THE ADJOINT SEMI-GROUP

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Introduction. The purpose of this paper is to develop a general theory for the adjoint semi-group of operators which fits into the framework of the present theory of semi-groups. To each semi-group of linear bounded operators $[T(s)]$ defined on a Banach space $\mathfrak{X}$ to itself and possessing suitable continuity properties, we shall assign an adjoint semi-group with like continuity properties, defined on an "adjoint" Banach space $\mathfrak{X}^+$ which is in general a proper subspace of the adjoint space $\mathfrak{X}^*$. The usefulness of the adjoint semi-group has already been demonstrated by W. Feller [3] in his treatise on the parabolic differential equation.[1]

In our theory of the adjoint semi-group, the choice of the subspace $\mathfrak{X}^+ \subset \mathfrak{X}^*$ is decisive. We have been led to $\mathfrak{X}^+$ by two independent considerations. In the first place $\mathfrak{X}^+$ is the largest domain over which the ordinary adjoint $T^*(s)$ has suitable continuity properties. It should be noted, however, that a rather extensive theory of semi-groups has been developed by W. Feller [4] which has no such continuity requirements. The more compelling reason for our choice of $\mathfrak{X}^+$ has to do with the infinitesimal generator. In most applications of the theory of semi-groups one starts with an infinitesimal generator A and it is desired to establish the existence of a semi-group of operators generated by A. It is natural to expect the behavior of the semi-group operators $T(s)$ to be uniquely determined on the domain of A (in symbols $\mathfrak{D}(A)$); and since $T(s)$ is required to be bounded, there will exist a unique extension to the smallest closed subspace containing $\mathfrak{D}(A)$, namely $\overline{\mathfrak{D}(A)}$. *Further extensions are not uniquely determined by A and should not be associated with the operator A.* A reasonable approach to the adjoint semi-group would be to require that its infinitesimal generator be the adjoint A^* of the infinitesimal generator A of the original semi-group. In accordance with the above remarks, the proper domain for the adjoint semi-group

[1] It is remarkable that Feller actually obtained the entire adjoint semi-group without employing a precise notion for the adjoint to an unbounded operator such as the infinitesimal generator. For without this, the general formulation loses much of its significance.

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would then be $\overline{\mathfrak{D}(A^*)}$. Now $\mathfrak{X}^+$ is precisely $\overline{\mathfrak{D}(A^*)}$; however the infinitesimal generator A^+ of the adjoint semi-group turns out to be the maximal restriction of A^* with domain and range in $\overline{\mathfrak{D}(A^*)} = \mathfrak{X}^+$.

As in the ordinary theory of adjoint spaces, it is possible to develop an entire hierarchy of "adjoint" spaces for a given semi-group of operators.[2] However it can happen that the second "adjoint" is equal to the original space (under the natural mapping); in this case nothing new is achieved by going beyond the first "adjoint." This situation occurs not only when $\mathfrak{X}$ is reflexive in the usual sense but, more generally, when the resolvent of A is weakly compact (as in the case of most nonsingular problems of mathematical physics).

1. The adjoint transformation. We take $\mathfrak{X}$ and $\mathfrak{Y}$ to be Banach spaces over the real (or complex) scaler field. The transformation $y = T(x)$ is taken to be linear with domain $\mathfrak{D} \subset \mathfrak{X}$ and range $\mathfrak{R} \subset \mathfrak{Y}$, and it is assumed that $\mathfrak{D}$ is a linear subspace of $\mathfrak{X}$.

DEFINITION 1. Let $y = T(x)$ be defined on a domain $\mathfrak{D}$ dense in $\mathfrak{X}$ to $\mathfrak{Y}$, and let $\mathfrak{X}^*$ and $\mathfrak{Y}^*$ be the adjoint spaces to $\mathfrak{X}$ and $\mathfrak{Y}$ respectively. The *adjoint transformation* T^* of T is defined as follows: Its domain $\mathfrak{D}(T^*)$ consists of the set of all $y^* \in \mathfrak{Y}^*$ for which there exists an $x^* \in \mathfrak{X}^*$ such that $y^*[T(x)] = x^*(x)$ for all $x \in \mathfrak{D}$; for such a y^* we define $T^*(y^*) = x^*$.

It is clear that the density of $\mathfrak{D}$ in $\mathfrak{X}$ is required in order that T^* be single-valued. Further it is easy to show that T^* is a closed linear transformation on $\mathfrak{D}(T^*)$ to $\mathfrak{X}^*$. On the other hand the second adjoint is not always well defined since $\mathfrak{D}(T^*)$ is in general not dense in $\mathfrak{Y}^*$. In this connection we have:

THEOREM 1.1. *If T is a closed linear transformation with domain $\mathfrak{D}$ dense in $\mathfrak{X}$, then $\mathfrak{D}(T^*)$ is weakly* dense in $\mathfrak{Y}^*$. In particular, if $\mathfrak{Y}$ is reflexive then $\mathfrak{D}(T^*)$ is strongly dense in $\mathfrak{Y}^*$.*

Proof. If $\mathfrak{D}(T^*)$ were not weakly* dense in $\mathfrak{Y}^*$, then the weak* closure of $\mathfrak{D}(T^*)$ would be regularly closed [**1**] so that there would exist a $y_0 \in \mathfrak{Y}$, $y_0 \neq 0$, such that $y^*(y_0) = 0$ for all $y^* \in \mathfrak{D}(T^*)$. Now $(0, y_0)$ does not belong to the graph $\mathfrak{G}$ of T, and $\mathfrak{G}$ is a closed linear subspace of $\mathfrak{X} \oplus \mathfrak{Y}$. Hence by a theorem

