CONSTRUCTIONS FOR POLES AND POLARS IN n-DIMENSIONS

A. P. DEMPSTER AND S. SCHUSTER

1. Introduction. As far back as 1847, von Staudt [2, p. 131-136] introduced the notion of handling a symmetric polarity (that is, a nonnull polarity) by means of a self-polar simplex and an additional pair of corresponding elements. In projective space of two dimensions (S_2) such a polarity is completely determined by a self-polar triangle $A_1A_2A_3$, a point P, and its polar line p. We write this polarity as $(A_1A_2A_3)(Pp)$. In S_3, the polarity is determined by a self-polar tetrahedron $A_1A_2A_3A_4$, a point P, and its polar plane π. We write it $(A_1A_2A_3A_4)(P\,\pi)$. In general, we have a polarity in S_n determined by the self-polar simplex $A_1A_2\cdots A_{n+1}$, a point P, and its corresponding polar prime or hyperplane π. We write it $(A_1A_2\cdots A_{n+1})(P\,\pi)$.

Left unanswered by von Staudt and his followers is the following question: Given an arbitrary point X, how can we construct the polar prime χ of X? And, conversely, given the prime χ, how do we actually find its pole, the point X?

2. Construction. The construction of the polar line x of an arbitrary point X for the polarity $(A_1A_2A_3)(Pp)$ in S_2 was given by Coxeter [1, 64]. We give a direct generalization of this to n dimensions: to find the polar prime χ of an arbitrary point X relative to $(A_1A_2\cdots A_{n+1})(P\,\pi)$.

Consider first the point X not in any face of $A_1A_2\cdots A_{n+1}$. Let α_i denote face $A_1A_2\cdots A_{i-1}A_{i+1}\cdots A_{n+1}$, and let

$$A_i' = PX\cdot\alpha_i,\quad P_i = XA_i\cdot\pi,\quad \text{and}\quad X^i = PA_i\cdot P_iA_i'.$$

In the plane PXA_i we have pairs P, P_i and A_i, A_i' conjugate under the induced plane polarity. By Hesse's theorem in the plane [1, pp. 60-61], X and X^i are conjugate for the induced polarity, and hence for the given polarity. In this manner we determine $n+1$ points $X^1, X^2, \cdots, X^{n+1}$ lying in χ. The points $X^1, X^2, \cdots, X^n$ determine χ since otherwise they must lie in an $(n-2)$-flat which implies that the flat determined by $P, X^1, \cdots, X^n$ is of at most $(n-1)$ dimensions, which is impossible since the space contains $P, A_1, A_2, \cdots, A_n$. It

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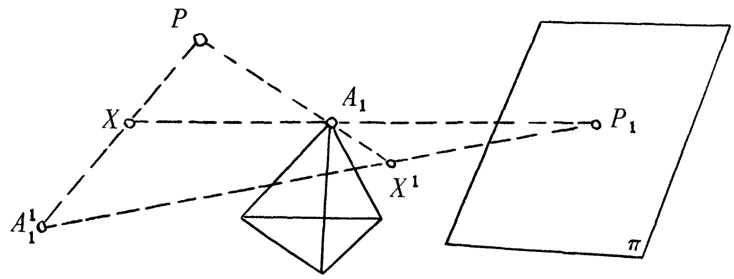
follows that χ is determined by any $(n-1)$ of the points X^i. This completes the construction in S_n for general X. This is illustrated for $n = 3$, and is easily seen to yield Coxeter's construction for $n = 2$.



A second approach is to reduce the question of finding χ in S_n to two analogous constructions in $(n-1)$ dimensions, namely in any two faces α_i. Under the polarity induced in α_i the point $X_i = XA_i \cdot \alpha_i$ maps into an $(n-2)$-flat x_i consisting of points conjugate to X. For the general X considered, no two x_i coincide; hence, any two of them determine an $(n-1)$-flat of points conjugate to X. This can only be χ. Using this idea we can reduce the construction in S_n to 2^r analogous constructions in $n-r$ dimensions, and at any stage of this induction on r, we may use the first method to solve the question completely.

In particular, if $n = 2$ we can construct directly by the first method or use the construction for corresponding points in two involutions on the sides of $A_1A_2A_3$. If $n = 3$ we can use the first method, or carry out constructions in two faces of $A_1A_2A_3A_4$, or carry out constructions in four edges of $A_1A_2A_3A_4$.

Going back to n dimensions, suppose X is not of general position; that is, X lies in a face α_i. If X lies in r such faces we may name these $\alpha_1, \cdots, \alpha_r$. Then χ contains $A_1, \cdots, A_r$. Considering the $(n-r)$-flat determined by simplex $A_{r+1} \cdots A_{n+1}$, we see that the polarity induced in this space has $A_{r+1} \cdots A_{n+1}$ as a self-polar simplex and X belongs to the space but is not on a face of $A_{r+1} \cdots A_{n+1}$. Thus, we can use the first method to determine the polar prime χ' of X in this space. Then $A_1, \cdots, A_r$, and χ' generate an $(n-1)$-flat of points conjugate to X. This $(n-1)$-flat is χ.

The problem of finding X when given χ is solved by dualizing the foregoing procedures.

REFERENCES

1. H. S. M. Coxeter, *The real projective plane*, New York, 1949.
2. C. G. C. von Staudt, *Geometrie der Lage*, Nuremberg, 1847.

UNIVERSITY OF TORONTO
POLYTECHNIC INSTITUTE OF BROOKLYN