SIMPLE PROOF OF A THEOREM OF P. KIRCHBERGER

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In their paper Rademacher and Schoenberg [2] give a simple proof of a theorem due to P. Kirchberger [1] on separation of sets in the Euclidean n-space by means of hyperplanes. Their proof utilizes Helly's theorem on convex sets and is one instance of several important applications of that theorem given in their paper.

The object of this note is to give a simple direct proof of Kirchberger's theorem. Our proof does not use Helly's theorem and is based on the well-known theorem of Carathéodory on convex sets (which is also utilized by Rademacher and Schoenberg in their derivation of Helly's theorem).

The statement and proof of the theorem follow.

THEOREM OF P. KIRCHBERGER. Let S and T be two finite sets of points in the Euclidean n-space, E^n. If any $k+1$ points of S may be separated from any $l+1$ points of T by a hyperplane of E^n ($k+l \leq n$), then S may be separated from T by a hyperplane of E^n.

Proof. We first prove our theorem under the (seemingly) stronger condition that the assumption of the theorem holds for any k, l satisfying $k \leq n$, $l \leq n$, instead of $k + l \leq n$.

It is well-known that two compact convex sets in E^n having no point in common may be separated by a hyperplane of E^n. The assertion of the theorem is therefore equivalent to the following: The convex hulls $H(S)$ and $H(T)$ of S and T respectively have no point in common. If this were not the case, any point of $H(S) \cap H(T)$ would, by Carathéodory's theorem, belong to two simplexes of dimensions not exceeding n with vertices in S, T respectively. But, according to our assumption, the vertices of the one simplex may be separated from those of the other by a hyperplane of E^n, and therefore their convex hulls may be similarly separated. The contradiction obtained proves the validity of our assertion.

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Next we show that if there are two subsets of S, T consisting of $k+1$, $l+1$ points resp. with $k \leq n$, $l \leq n$, which cannot be separated by a hyperplane, then there necessarily exist two subsets consisting of k_1+1, l_1+1 points resp. of S, T, with $k_1+l_1 \leq n$, which also cannot be separated by a hyperplane.

Indeed let A, B be two sets in E^n, consisting of $k+1$, $l+1$ points respectively $(k \leq n,\ l \leq n)$ which cannot be separated by a hyperplane. The set $H(A) \cap H(B)$ is non-void; let P_0 be one of its points and $\sigma^{k'}$, $\tau^{l'}$ two simplexes with vertices in A and B respectively containing P_0 in their (relative, that is, k'-or l'-dimensional) interior. We denote by $E^{k'}$, $F^{l'}$ the "minimal flats" (linear manifolds of minimum dimension) containing $\sigma^{k'}$, $\tau^{l'}$ respectively. In case $k'+l' > n$, the flat $E^{k'} \cap F^{l'}$ is at least l-dimensional and so contains a ray issuing from P_0. If P_1 is the first point on that ray which does not belong to the (relative) interior of both $\sigma^{k'}$ and $\tau^{l'}$, then P_1 is an interior point of some faces (at least one of them proper) $\sigma^{k''}$ and $\tau^{l''}$ of $\sigma^{k'}$ and $\tau^{l'}$ resp., where $k''+l'' < k'+l'$. By repeating this process of reducing the dimension-sum we can ultimately make it smaller than or equal to n, which gives the required result.

This completes the proof of our theorem.

REFERENCES

1. P. Kirchberger, *Über Tschebyscheffsche Annäherungsmethoden*, Math. Ann., **57** (1903), 509-540.

2. H. Rademacher, I. J. Schoenberg, *Helly's theorem on convex domains and Tchebycheff's approximation problem*, Can. Journ. Math. **2** (1950), 245-256.

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