[2] For example if $X = C_0(-\infty,\infty)$, the space of continuous functions $f(\xi)$ on $(-\infty,\infty)$ such that $\lim_{|\xi|\to 0} f(\xi) = 0$ and $\|f\| = \sup|f(\xi)|$, and if $A(f) = f'$, $D(A) = [f;\ f$ continuously differentiable, f and $f' \in C_0]$, then $X^+ = L_1(-\infty,\infty)$, $(X^+)^+ =$ space of all functions $f(\xi)$ uniformly continuous and bounded on $(-\infty,\infty)$ with $\|f\| = \sup|f(\xi)|$, and so on.

due to H. Hahn [5, Theorem 2.9.4], there exists an

$$(x_0^*, y_0^*) \in (\mathfrak{X} \oplus \mathfrak{Y})^* = \mathfrak{X}^* \oplus \mathfrak{Y}^*$$

such that

$$x_0^*(x) + y_0^*[T(x)] = 0 \quad \text{for all } x \in \mathfrak{D} \text{ and } x_0^*(0) + y_0^*(y_0) \neq 0.$$

It follows that

$$y_0^* \in \mathfrak{D}(T^*), \quad T^*(y_0^*) = -x_0^*, \text{ and yet } y_0^*(y_0) \neq 0,$$

which is impossible. In case $\mathfrak{Y}$ is reflexive we conclude that $\mathfrak{D}(T^*)$ is weakly dense and hence strongly dense in $\mathfrak{Y}^*$ (the latter conclusion follows from the above-mentioned Hahn theorem).

We turn now to the relation between a transformation, its adjoint, and their inverses.

THEOREM 1.2. *Let T be a linear transformation with $\overline{\mathfrak{D}} = \mathfrak{X}$. Then $(T^*)^{-1}$ exists if and only if $\overline{\mathfrak{R}} = \mathfrak{Y}$. More generally, $\overline{\mathfrak{R}}$ consists of the set of all points y such that $T^*(y^*) = 0$ implies $y^*(y) = 0$.*

Proof. If $T^*(y_0^*) = 0$, then

$$[T^*(y_0^*)](x) = y_0^*[T(x)] = 0$$

for all $x \in \mathfrak{D}$, and hence $y_0^*(\overline{\mathfrak{R}}) = 0$. In particular, $\overline{\mathfrak{R}} = \mathfrak{Y}$ implies that $y_0^* = 0$, and hence that T^* has an inverse. On the other hand if $y_0 \notin \overline{\mathfrak{R}}$, then by the Hahn theorem there exists a functional $y_0^* \in \mathfrak{Y}^*$ such that $y_0^*(y_0) = 1$ and $y_0^*(\overline{\mathfrak{R}}) = 0$. Thus $y_0^*[T(x)] = 0$ for all $x \in \mathfrak{D}$; it follows that $y_0^* \in \mathfrak{D}(T^*)$ and $T^*(y_0^*) = 0$; whereas $y_0^*(y_0) \neq 0$. In particular we see that if $\overline{\mathfrak{R}} \neq \mathfrak{Y}$, then T^* cannot have an inverse.

THEOREM 1.3. *Let T be a linear transformation with $\overline{\mathfrak{D}} = \mathfrak{X}$.* If $\mathfrak{R}(T^*)$ *is weakly* dense in $\mathfrak{X}^*$, then T has an inverse.*

Proof. Suppose that T has no inverse; then there is an $x_0 \neq 0$ such that $T(x_0) = 0$. Consequently

$$[T^*(y^*)](x_0) = y^*[T(x_0)] = 0$$

for all $y^* \in \mathfrak{D}(T^*)$, and this shows that the weak* closure of $\mathfrak{R}(T^*)$ is a proper subspace of $\mathfrak{X}^*$, contrary to assumption.

THEOREM 1.4. *Let T be a linear transformation with an inverse and such that $\overline{\mathfrak{D}} = \mathfrak{X}$ and $\overline{\mathfrak{R}} = \mathfrak{Y}$. Then $(T^*)^{-1} = (T^{-1})^*$; further T^{-1} is bounded if and only if $(T^*)^{-1}$ is bounded on $\mathfrak{X}^*$.*

Proof. In the first place $(T^{-1})^*$ exists because $\mathfrak{R} = \mathfrak{D}(T^{-1})$ is dense in $\mathfrak{Y}$, and $(T^*)^{-1}$ exists by Theorem 1.2. If $y \in \mathfrak{R}$ and $y^* \in \mathfrak{D}(T^*)$, then

$$y^*(y) = y^*\{T[T^{-1}(y)]\} = [T^*(y^*)][T^{-1}(y)].$$

This implies that $\mathfrak{R}(T^*) \subset \mathfrak{D}[(T^{-1})^*]$ and

$$(T^{-1})^*[T^*(y^*)] = y^*$$

for all $y^* \in \mathfrak{D}(T^*)$. Thus $(T^{-1})^*$ is an extension of $(T^*)^{-1}$. On the other hand if $x \in \mathfrak{D}$, then

