TRANSFORMATIONS OF SERIES OF THE TYPE ${}_3\Psi_3$

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1. **Sears** [3] has given relations between series of the type ${}_3\Phi_2$. Generalizations of some of these results are included in, or may be obtained from, the following two formulae established by Slater [4]:

$$\prod_{r=0}^{\infty} \frac{(1-x\xi q^r)(1-q^{r+1}/x\xi)(1-b_1q^r)\cdots(1-b_Mq^r)}{(1-a_1q^r)\cdots(1-a_Mq^r)}$$

$$\times \frac{(1-q^{r+1}/a_{M+2})\cdots(1-q^{r+1}/a_{2M+1})}{(1-q^{r+1}/a_1)\cdots(1-q^{r+1}/a_M)} \quad {}_M\Psi_M\left[\begin{matrix} a_{M+2},\cdots,a_{2M+1}\ ;\ x \\ b_1,\cdots,b_M \end{matrix}\right]$$

$$= q/a_1 \prod_{r=0}^{\infty}\left[\frac{(1-a_1x\xi q^{r-1})(1-q^{r+2}/a_1x\xi)(1-b_1q^{r+1}/a_1)\cdots}{(1-a_1q^r)(1-q^{r+1}/a_1)(1-a_1q^r/a_2)\cdots}\right.$$

$$\tag{1.1} \left.\times\cdots\frac{(1-b_Mq^{r+1}/a_1)(1-a_1q^r/a_{M+2})\cdots(1-a_1q^r/a_{2M+1})}{(1-a_1q^r/a_M)(1-a_2q^{r+1}/a_1)\cdots(1-a_Mq^{r+1}/a_1)}\right]$$

$$\times\ {}_M\Psi_M\left[\begin{matrix} qa_{M+2}/a_1,\cdots,qa_{2M+1}/a_1\ ;\ x \\ qb_1/a_1,\cdots,qb_M/a_1 \end{matrix}\right]$$

$+ (M-1)$ similar terms obtained by interchanging a_1 with $a_2, a_3, \cdots, a_M$,

$$= q/a_1 \prod_{r=0}^{\infty}\left[\frac{(1-a_1x\xi q^{r-1})(1-q^{r+2}/a_1x\xi)(1-b_1q^{r+1}/a_1)\cdots}{(1-a_1q^r)(1-q^{r+1}/a_1)(1-a_1q^r/a_2)\cdots}\right.$$

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$$(1.2)\qquad \times \frac{(1-b_Mq^{r+1}/a_1)(1-a_1q^r/a_{M+2})\cdots(1-a_1q^r/a_{2M+1})}{(1-a_1q^r/a_M)(1-a_2q^{r+1}/a_1)\cdots(1-a_Mq^{r+1}/a_1)}\Bigg]$$

$$\times {}_M\Psi_M\left[\begin{matrix} a_1/b_1, \cdots, a_1/b_M; \\ a_1/a_{M+2}, \cdots, a_1/a_{2M+1} \end{matrix} \quad \frac{b_1\cdots b_M}{xa_{M+2}\cdots a_{2M+1}}\right]$$

$$+ (M-1) \text{ similar terms obtained as in } (1.1),$$

where

$$M \geq 1, \ \xi = \frac{a_{M+2}\cdots a_{2M+1}}{a_1\cdots a_M}, \ |x| < 1, \text{ and } |q| < 1.$$

In particular we see that (1.2), with $M = 3$, is a generalization of the basic analogue of the fundamental three-term relation [3, § 10, result IV a] for ${}_3F_2$ to which it reduces if we take $a_1 = aq$, $a_2 = bq$, $a_3 = cq$, $a_5 = a$, $a_6 = b$, $a_7 = c$, $b_1 = q$, $b_2 = e$, $b_3 = f$, and $x = ef/abc$. Similarly, (1.1) and (1.2) may be used to obtain many more of the relations given by Sears. It will be noted, however, that the parameters occurring in the Ψ series in (1.1) and (1.2) are related in a very symmetrical way, and consequently these formulae can only be expected to provide generalizations of the two-, three-, and four-term relations between ${}_3\Phi_2$ which are of a symmetrical nature; in particular, they do not provide a generalization of the basic analogue of the fundamental two-term relation [3, § 10, 1]. In this paper, one such generalization is obtained which, when used in conjunction with (1.1), will yield generalizations of all Sears' formulae and provide basic analogues of known transformations [2] of ${}_3H_3$.

