line segments in I. The set E shall consist of all functions f having the properties:

f is absolutely continuous along each line segment in I;

$\|f\|_E < \infty$;

if p is a boundary point of I, then $f(p) = 0$.

We let $\rho(f, g) = \|f - g\|_E$. The space E is metric, complete (to be proved in the following section), and linear, and convergence in this metric implies convergence in the classical metric for the space C. In view of 9.1.3 and 9.2, it is clear that $B_n \subset P \cap E$. More precisely, if $f \in B_n$ then

$$\|f\|_E \leq 2Q(n)/Q(n+1) = 2/Q(n).$$

If we let

$$\|f\|_{P \cap E} = \|f\|_E + P(f),$$

we have a proper norm in $P \cap E$. Using this norm and corresponding metric, we need only make appropriate minor changes in the preceding sections to obtain analogous results in $P \cap E$ culminating in the following theorem.

11.1. THEOREM. *In the space $P \cap E$ there exists a residual set of functions* *each element of which is nowhere totally differentiable.*

12. Completeness of E. In this and the next section we shall show that $P \cap E$ is of first category in E with respect to the metric used by Saks and which we have defined in the preceding section. As before we use the theorem of Baire.

12.1. THEOREM. *E is complete.*

Proof. If s is a line segment in I and f any function in E, then by $T_s(f)$ we mean the total variation of f along the segment s. If $\{f_n\}$ is a sequence of functions of E with

$$\lim_{m,n \to \infty} \|f_m - f_n\|_E = 0,$$

then for each positive ϵ there exists a positive integer N such that for each $n > N$, for each $k > 0$, and for each line segment s in I,

$$T_s(f_n - f_{n+k}) < \epsilon .$$

Since each function in E vanishes on the boundary of I, this implies that for such n and k and for each $p \in I$,

$$|f_n(p) - f_{n+k}(p)| < \epsilon .$$

It follows that there exists a continuous function f such that for each n,

$$\lim_{k \to \infty} \|f_n - f_{n+k}\|_C = \|f_n - f\|_C .$$

In view of the well-known semi-continuity property of the total variation for functions of one real variable, we have for each segment s

$$\liminf_{k \to \infty} T_s(f_n - f_{n+k}) \geq T_s(f_n - f).$$

Since

$$\lim_{n,k \to \infty} \|f_n - f_{n+k}\|_E = 0,$$

it follows that

12.1.1 $$\lim_{n \to \infty} \|f_n - f\|_E = 0.$$

We now need only show that f is absolutely continuous along each segment in I, that $\|f\|_E < \infty$, and that f vanishes on the boundary of I. Let s be a line segment in I; let $\epsilon > 0$ be given. In view of 12.1.1, there exists N such that for $n > N$,

12.1.2 $$T_s(f_n - f) < \epsilon/2.$$

Also since f_{N+1} is absolutely continuous on s, there exists $\delta > 0$ such that if $\{I_i\}$ $(i = 1, 2, \cdots, m)$ is any set of intervals on s, the sum of whose lengths does not exceed δ, then

$$\sum_{i=1}^{m} T_{I_i}(f_{N+1}) < \epsilon/2.$$

Also in view of 12.1.2,

$$\sum_{i=1}^{m} T_{I_i}(f_{N+1} - f) \leq T_s(f_{N+1} - f) < \epsilon/2.$$

We then have

$$\sum_{i=1}^{m} T_{I_i}(f) \leq \sum_{i=1}^{m} T_{I_i}(f_{N+1} - f) + \sum_{i=1}^{m} T_{I_i}(f_{N+1}) < \epsilon/2 + \epsilon/2 = \epsilon.$$

Since this holds for each set of intervals the sum of whose lengths is less than δ, we conclude that f is absolutely continuous along s. Also since $\lim_{n \to \infty} T(f_n)$ clearly exists, we conclude that $\|f\|_E < \infty$; and since f clearly must vanish on the boundary, our theorem is proved.

13. Category of $P \cap E$ in E.

13.1. DEFINITION. For each positive integer N, let

$$W_N = E \cap [f \mid P(f) \leq N].$$

Clearly

$$P \cap E = \bigcup_{N=1}^{\infty} W_N.$$

13.2. LEMMA. *For each* N, $W_N = \overline{W_N}$.

Proof. Let f_n be a sequence of functions of W_N convergent in our metric. For each λ and ν, and $f, f_n \in P$,