$$x^*(x) = x^*\{T^{-1}[T(x)]\} = [(T^{-1})^*(x^*)][T(x)],$$

for all $x^* \in \mathfrak{D}[(T^{-1})^*]$. It follows that $\mathfrak{R}(T^*) \supset \mathfrak{D}[(T^{-1})^*]$. Therefore

$$\mathfrak{D}[(T^{-1})^*] = \mathfrak{R}(T^*) = \mathfrak{D}[(T^*)^{-1}],$$

and hence $(T^{-1})^* = (T^*)^{-1}$. If, in addition, T^{-1} is bounded, then it is clear that $(T^{-1})^*$ is also bounded. Conversely if $(T^*)^{-1}$ is bounded on $\mathfrak{X}^*$, then for all $x \in \mathfrak{R}$ and $x^* \in \mathfrak{X}^*$ we have

$$|x^*[T^{-1}(x)]| = |[(T^{-1})^*(x^*)](x)| \leq ||(T^*)^{-1}||\,||x^*||\,||x||.$$

It follows that T^{-1} is bounded.

If T is a linear operator with both domain and range in $\mathfrak{X}$, $\overline{\mathfrak{D}} = \mathfrak{X}$, then the adjoint transformation T^* has its domain and range in $\mathfrak{X}^*$. It is easy to show for an arbitrary bounded operator B on $\mathfrak{X}$ to itself, that

$$(B + T)^* = B^* + T^* \quad \text{and} \quad \mathfrak{D}[(B + T)^*] = \mathfrak{D}(T^*).$$

We are especially interested in the combination $\lambda I - T$, where I is the identity operator and λ is a real (or complex) number. If $\lambda I - T$ has a bounded inverse with domain dense in $\mathfrak{X}$, then λ is said to belong to $\rho(T)$, the resolvent set of T, and

$$(\lambda I - T)^{-1} \equiv R(\lambda; T)$$

is called the resolvent of T.

THEOREM 1.5. *If T is a linear operator with $\overline{\mathfrak{D}} = \mathfrak{X}$ and $\mathfrak{R} \subset \mathfrak{X}$, then*

$$\rho(T) = \rho(T^*) \text{ and } [R(\lambda; T)]^* = R(\lambda; T^*).$$

Proof. If $\lambda \in \rho(T)$, then, according to Theorem 1.4, $\lambda \in \rho(T^*)$ and

$$[R(\lambda; T)]^* = R(\lambda; T^*).$$

On the other hand if $\lambda \in \rho(T^*)$, then Theorem 1.3 shows that T has an inverse, Theorem 1.2 shows that $\overline{\mathfrak{R}} = \mathfrak{X}$, and Theorem 1.4 then implies that $\lambda \in \rho(T)$.

2. The adjoint semi-group. We now apply the previous results to semi-groups of linear bounded operators (cf. [5]). Let $\mathfrak{E}(\mathfrak{X})$ be the Banach algebra of endomorphism of $\mathfrak{X}$, and let $[T(s)]$ be a one-parameter family of operators in $\mathfrak{E}(\mathfrak{X})$ defined for $s \in [0, \infty)$ and satisfying:

(i) $T(s_1 + s_2) = T(s_1)T(s_2)$ for all $s_1, s_2 \geq 0$, $T(0) = I$;

(ii) for each $x \in \mathfrak{X}$, $T(s)x$ is continuous for $s > 0$;

(iii) $\int_0^1 ||T(\sigma)x|| d\sigma < \infty$ for each $x \in \mathfrak{X}$.

If T satisfies the additional condition

(iv) $\lim_{\lambda \to \infty} \lambda \int_0^\infty \exp(-\lambda\sigma) T(\sigma) x d\sigma = x$ for each $x \in \mathfrak{X}$,

then $T(s)$ is said to be *of class* $(0, A)$. If, instead of (iv), $T(s)$ satisfies the stronger condition

(v) $\lim_{\tau \to 0} \tau^{-1} \int_0^\tau T(\sigma) x d\sigma = x$ for each $x \in \mathfrak{X}$,

then $T(s)$ is said to be *of class* $(0, C)$. Finally if $T(s)$ satisfies (i), (ii), (iii), and the still stronger continuity condition

(vi) $\lim_{s \to 0} T(s)x = x$ for each $x \in \mathfrak{X}$,

then $T(s)$ is said to be *of class* C.

The domain $\mathfrak{D}(A)$ of the infinitesimal generator A is the set of elements x for which

$$\lim_{\tau \to 0} \tau^{-1}[T(\tau) - I]x$$

exists, and this limit is defined to be Ax. It follows from (iv) (and hence (v) or (vi)) that $\mathfrak{D}(A)$ is dense in $\mathfrak{X}$ (cf. [5, Theorem 9.3.1]). We have previously shown [6] that A is closed if and only if $T(s)$ is of class $(0, C)$. However, even when $T(s)$ is of class $(0, A)$, the infinitesimal generator has a smallest closed extension, called the complete infinitesimal generator (c.i.g.) and denoted by $\overline{A}$. For each $x_0 \in \mathfrak{D}(\overline{A})$ there is a sequence $\{x_n\} \subset \mathfrak{D}(A)$ such that $x_n \longrightarrow x_0$ and $Ax_n \longrightarrow \overline{A}x_0$. It follows that $R(\lambda; \overline{A})$ is an extension of $R(\lambda; A)$, that $\rho(A) = \rho(\overline{A})$, that $A^* = (\overline{A})^*$, and that

$$[R(\lambda; A)]^* = [R(\lambda; \overline{A})]^*. \text{ [3]}$$

It can be shown that

$$\omega_0 = \inf_{s > 0} \log \|T(s)\|/s = \lim_{s \to \infty} \log \|T(s)\|/s. \tag{2.1}$$

Each $\lambda > \omega_0$ belongs to the resolvent set for $\overline{A}$, and the resolvent is given by

$$R(\lambda; \overline{A})x = \int_0^\infty \exp(-\lambda\sigma) T(\sigma) x d\sigma; \tag{2.2}$$

see [6].

