GENERALIZATIONS OF THE ROGERS-RAMANUJAN IDENTITIES

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1. Introduction. The first of the two Rogers-Ramanujan identities [**1**, Chap. 19] states that

$$(1) \qquad \prod_{\nu=0}^{\infty} \frac{1}{(1-x^{5\nu+1})(1-x^{5\nu+4})} = \sum_{\mu=0}^{\infty} \frac{x^{\mu^2}}{(1-x)(1-x^2)\cdots(1-x^{\mu})},$$

where the left side is the generating function for the number of partitions into parts not congruent to 0, $\pm 2 \pmod 5$. This paper shows that as a generalization of (1) the generating function for the number of partitions into parts not congruent to 0, $\pm k \pmod{2k+1}$, where k is any positive integer, can be expressed as a sum similar to the one appearing in (1); in fact in general the x^{μ^2} are replaced by polynomials $G_{k,\mu}(x)$, so that we have the following theorem:

THEOREM 1. *The following identity holds*:

$$(2) \qquad \prod_{\nu=0}^{\infty} \frac{(1-x^{(2k+1)\nu+k})(1-x^{(2k+1)\nu+k+1})}{(1-x^{(2k+1)\nu+1})(1-x^{(2k+1)\nu+2})\cdots(1-x^{(2k+1)\nu+2k})}$$

$$= \sum_{\mu=0}^{\infty} \frac{G_{k,\mu}(x)}{(1-x)(1-x^2)\cdots(1-x^{\mu})},$$

where the left side is the generating function for the number of partitions into parts not congruent to 0, $\pm k \pmod{2k+1}$. *The* $G_{k,\mu}(x)$ *are polynomials in* x *and reduce to the monomial* x^{μ^2} *for* $k = 2$, *that is, for the Rogers-Ramanujan case.*

While the right side of (1) is the generating function for the number of

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partitions into parts differing by at least 2, no similar interpretation of the right hand of (2) is possible. In particular, it follows from a theorem of the author [2] that the right side of (2) cannot be interpreted as the generating function for the number of partitions of n into parts differing by at least d, each part being greater than or equal to m, unless $d = 2$, $m = 1$, that is, unless we have the Rogers-Ramanujan identity (1).

As a generalization of the second of the Rogers-Ramanujan identities:

$$(3) \quad \prod_{\nu=0}^{\infty} \frac{1}{(1-x^{5\nu+2})(1-x^{5\nu+3})} = \sum_{\mu=0}^{\infty} \frac{x^{\mu^2+\mu}}{(1-x)(1-x^2)\cdots(1-x^{\mu})},$$

we have again that not only the generating function for the number of partitions into parts not congruent to 0, $\pm 1 \pmod 5$, but in general the one for the number of partitions into parts not congruent to 0, $\pm 1 \pmod{2k+1}$ can be expressed as a sum; in fact again the x^{μ^2} are replaced by the same polynomials $G_{k,\mu}(x)$ appearing in (2), so that we have the following theorem:

THEOREM 2. *The following identity holds*:

$$(4) \quad \prod_{\nu=0}^{\infty} \frac{1}{(1-x^{(2k+1)\nu+2})(1-x^{(2k+1)\nu+3})\cdots(1-x^{(2k+1)\nu+2k-1})}$$

$$= \sum_{\mu=0}^{\infty} \frac{G_{k,\mu}(x)\, x^{\mu}}{(1-x)(1-x^2)\cdots(1-x^{\mu})}.$$

More generally, it can be shown that identities involving the generating function for the number of partitions into parts not congruent to 0, $\pm(k-r) \pmod{2k+1}$, where $0 \le r \le k-1$, can be obtained, of which (2) is the particular case where $r = k-1$, that is, for each modulus $2k+1$ there are k identities.