2. **To obtain** the required generalization, we establish the basic analogue of the formula [2, § 2.1] which was used to obtain the generalization of the fundamental two-term relation between ${}_3F_2$. The method by which this result can be obtained has been indicated by Bailey [1], who obtained a particular case of the following formula (2.1). We use the fact that a basic bilateral series ${}_8\Psi_8$ which terminates below can be expressed in terms of an ${}_8\Phi_7$, which can in turn be transformed into two series ${}_4\Phi_3$, one of which can be replaced by a ${}_4\Psi_4$ which terminates below. Then, proceeding to the limit, we obtain a transformation which can be restated in the form (2.1). The analysis is straightforward, though rather lengthy, so we just state the result:

$$(2.1)\quad \sum_{n=-\infty}^{\infty}\left[\frac{(q\sqrt{a})_n(-q\sqrt{a})_n(b)_n(c)_n(d)_n}{(\sqrt{a})_n(-\sqrt{a})_n(aq/b)_n(aq/c)_n(aq/d)_n}\times\frac{(e)_n(f)_n(-1)^n q^{n^2/2+n}}{(aq/e)_n(aq/f)_n}\left(\frac{a^3}{bcdef}\right)^n\right]$$

$$=\prod_{r=0}^{\infty}\frac{(1-aq^{r+1})(1-q^{r+1}/a)(1-aq^{r+1}/bc)}{(1-q^{r+1}/d)(1-q^{r+1}/e)(1-q^{r+1}/f)(1-aq^{r+1}/b)(1-aq^{r+1}/c)}$$

$$\times\left\{\prod_{r=0}^{\infty}\frac{(1-aq^{r+1}/de)(1-aq^{r+1}/ef)(1-aq^{r+1}/df)}{(1-a^2q^{r+1}/def)}\right.$$

$$\times\ {}_3\Psi_3\left[\begin{matrix}b,\ c,\ a^2q/def; & \dfrac{aq}{bc}\\ aq/d, aq/e, aq/f & \end{matrix}\right]$$

$$+\prod_{r=0}^{\infty}\frac{(1-dq^r/a)(1-eq^r/a)(1-fq^r/a)}{(1-q^{r+1}/b)(1-q^{r+1}/c)}$$

$$\times\frac{(1-a^2q^{r+2}/bdef)(1-a^2q^{r+2}/cdef)(1-q^{r+1})}{(1-a^2q^{r+2}/def)(1-defq^{r-1}/a^2)}$$

$$\left.\times\ {}_3\Phi_2\left[\begin{matrix}aq/ef,\ aq/df,\ aq/de\ ;\ q\\ a^2q^2/bdef,\ a^2q^2/cdef\end{matrix}\right]\right\}.$$

We obtain a generalization of the basic analogue of the fundamental two-term relation by interchanging both b and d and c and e in (2.1), then replacing a by def/aq^2, d by ef/aq, e by df/aq, f by de/aq, leaving b and c unaltered, and replacing $def/abcq$ by σ, we obtain:

$$\prod_{r=0}^{\infty}\frac{(1-\sigma q^r)}{(1-aq^{r+2}/ef)\,(1-aq^{r+2}/df)(1-\sigma cq^r)(1-\sigma bq^r)}$$

$$\times \left\{ \prod_{r=0}^{\infty} \frac{(1-aq^{r+1}/d)(1-aq^{r+1}/e)(1-aq^{r+1}/f)}{(1-aq^r)} \; {}_3\Psi_3\left[\begin{matrix} a,\ b,\ c\ ; & \dfrac{def}{abcq} \\ d,\ e,\ f & \end{matrix}\right] \right.$$

$$+ \prod_{r=0}^{\infty} \frac{(1-q^{r+1}/d)(1-q^{r+1}/e)(1-q^{r+1}/f)}{(1-q^{r+1}/b)(1-q^{r+1}/c)}$$

$$\left. \times \frac{(1-aq^r/b)(1-aq^r/c)(1-q^{r+1})}{(1-aq^{r+1})(1-q^r/a)} \; {}_3\Phi_2\left[\begin{matrix} aq/d,\ aq/e,\ aq/f;\ q \\ aq/b,\ aq/c \end{matrix}\right] \right\}$$

$$(2.2) \qquad = \prod_{r=0}^{\infty} \frac{(1-aq^{r+1}/f)}{(1-q^{r+1}/b)(1-q^{r+1}/c)(1-dq^r)(1-eq^r)}$$

$$\times \left\{ \prod_{r=0}^{\infty} \frac{(1-\sigma q^r)(1-fq^r/b)(1-fq^r/c)}{(1-f\sigma q^{r-1})} \; {}_3\Psi_3\left[\begin{matrix} ef/aq,\ df/aq,\ f/q\ ; & \dfrac{aq}{f} \\ b,\quad c,\quad f & \end{matrix}\right] \right.$$