DEFINITION 2.1. The semi-group $T(s)$ is said to be *of class* $(0, A)^*$, $(0, C)^*$, or C^* if it is of class $(0, A)$, $(0, C)$, or C, respectively, and if in addition $\|T^*(s)x^*\|$, $0 \leq s \leq 1$, is majorized by integrable function for each $x^* \in \mathfrak{X}^*$.[4]

DEFINITION 2.2. Let $T(s)$ be a semi-group of class $(0, A)$ with infinitesimal generator A. We define the *adjoint semi-group* to be the restriction of $T^*(s)$ to $\mathfrak{X}^+ = \overline{\mathfrak{D}(A^*)}$ and denote it by $T^+(s)$. We denote the infinitesimal generator of $T^+(s)$ by A^+.

[3] For $\lambda \in \rho(A)$, the resolvent $R(\lambda; A)$ has a unique bounded linear extension $R(\lambda; A)_1$ on $\mathfrak{X}$. If $\{x_n\} \subset \mathfrak{D}(A)$, $x_n \longrightarrow x_0 \in \mathfrak{D}(\overline{A})$, and $Ax_n \longrightarrow \overline{A}x_0$, then $R(\lambda; A)(\lambda I - A)x_n = x_n$ implies that $R(\lambda; A)_1(\lambda I - \overline{A})x_0 = x_0$. Likewise for $\{y_n\} \subset \mathfrak{R}(\lambda I - A)$ and $y_n \longrightarrow y_0$, the relation $(\lambda I - A)R(\lambda; A)y_n = y_n$ implies that $(\lambda I - \overline{A})R(\lambda; A)_1 y_0 = y_0$. It follows that $R(\lambda; \overline{A})$ exists and is identical with $R(\lambda; A)_1$. This shows that $\rho(A) \subset \rho(\overline{A})$. A similar argument can be used to prove $A^* = \overline{A}^*$, and the last relation is obvious.

[4] This condition is automatically satisfied if $\int_0^1 \|T(\sigma)\| d\sigma < \infty$ or if $T(s)$ if of class C.

THEOREM 2.1. *If $T(s)$ is a semi-group of class $(0,A)^*$, $(0,C)^*$, or C^*, then the adjoint semi-group is of class $(0,A)$, $(0,C)$ or C, respectively. The* c.i.g. $\overline{A^+}$ *is the largest restriction of A^* with domain and range in $\mathfrak{X}^+$.*

Proof. According to Theorem 1.5,

$$R(\lambda; A^*) = R(\lambda; \overline{A^*}) = R^*(\lambda; A)$$

and hence $\mathfrak{D}(A^*)$ is simply the range of $R^*(\lambda; A)$. For $\lambda > \omega_0$, $R^*(\lambda; A)$ can be expressed by means of a Dunford integral [2] as

$$(2.3) \qquad R^*(\lambda; A)x^* = \int_0^\infty \exp(-\lambda\sigma)\, T^*(\sigma)x^*\, d\sigma\, .$$

It is clear from this that

$$T^*(s)R^*(\lambda; A) = R^*(\lambda; A)\, T^*(s),$$

so that $T^*(s)$ takes $\mathfrak{D}(A^*)$ into $\mathfrak{D}(A^*)$. Since $T^*(s)$ is bounded, it follows that $T^*(s)(\mathfrak{X}^+) \subset \mathfrak{X}^+$; that is, $T^+(s) \in \mathfrak{E}(\mathfrak{X}^+)$. It is obvious that $T^*(s)$ and hence $T^+(s)$ satisfies (i).

In order to establish continuity we first note that

$$(2.4) \qquad [T^*(\tau) - I^*]R^*(\lambda; A)x^* = [\exp(\lambda\tau) - 1] \int_0^\infty \exp(-\lambda\sigma)\, T^*(\sigma)x^*\, d\sigma$$
$$- \exp(\lambda\tau) \int_0^\tau \exp(-\lambda\sigma)\, T^*(\sigma)x^*\, d\sigma.$$