2. Proof of Theorem 1: If we replace, in Jacobi's identity,

$$(5) \quad \prod_{\nu=0}^{\infty} (1-y^{2\nu+2})\,[1+(z+z^{-1})y^{2\nu+1}+y^{4\nu+2}] = \sum_{\mu=-\infty}^{\infty} y^{\mu^2} z^{\mu},$$

y by $x^{(2k+1)/2}$ and z by $-x^{1/2}$, we have

$$(6)\qquad \prod_{\nu=0}^{\infty} (1 - x^{(2k+1)\nu+k})(1 - x^{(2k+1)\nu+k+1})(1 - x^{(2k+1)\nu+(2k+1)})$$

$$\sum_{\mu=-\infty}^{\infty} (-1)^{\mu} x^{((2k+1)\mu^2+\mu)/2} ,$$

so that, dividing both sides of (6) by $(1-x)(1-x^2)(1-x^3)\cdots$, we obtain

$$(7)\qquad \prod_{\nu=0}^{\infty} \frac{(1-x^{(2k+1)\nu+k})(1-x^{(2k+1)\nu+k+1})}{(1-x^{(2k+1)\nu+1})(1-x^{(2k+1)\nu+2})\cdots(1-x^{(2k+1)\nu+2k})}$$

$$= \frac{\sum_{\mu=-\infty}^{\infty} (-1)^{\mu} x^{((2k+1)\mu^2+\mu)/2}}{\prod_{s=1}^{\infty}(1-x^s)} .$$

To prove Theorem 1, we therefore have to show that the right side of (7) is the same as the right side of (2).

We use the auxiliary function

$$(8)\qquad C_{k,i}(y) = 1 - y^i x^i + \sum_{\mu=1}^{\infty} (-1)^{\mu} y^{k\mu} x^{(2k+1)(\mu^2+\mu)/2 - i\mu}$$

$$(1 - y^i x^{(2\mu+1)i}) \frac{(1-yx)(1-yx^2)\cdots(1-yx^{\mu})}{(1-x)(1-x^2)\cdots(1-x^{\mu})} ,$$

which was first used by Selberg [3] and is a generalization of the function used in some proofs of the Rogers-Ramanujan identities [**1**, Chap. 19]. The function (8) converges if $|y| < 1$ and if k is real and $> -1/2$. In our case k and i will be nonnegative integers. For $i = k$ and $y = 1$, (8) reduces to

$$(9)\qquad C_{k,k}(1) = \sum_{\mu=-\infty}^{\infty} (-1)^{\mu} x^{((2k+1)\mu^2+\mu)/2} .$$

Since the $C_{k,i}(y)$ satisfy the equation

$$C_{k,i}(y) = C_{k,i-1}(y) + y^{i-1}x^{i-1}(1-yx)C_{k,k-i+1}(yx),$$

it is easily seen that we can find a functional equation for the $C_{k,k}(y)$, which can be found to be of the form

$$(10) \qquad C_{k,k}(y) = \sum_{\mu=1}^{k} A_{k,\mu}(y,x)(1-yx^{\mu})C_{k,k}(yx^{\mu}).$$

If we let

$$(11) \qquad Q_k(y) = \frac{C_{k,k}(y)}{\prod_{s=1}^{\infty}(1-yx^s)},$$

(10) reduces to

$$(12) \qquad Q_k(y) = \sum_{\mu=1}^{k} A_{k,\mu}(y,x)Q_k(yx^{\mu}).$$

If, for instance, $k = 3$, (12) becomes

$$(13) \qquad Q_3(y) = (1+yx)Q_3(yx) + y^2x^2Q_3(yx^2) - y^3x^5Q_3(yx^3),$$

while for $k = 4$ we would have

$$(14) \qquad Q_4(y) = (1+yx)Q_4(yx) + y^2x^2(1+yx+yx^2)Q_4(yx^2) - y^4x^7Q_4(yx^3) - y^6x^{13}Q_4(yx^4).$$

In order to solve (12) for $Q_k(y)$ we try a solution of the form

$$(15) \qquad Q_k(y) = \sum_{\mu=0}^{\infty} B_{k,\mu}(x)y^{\mu},$$

where $B_{k,0}(x) = Q_k(0) = 1$ by use of (11) and (8).