13.2.1 $$\omega_\nu^\lambda(f) = \sup_{p,p' \in I(\nu)} |f(p) - f(p')|$$

$$= \sup_{p,p' \in I(\nu)} |f(p) - f_n(p) + f_n(p) - f_n(p') + f_n(p') - f(p')|$$

$$\leq \sup_{p,p' \in I(\nu)} |f_n(p) - f_n(p')| + \sup_{p \in I(\nu)} |f(p) - f_n(p)|$$

$$+ \sup_{p' \in I(\nu)} |f(p') - f_n(p')|$$

$$= \omega_\nu^\lambda(f) + 2 \cdot \sup_{p \in I(\nu)} |f(p) - f_n(p)|.$$

As noted in the proof of the preceding theorem, there exists $f \in E$ such that

$$\lim_{n \to \infty} ||f_n - f||_E = 0.$$

This implies that

$$\lim_{n \to \infty} ||f_n - f||_C = 0,$$

and hence if a positive integer λ and a positive number ϵ be given, there exists M such that for each $n > M$ and each $p \in I$,

$$|f_n(p) - f(p)| < \epsilon/2\lambda.$$

Also since $f_n \in W_N$ for each n, we have

$$P_\lambda(f_n) \leq P(f_n) \leq N.$$

Hence if $n > M$ we have, recalling 13.2.1,

$$P_\lambda(f) = \sum_{\nu=1}^{\lambda^2} \omega_\nu^\lambda(f)/\lambda \leq \sum_{\nu=1}^{\lambda^2} \omega_\nu^\lambda(f_n)/\lambda + \sum_{\nu=1}^{\lambda^2} 2 \sup_{p \in I(\nu)} |f(p) - f_n(p)|/\lambda$$

$$< P_\lambda(f_n) + \sum_{\nu=1}^{\lambda^2} (2/\lambda)(\epsilon/2\lambda) = P_\lambda(f_n) + \epsilon \leq N + \epsilon.$$

It follows that

$$P(f) = \sup_\lambda P_\lambda(f) \leq N,$$

and hence that $f \in W_N$, completing the proof.

13.3. LEMMA. *For each* N, $\overline{E - W_N} = E$.

Proof. Let λ be a fixed positive integer. Let the λ-net be constructed and its cells denoted by $I(\nu)$ $(\nu = 1, 2, \cdots, \lambda^2)$. For each ν, let $C(\nu)$ be an open circular region entirely within $I(\nu)$, and in addition let these λ^2-circular regions be so constructed that no straight line shall have points in common with more than two of them. Let f be an arbitrary function of E, and let

$$\sigma_\lambda(f) = \sup_\nu \omega_\nu^\lambda(f).$$

Let g_λ be a function whose graph is as follows: Over each circle $C(\nu)$ it is a right circular cone of height $\sigma_\lambda(f) + \lambda^{-1/2}$; at all other points its value is zero. Let $f_\lambda(p) = f(p) + g_\lambda(p)$ for each $p \in I$; then

$$||f_\lambda||_E \leq ||f||_E + ||g_\lambda||_E \leq ||f||_E + 4(\sigma_\lambda(f) + \lambda^{-1/2}).$$

Clearly $f_\lambda \in E$. Also,

$$||f_\lambda - f||_E = ||g_\lambda||_E = 4(\sigma_\lambda(f) + \lambda^{-1/2}).$$

Therefore,

$$\lim_{\lambda \to \infty} ||f_\lambda - f||_E = 0.$$

For each ν,

$$\omega_\nu^\lambda(f_\lambda) \geq \omega_\nu^\lambda(g_\lambda) - \omega_\nu^\lambda(f) \geq \lambda^{-1/2}.$$

Hence

$$P_\lambda(f_\lambda) \geq \lambda^{1/2}.$$

Let N be any integer; then if $\lambda > N^2$, we have $P_\lambda(f_\lambda) > N$, and hence $f_\lambda \in E - W_N$.

It follows that each set $E - W_N$ is dense in E.

13.4. THEOREM. *$P \cap E$ is of first category in E.*

Proof. In view of 13.1,

$$P \cap E = \bigcup_{N=1}^{\infty} W_N.$$

By 13.2, for each N,

$$W_N = \overline{W_N}.$$

By 13.3,

$$\overline{E - W_N} = E.$$

It follows that $P \cap E$ is a union of closed sets which are nowhere dense in E, and our proof is complete.

REFERENCES

1. C. R. Adams and J. A. Clarkson, *Properties of functions $f(x,y)$ of bounded variation*, Trans. Amer. Math. Soc. 36 (1934), 711-730; 46 (1939), 468.

2. J. C. Burkill and V. S. Haslam-Jones, *Notes on the differentiability of functions of two variables*, J. London Math. Soc. 7 (1932), 297-305.

3. C. Carathéodory, *Vorlesungen über reelle Funktionen*, Leipzig, 1918.

4. J. A. Clarkson and C. R. Adams, *On definitions of bounded variation for functions of two variables*, Trans. Amer. Math. Soc. 35 (1933), 824-854.

5. H. Hahn, *Theorie der reellen Funkionen*, Berlin, 1921.

6. S. Saks, *On the surfaces without tangent planes*, Ann. Math. 34 (1933), 114-124.

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