The first term in the right member is simply $[\exp(\lambda\tau) - 1]\, R^*(\lambda; A)x^*$, and it clearly converges to zero with τ; further the assumption that $\|T^*(\sigma)x^*\|$ is majorized by a function in $L_1(0,1)$ implies that the second term also goes to zero with τ. Thus

$$\lim_{s \to 0} T^*(s)y^* = y^*$$

for all $y^* \in \mathfrak{D}(A^*)$. It follows from this (cf. [5, Theorem 9.4.1]) that $T^*(s)y^*$ is strongly continuous for $s \geq 0$, $y^* \in \mathfrak{D}(A^*)$. Further since $\|T^*(s)\| = \|T(s)\|$ is uniformly bounded in each interval of the form $(\delta, 1/\delta)$, we see that $T^*(s)x^*$ is strongly continuous for $s > 0$ and all $x^* \in \mathfrak{X}^+$. Thus $T^+(s)$ satisfies (i), (ii), and (iii). Again, for each $x^* \in \mathfrak{D}(A^*)$,

$$T^{+}(s)x^{*} \longrightarrow x^{*} \text{ as } s \longrightarrow 0$$

and *a fortiori*

$$\tau^{-1}\int_0^{\tau} T^{*}(\sigma)x^{*}\,d\sigma \longrightarrow x^{*} \text{ as } \tau \longrightarrow 0$$

and

$$\lambda R^{*}(\lambda; A)x^{*} \longrightarrow x^{*} \text{ as } \lambda \longrightarrow \infty.$$

Now if $T(s)$ is of class C, then $\|T^{*}(s)\| = O(1)$; if $T(s)$ is of class $(0, C)$ then

$$\|[\tau^{-1}\int_0^{\tau} T(\sigma)\,d\sigma]^{*}\| = O(1);$$

and if $T(s)$ is of class $(0, A)$ then $\|\lambda R^{*}(\lambda; A)\| = O(1)$. It now follows from the Banach-Steinhaus theorem that $T^{+}(s)$ will satisfy (vi), (v), or (iv) with $T(s)$.

Finally, the c.i.g. $\overline{A^{+}}$ of $T^{+}(s)$ is determined by its resolvent (cf. [**6**]), which for $\lambda > \omega_0$ can be expressed by the Bochner integral

$$R(\lambda; \overline{A^{+}})x^{*} = \int_0^{\infty} \exp(-\lambda\sigma)\,T^{+}(\sigma)x^{*}\,d\sigma \qquad (x^{*} \in \mathfrak{X}^{+}).$$

According to formula (2.3) this is simply the restriction of $R(\lambda; A^{*})$ to $\mathfrak{X}^{+}$; thus $\overline{A^{+}}$ is a restriction of A^{*}. Now if $x^{*} \in \mathfrak{D}(A^{*})$ and $A^{*}(x^{*}) \in \mathfrak{X}^{+}$, then $(\lambda I^{*} - A^{*})x^{*} \in X^{+}$ and hence

$$R(\lambda; A^{*})(\lambda I^{*} - A^{*})x^{*} = x^{*} \in \mathfrak{D}(\overline{A^{+}}).$$

Conversely if $x^{*} \in \mathfrak{D}(\overline{A^{+}})$, then $x^{*} \in \mathfrak{D}(A^{*})$ and $A^{*}x^{*} = \overline{A^{+}}x^{*} \in \mathfrak{X}^{+}$. In other words, $\overline{A^{+}}$ is the maximal restriction of A^{*} which maps $\mathfrak{X}^{+}$ into $\mathfrak{X}^{+}$. This concludes the proof.

COROLLARY. *If $\lambda \in \rho(\overline{A})$, then $\lambda \in \rho(\overline{A^{+}})$ and $R(\lambda; \overline{A^{+}})$ equals the restriction of $R(\lambda; A^{*})$ to $\mathfrak{X}^{+}$.*

Proof. If $\lambda \in \rho(A)$, then $R(\lambda; A^{*})$ exists. Let $R(\lambda; A^{*})_0$ be the restriction of $R(\lambda; A^{*})$ to $\mathfrak{X}^{+}$. For $x^{*} \in \mathfrak{D}(\overline{A^{+}})$, we have

$$(\lambda I^{+} - \overline{A^{+}})x^{*} = (\lambda I^{*} - A^{*})x^{*}$$

and hence $R(\lambda; A^*)_0$ is a left inverse for $\lambda I^+ - \overline{A^+}$. On the other hand if $x^* \in \mathfrak{X}^+$, then

$$(\lambda I^* - A^*) R(\lambda; A^*)_0 x^* = x^*.$$

Since $R(\lambda; A^*)_0 x^* \in \mathfrak{D}(A^*) \subset \mathfrak{X}^+$ we also have $A^* R(\lambda; A^*)_0 x^* \in \mathfrak{X}^+$ and hence by the above theorem $R(\lambda; A^*)_0 x^* \in \mathfrak{D}(\overline{A^+})$. It follows that $R(\lambda; A^*)_0$ is also the right inverse for $\lambda I^+ - \overline{A^+}$ so that $\lambda \in \rho(\overline{A^+})$.

A converse to the above corollary is obtained in Theorem 3.2 where it is shown that $\rho(\overline{A}) = \rho(\overline{A^+})$.

COROLLARY. *If $\mathfrak{X}$ is reflexive, then $\mathfrak{X}^+ = \mathfrak{X}^*$.*

Proof. If $\mathfrak{X}$ is reflexive, then, according to Theorem 1.1, $\mathfrak{D}(A^*)$ is dense in $\mathfrak{X}^*$. Hence $\mathfrak{X}^+ = \overline{\mathfrak{D}(A^*)} = \mathfrak{X}^*$.

We conclude this section with two other characterizations of $\mathfrak{X}^+$.

THEOREM 2.2. *For a semi-group $T(s)$ of class $(0, A)^*$, let*

$$\Gamma = [x^*;\ T^*(s)x^* \longrightarrow x^* \text{ as } s \longrightarrow 0].$$

Then $\mathfrak{X}^+ = \overline{\Gamma}$.