Putting (15) into (12) we obtain a difference equation for the $B_{k,\mu}(x)$. It can easily be verified that the $B_{k,\mu}(x)$ are of the form

$$(16) \qquad B_{k,\mu}(x) = \frac{G_{k,\mu}(x)}{(1-x)(1-x^2)\cdots(1-x^{\mu})},$$

where the $G_{k,\mu}(x)$ are polynomials in x and reduce to the monomial x^{μ^2} for $k = 2$. In general these polynomials do not seem to possess any striking properties, even for small values of k and μ, as shall be illustrated below for $k = 3$ and $k = 4$.

Substituting now (16) into (15), and remembering (11), we obtain

$$(17) \quad Q_k(y) = \sum_{\mu=0}^{\infty} \frac{G_{k,\mu}(x) y^{\mu}}{(1-x)(1-x^2)\cdots(1-x^{\mu})} = \frac{C_{k,k}(y)}{\prod_{s=1}^{\infty}(1-yx^s)},$$

so that we have, in view of (9),

$$(18) \quad \frac{C_{k,k}(1)}{\prod_{s=1}^{\infty}(1-x^s)} = \frac{\sum_{\mu=-\infty}^{\infty}(-1)^{\mu} x^{((2k+1)\mu^2+\mu)/2}}{\prod_{s=1}^{\infty}(1-x^s)}$$

$$= \sum_{\mu=0}^{\infty} \frac{G_{k,\mu}(x)}{(1-x)(1-x^2)\cdots(1-x^{\mu})},$$

which completes the proof of the theorem.

In case $k = 3$, the difference equation for the $B_{3,\mu}(x)$, which can easily be obtained from (13), is the following:

$$(19) \quad B_{3,\mu}(x)(1-x^{\mu}) = B_{3,\mu-1}(x)x^{\mu} + B_{3,\mu-2}(x)x^{2\mu-2} - B_{3,\mu-3}(x)x^{3\mu-4},$$

from which we calculate, remembering that $B_{3,0}(x) = 1$:

$G_{3,1}(x) = x,$

$G_{3,2}(x) = x^2,$

$G_{3,3}(x) = x^5 + x^6 - x^8,$

$G_{3,4}(x) = x^8 + x^{10} - x^{14},$

$G_{3,5}(x) = x^{13} + x^{14} + x^{15} - x^{18} - x^{19},$

$G_{3,6}(x) = x^{18} + x^{20} + x^{21} + x^{22} - x^{25} - x^{26} - x^{27} - x^{28} + x^{31},$

$G_{3,7}(x) = x^{25} + x^{26} + x^{27} + x^{28} + x^{29} - x^{32} - x^{33} - x^{34} - x^{35} - x^{36} + x^{42},$

and so on.

It can easily be verified by induction that the degree of the $G_{3,\mu}(x)$ is equal to

$$\frac{5\mu^2+\mu}{6} \quad \text{if } \mu \equiv 0 \text{ or } 1 \pmod 3,$$

and is less than or equal to

$$\frac{5\mu^2-\mu-6}{6} \quad \text{if } \mu \equiv 2 \pmod 3.$$

Similarly, it can be shown that the term with smallest exponent in each polynomial $G_{3,\mu}(x)$ is $x^{[(\mu^2+1)/2]}$, so that each polynomial has this power of x as a divisor and no higher power.