$$+ \prod_{r=0}^{\infty} \frac{(1-q^{r+1}/c\sigma)(1-q^{r+1}/b\sigma)(1-q^{r+1})}{(1-aq^{r+2}/ef)(1-aq^{r+2}/df)}$$

$$\left. \times \frac{(1-q^{r+1}/f)(1-dfq^r/bc)(1-efq^r/bc)}{(1-\sigma fq^r)(1-q^{r+1}/f\sigma)} \; {}_3\Phi_2\left[\begin{matrix} f/c,\ f/b,\ \sigma\ ;\ q \\ df/bc,\ ef/bc \end{matrix}\right] \right\}.$$

The two ${}_3\Phi_2$ which occur in this formula are not connected by a two-term relation, and it would appear therefore that (2.2) is probably the simplest generalization of the fundamental two-term relation for ${}_3\Phi_2$ to which it reduces when $f = q$. This is the only relation between ${}_3\Phi_2$ which can be obtained from (2.2).

There are some relations involving ${}_3\Psi_3$, which generalize more than one ${}_3\Phi_2$ transformation. Such a formula can be obtained from (2.1) by interchanging the parameters b and d, then replacing a by def/aq^2, d by ef/aq, e by df/aq, f by de/aq, but leaving b and c unaltered:

$${}_3\Psi_3\left[\begin{matrix} a,\ b,\ c\ ; & \dfrac{def}{abcq} \\ d,\ e,\ f & \end{matrix}\right]$$

$$(2.3) \qquad = \prod_{r=0}^{\infty} \frac{(1-aq^r)(1-aq^{r+2}/ef)(1-\sigma c q^r)}{(1-q^{r+1}/b)(1-dq^r)(1-\sigma q^r)}$$

$$\times \frac{(1-dq^r/c)(1-eq^r/b)(1-fq^r/b)}{(1-aq^{r+1}/e)(1-aq^{r+1}/f)(1-efq^{r-1}/b)} \; {}_3\Psi_3\left[\begin{matrix} c,\ ef/aq,\ ef/bq\ ; & \frac{d}{c} \\ \sigma c,\ e,\ f & \end{matrix}\right]$$

$$+ \prod_{r=0}^{\infty} \frac{(1-q^{r+1}/e)(1-q^{r+1}/f)(1-aq^{r+1}/b)}{(1-aq^{r+1}/d)(1-q^{r+1}/b)(1-q^{r+1}/c)}$$

$$\times \frac{(1-q^{r+1})(1-aq^r)(1-c\sigma q^r)}{(1-aq^{r+1}/e)(1-aq^{r+1}/f)(1-\sigma q^r)}$$

$$\times \left\{ \prod_{r=0}^{\infty} \frac{(1-q^{r+1}/c\sigma)(1-efq^r/bc)(1-dq^r/c)}{(1-efq^r/b)(1-bq^{r+1}/ef)(1-dq^r)} \; {}_3\Phi_2\left[\begin{matrix} aq/d,\ f/b,\ e/b\ ;\ q \\ aq/b,\ ef/bc \end{matrix}\right] \right.$$

$$\left. - \prod_{r=0}^{\infty} \frac{(1-\sigma q^r)(1-q^{r+1}/d)(1-aq^{r+1}/c)}{(1-c\sigma q^r)(1-aq^{r+1})(1-q^r/a)} \; {}_3\Phi_2\left[\begin{matrix} aq/d,\ aq/e,\ aq/f\ ;\ q \\ aq/b,\ aq/c \end{matrix}\right] \right\}.$$

If e (or f) $= q$, (2.3) reduces to a two-term relation; but it reduces to a four-term relation between $_3\Phi_2$ when $c = 1$. This particular result is not stated explicitly by Sears but can be deduced from his results.

It will be seen that the $_3\Psi_3$ transformations are more complicated than the analogous $_3H_3$ transformations. For this reason, no more such results are given, but they can all be obtained from (1.1) and (2.2).

3. Corrigenda. In (2.3) and (2.4) of [2], the terms $\Gamma(1+b-\sigma)$, $\Gamma(1+c-\sigma)$ should be $\Gamma(1-b-\sigma)$, $\Gamma(1-c-\sigma)$, in (5.1) the factor $\Gamma(d-c)$ on the left should be in the denominator of the first term on the right, and there should be a factor $\Gamma(d)$ in the denominator on the left.

References

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