Proof. It is clear that $\mathfrak{D}(A^*) \subset \Gamma$; and since $\mathfrak{D}(A^*)$ is dense in $\mathfrak{X}^+$, we have $\mathfrak{X}^+ \subset \overline{\Gamma}$. On the other hand if $x^* \in \Gamma$, then a direct calculation shows that

$$\lambda R(\lambda; A^*) x^* = \lambda \int_0^\infty \exp(-\lambda\sigma) T^*(\sigma) x^* d\sigma \longrightarrow x^* \qquad \text{as } \lambda \longrightarrow \infty.$$

Consequently $x^* \in \overline{\mathfrak{D}(A^*)} = \mathfrak{X}^+$.

THEOREM 2.3. *For a semi-group $T(s)$ of class $(0, A)^*$ let*

$$\Gamma_0 = [y^*_{\alpha\beta};\ y^*_{\alpha\beta} = \int_\alpha^\beta T^*(\sigma) x^* d\sigma,\ x^* \in \mathfrak{X}^*,\ 0 \le \alpha < \beta].$$

Then $\mathfrak{X}^+ = \overline{\Gamma_0}$.

Proof. An easy calculation shows that $\Gamma_0 \subset \Gamma$. On the other hand if $x^* \in \Gamma$ then

$$\tau^{-1} \int_0^\tau T^*(\sigma) x^* d\sigma \longrightarrow x^* \qquad \text{as } \tau \longrightarrow 0$$

and belongs to Γ_0; thus $\overline{\Gamma}_0 \supset \Gamma$ and therefore $\overline{\Gamma}_0 = \overline{\Gamma} = \mathfrak{X}^+$.

3. The adjoint space. We shall call $\mathfrak{X}^+$ the *adjoint space to* $\mathfrak{X}$ *relative to the semi-group* $[T(s)]$, or simply, the *adjoint space*; and we shall denote the generic element of $\mathfrak{X}^+$ by x^+. To avoid confusion we shall hereafter refer to $\mathfrak{X}^*$ as the *full adjoint space*. This section is devoted to a study of the hierarchy of adjoint spaces which arise from a given semi-group of operators of class $(0, A)^*$.

It will be observed that whereas

$$||x^*|| = \sup[|x^+(x)|;\ ||x|| \leq 1,\ x \in \mathfrak{X}],$$

it is not in general true that $||x||$ can be obtained in like manner as

$$(3.1) \qquad ||x||' = \sup[|x^+(x)|;\ ||x^+|| \leq 1,\ x^+ \in \mathfrak{X}^+].$$

All that can be asserted here is that $||x||' \leq ||x||$. If $\mathfrak{X}^+$ is equal to the full adjoint space, then it is clear that $||x||' = ||x||$. This occurs when $\mathfrak{X}$ is reflexive or when A is bounded. In any case we see that the function $||x||'$ satisfies the postulates of a pseudo-norm. However, more is true:

THEOREM 3.1. *The norm* $||x||'$ *defines an equivalent topology for* $\mathfrak{X}$; *in fact, there exists an* $m > 0$ *such that*

$$||x|| \geq ||x||' \geq m\ ||x||$$

for all $x \in \mathfrak{X}$. *In particular if*

$$\liminf_{\lambda \to \infty} ||\lambda R(\lambda; \overline{A})|| = 1,$$

then $||x|| \equiv ||x||'$.

Proof. For a fixed $x \in \mathfrak{X}$ there exists an $x^* \in \mathfrak{X}^*$, $||x^*|| = 1$, such that $x^*(x) = ||x||$. It follows from (iv) that

$$[\lambda R^*(\lambda; \overline{A})x^*](x) = x^*[\lambda R(\lambda; \overline{A})x] \longrightarrow x^*(x) \qquad \text{as } \lambda \longrightarrow \infty,$$

and from (iv) together with the uniform boundedness theorem that

$$\overline{\lim_{\lambda \to \infty}}\ ||\lambda R(\lambda; \overline{A})|| = M < \infty.$$

Consequently, given $\epsilon > 0$, there is a λ_ϵ with

$$||\lambda_\epsilon R^*(\lambda_\epsilon;\overline{A})|| \leq M+\epsilon \quad \text{and} \quad |[\lambda_\epsilon R^*(\lambda_\epsilon;\overline{A})x^*](x) - ||x||| < \epsilon.$$

Now

$$y_\epsilon^* \equiv \lambda_\epsilon R^*(\lambda_\epsilon; A)x^* \in \mathfrak{X}^+ \quad \text{and} \quad ||y_\epsilon^*|| \leq M+\epsilon.$$

Hence

$$\frac{|y_\epsilon^*(x)|}{||y_\epsilon^*||} \geq \frac{||x||-\epsilon}{M+\epsilon};$$

and since ϵ is arbitrary this gives the desired result with $m = 1/M$. In particular if $M = 1$, then $||x|| = ||x||'$.

THEOREM 3.2. *If* $[T(s)]$ *is a semi-group of operators of class* $(0, A)^*$, *then* $\rho(\overline{A}) = \rho(\overline{A^+})$.