For $k = 4$, we obtain the difference equation for the $B_{4,\mu}(x)$ from (14):

$$(20) \quad B_{4,\mu}(x)(1-x^{\mu}) = B_{4,\mu-1}(x)x^{\mu} + B_{4,\mu-2}(x)x^{2\mu-2}$$
$$+ B_{4,\mu-3}(x)x^{2\mu-3}(x+1) - B_{4,\mu-4}(x)x^{3\mu-5} - B_{4,\mu-6}(x)x^{4\mu-11},$$

so that we obtain:

$G_{4,0}(x) = 1,$

$G_{4,1}(x) = x,$

$G_{4,2}(x) = x^2,$

$G_{4,3}(x) = x^3,$

$G_{4,4}(x) = x^6 + x^7 + x^8 - x^9 - x^{10} - x^{11} + x^{13},$

$G_{4,5}(x) = x^9 + x^{10} + x^{11} - x^{14} - x^{15} - x^{16} + x^{20},$

$G_{4,6}(x) = x^{12} + x^{14} + x^{15} + x^{16} - x^{19} - 2x^{20} - x^{21} - x^{22} + x^{25} + x^{26},$

$G_{4,7}(x) = x^{17} + x^{18} + 2x^{19} + x^{20} + x^{21} - x^{23} - 2x^{24} - 2x^{25} - 2x^{26} - x^{27} + x^{30}$
$+ x^{31} + x^{32},$

and so on.

In this case the term with smallest exponent can be shown to equal $x^{[(\mu^2+2)/3]}$, while for $G_{5,\mu}(x)$ we would find the corresponding term to be $x^{[(\mu^2+3)/4]}$ for $\mu > 2$, and so on.

3. Proof of Theorem 2. From the definition of $C_{k,i}(y)$ we find

$$(21) \qquad (1-x)C_{k,k}(x) = \sum_{\mu=-\infty}^{\infty} (-1)^{\mu} x^{((2k+1)\mu^2+(2k-1)\mu)/2}.$$

Substituting now, in Jacobi's identity (5), $x^{(2k+1)/2}$ for y and $-x^{(2k-1)/2}$ for z, and dividing at the same time both sides by $(1-x)(1-x^2)(1-x^3)\cdots$, we obtain

$$(22) \qquad \prod_{\nu=0}^{\infty} \frac{1}{(1-x^{(2k+1)\nu+2})(1-x^{(2k+1)\nu+3})\cdots(1-x^{(2k+1)\nu+2k-1})}$$

$$= \frac{\sum_{\mu=-\infty}^{\infty}(-1)^{\mu}x^{((2k+1)\mu^2+(2k-1)\mu)/2}}{\prod_{s=1}^{\infty}(1-x^s)}$$

$$= \frac{(1-x)C_{k,k}(x)}{\prod_{s=1}^{\infty}(1-x^s)} = Q_k(x) = \sum_{\mu=0}^{\infty} \frac{G_{k,\mu}(x)x^{\mu}}{(1-x)(1-x^2)\cdots(1-x^{\mu})},$$

if we recall (11), (15), and (16).

Identities involving the generating function for the number of partitions into parts not congruent to 0, $\pm(k-r) \pmod{2k+1}$, where $0 \le r \le k-1$, can be obtained by noting that, using Jacobi's identity with $y = x^{(2k+1)/2}$ and $z = -x^{(2r+1)/2}$, we obtain

$$\prod_{\nu=0}^{\infty} \left[(1-x^{(2k+1)\nu+k-r})(1-x^{(2k+1)\nu+k+r+1})(1-x^{(2k+1)\nu+(2k+1)})\right]$$

$$= \sum_{\mu=-\infty}^{\infty} (-1)^{\mu} x^{((2k+1)\mu^2+(2r+1)\mu)/2},$$

where the right side, as can be verified, is expressible in terms of $C_{k,k}(y)$, which was shown already for $r = 0$ by Theorem 1 and for $r = k - 1$ by Theorem 2 and shall only be indicated here for $r = 1$, where we find

$$(23) \quad C_{k,k}(1) - x^{k-1}(1-x)(1-x^2)C_{k,k}(x^2) = \sum_{\mu=-\infty}^{\infty} (-1)^{\mu} x^{((2k+1)\mu^2+3\mu)/2}.$$

This method therefore allows us to find for each modulus $2k + 1$ exactly k identities, that is, one for each value of r in $0 \leq r \leq k - 1$.

REFERENCES

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