Proof. We have already shown in the first corollary to Theorem 2.1 that $\rho(\overline{A}) \subset \rho(\overline{A^+})$. If $\lambda \in \rho(\overline{A^+})$, then

$$\mathfrak{R}(\lambda I^* - \overline{A}^*) \supset \mathfrak{R}(\lambda I^+ - \overline{A^+}) = \mathfrak{X}^+.$$

Since, by Theorem 1.1, $\mathfrak{D}(\overline{A}^*) \subset \mathfrak{X}^+$ is weakly* dense in $\mathfrak{X}^*$, the same is true of $\mathfrak{R}(\lambda I^* - \overline{A}^*)$. It now follows from Theorem 1.3 that $\lambda I - \overline{A}$ has an inverse. Further, if

$$(\lambda I^* - \overline{A}^*)x_0^* = 0$$

then $x_0^* \in \mathfrak{D}(\overline{A}^*)$ and $\overline{A}^* x_0^* \in \mathfrak{D}(\overline{A}^*) \subset \mathfrak{X}^+$, so that $x_0^* \in \mathfrak{D}(\overline{A^+})$. Since $\overline{A^+}$ is a restriction of $\overline{A}^*$, this implies that $(\lambda I^+ - \overline{A^+})x_0^* = 0$ and hence that $x_0^* = 0$. Theorem 1.2 now asserts that $\mathfrak{R}(\lambda I - \overline{A})$ is dense in $\mathfrak{X}$. Finally for $x \in \mathfrak{R}(\lambda I - \overline{A})$ we have

$$\begin{aligned} ||(\lambda I - \overline{A})^{-1}x|| &\leq m^{-1}\, ||(\lambda I - \overline{A})^{-1}x||' \\ &= m^{-1} \sup\,[\,|x^+[(\lambda I - \overline{A})^{-1}x]\,|;\ ||x^+|| \leq 1,\ x^+ \in \mathfrak{X}^+] \\ &\leq m^{-1}\, ||R(\lambda; \overline{A^+})||\ ||x||; \end{aligned}$$

and this shows that $(\lambda I - \overline{A})^{-1}$ is bounded. It follows that $\lambda \in \rho(\overline{A})$.

We see from the above theorem that $\overline{A^+}$ has the same resolvent set as $\overline{A}^*$ (and $\overline{A}$) in spite of the fact that it is a restriction of $\overline{A}^*$.

Renorming $\mathfrak{X}$ by $||x||'$ has no effect on our determination of $\mathfrak{X}^+$; in fact, even the norm of the elements of $\mathfrak{X}^+$ remains the same. For

$$||x||' \leq ||x|| \text{ and } |x^+(x)| \leq ||x^+|| \, ||x||'$$

imply that

$$||x^+|| \leq \sup [\, |x^+(x)|;\ ||x||' \leq 1,\ x \in \mathfrak{X}] \leq ||x^+||.$$

Nevertheless, when we deal with the second adjoint space relative to a given semi-group $[T(s)]$, a slight advantage is obtained by renorming $\mathfrak{X}$ in this way.

THEOREM 3.3. *Suppose that both* $[T(s)]$ *and* $[T^+(s)]$ *are of class* $(0, A)^*$, *and let the norm of* $\mathfrak{X}$ *be given by* $||x||'$. *Then* $\mathfrak{X}$ *can be embedded in* $\mathfrak{X}^{++}$ *by means of the natural mapping.*

Proof. Each $x_0 \in \mathfrak{X}$ defines a unique bounded linear functional $F_0 \in (\mathfrak{X}^+)^*$, namely $F_0(x^+) = x^+(x_0)$. Further,

$$||F_0|| = \sup [\, |F_0(x^+)| = |x^+(x_0)|;\ ||x^+|| \leq 1,\ x^+ \in \mathfrak{X}^+] = ||x_0||'.$$

Hence $x_0 \longrightarrow F_0$ is a linear isometric mapping of $\mathfrak{X}$ onto a subspace of $(\mathfrak{X}^+)^*$. It remains to show that $\mathfrak{X} \subset (\mathfrak{X}^+)^+$ in the above sense. This in turn requires that $\mathfrak{X} \subset \mathfrak{D}[(\overline{A^+})^*]$. However, if $x_0 \longrightarrow F_0$ then

$$[R^*(\lambda; \overline{A^+}) F_0](x^+) = F_0[R(\lambda; \overline{A^+}) x^+] = [R(\lambda; \overline{A^+}) x^+](x_0) = x^+[R(\lambda; \overline{A}) x_0].$$

Hence

$$R(\lambda; \overline{A}) x_0 \longrightarrow R^*(\lambda; \overline{A^+}) F_0 .$$

Now

$$\lim_{\lambda \to \infty} \lambda R(\lambda; \overline{A}) x_0 = x_0$$

implies that

$$\lim_{\lambda \to \infty} \lambda R^*(\lambda; \overline{A^+}) F_0 = F_0 ;$$

and since

$$R^*(\lambda;\overline{A^+})F_0 \in \mathfrak{D}[(\overline{A^+})^*],$$

it follows that $x_0 \in \mathfrak{D}[(\overline{A^+})^*]$.

The space $\mathfrak{X}^{++}$ depends only on $T^+(s)$ and $\mathfrak{X}^+$. Further, the norm in $\mathfrak{X}^+$ is not effected by renorming $\mathfrak{X}$ with the norm $||x||'$; in fact

$$||x^+|| = \sup[|x^+(x)|;\ ||x||' \leq 1,\ x \in \mathfrak{X}].$$

Since $\mathfrak{X}$ with the norm $||x||'$ is a subset of $\mathfrak{X}^{++}$, it follows that

$$||x^+||' \equiv \sup[|x^{++}(x^+)|;\ ||x^{++}|| \leq 1, x^{++} \in \mathfrak{X}^{++}] = ||x^+||.$$

Thus it is only in the case of $\mathfrak{X}$ and $\mathfrak{X}^+$ that a nonsymmetric condition between norms may arise; for all other pairs of successive adjoint spaces the norms are symmetric. Even if $\mathfrak{X}$ is not renormed, $\mathfrak{X}$ will be isomorphic with its image in $\mathfrak{X}^{++}$ under the natural mapping.

DEFINITION 3.1. We define the (Γ)-*weak topology* in $\mathfrak{X}$ in the usual way be means of the generic neighborhood

$$N(x_0; x_1^*, \cdots, x_n^*; \epsilon) \equiv [x; |x_k^*(x - x_0)| < \epsilon, k = 1, \cdots, n],$$

where the $(x_1^*, \cdots, x_n^*)$ can be any finite subset of Γ and ϵ is an arbitrary positive number.

It is of interest to determine when, under the natural mapping, $\mathfrak{X} = \mathfrak{X}^{++}$; that is, under what conditions $\mathfrak{X}$ is reflexive relative to a given semi-group of operators $[T(s)]$. Here we assume that $\mathfrak{X}$ has been renormed with norm $||x||'$. If $\mathfrak{X}$ is a reflexive in the usual sense, then the second corollary to Theorem 2.1 asserts that $\mathfrak{X}^+ = \mathfrak{X}^*$, and likewise that

$$\mathfrak{X}^{++} = (\mathfrak{X}^+)^* = \mathfrak{X}^{**} = \mathfrak{X}.$$

More generally, we have:

THEOREM 3.4. *Suppose that both* $[T(s)]$ *and* $[T^+(s)]$ *are of class* $(0, A)^*$, *and let the norm of* $\mathfrak{X}$ *be given by* $||x||'$. *A necessary and sufficient condition for* $\mathfrak{X} = \mathfrak{X}^{++}$ *is that* $R(\lambda; \overline{A})$ *be* $(\mathfrak{X}^+)$-*weakly compact.*

Proof. Suppose first that $R(\lambda;\overline{A})$ is $(\mathfrak{X}^+)$-weakly compact; that is, the image of each bounded set is contained in an $(\mathfrak{X}^+)$-weakly compact subset of $\mathfrak{X}$. Let F_0 be an arbitrary element of $(\mathfrak{X}^+)^*$. Then by Helly's theorem, given a finite subset $\pi \subset \mathfrak{X}^+$, there exists an

$$x_\pi \in \mathfrak{X}, \quad ||x_\pi|| \leq 2\,||F_0||,$$

such that $F_0(x^+) = x^+(x_\pi)$ for all $x^+ \in \pi$. Ordering the π's by inclusion, we easily see that they form a directed set. Consequently,

$$[R^*(\lambda; \overline{A^+})F_0](x^+) = F_0[R(\lambda; \overline{A^+})x^+] = \lim_\pi [R(\lambda; \overline{A^+})x^+](x_\pi)$$

$$= \lim_\pi x^+[R(\lambda; \overline{A})x_\pi].$$

Since the $R(\lambda; \overline{A})$ image of any bounded set is contained in an $(\mathfrak{X}^+)$-weakly compact subset of $\mathfrak{X}$, it is easily shown that there exists an $x_0 \in \mathfrak{X}$ such that

$$\lim_\pi x^+[R(\lambda; A)x_\pi] = x^+(x_0)$$

for all $x^+ \in \mathfrak{X}^+$. Thus $R^*(\lambda; \overline{A^+})F_0$ is the image of x_0 under the natural mapping; in other words, $\mathfrak{X} \supset \mathfrak{D}[(\overline{A^+})^*]$. This together with Theorem 3.3 shows that $\mathfrak{X} = \mathfrak{X}^{++}$.

Conversely, suppose that $\mathfrak{X} = \mathfrak{X}^{++}$. Then $R^*(\lambda; \overline{A^+})[(\mathfrak{X}^+)^*]$ is contained in the images of $\mathfrak{X}$. Now $R^*(\lambda; \overline{A^+})$ is continuous in the usual weak* topology of $(\mathfrak{X}^+)^*$; hence the unit sphere, which is weakly* compact, maps onto a weakly* compact subset. Now this image lies in $\mathfrak{X}$ and the weak* topology in $\mathfrak{X} \subset (\mathfrak{X}^+)^*$ is the same as the $(\mathfrak{X}^+)$-weak topology for $\mathfrak{X}$. Hence $R(\lambda; \overline{A})$, which is essentially a restriction of $R^*(\lambda; \overline{A^+})$, takes bounded sets into $(\mathfrak{X}^+)$-weakly compact subsets of $\mathfrak{X}$. This concludes the proof.

COROLLARY *If $R(\lambda; \overline{A})$ is weakly compact relative to the usual weak topology of $\mathfrak{X}$, then $\mathfrak{X} = \mathfrak{X}^{++}$.*

Proof. It is clear that a weakly compact subset of $\mathfrak{X}$ is also weakly compact relative to any weaker topology such as the $(\mathfrak{X}^+)$-weak topology of $\mathfrak{X}$.

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