COLLECTIONS AND SEQUENCES OF CONTINUA IN THE PLANE

C. E. BURGESS

1. An inversion of the plane with respect to a closed circular disc. The inversion described here will be used in proving Theorem 5.

DEFINITION. Let S denote the plane, let K be a closed circular disc, and let T be a one-to-one transformation of S onto itself satisfying the following conditions:

(1) T is continuous over K, and $T(K) = K$;

(2) T is continuous over $S-K$, and $T(S-K) = S-K$; and

(3) if H is an unbounded subset of $S-K$ which does not have a limit point in K, then $T(H)$ is bounded and has a limit point in K.

The transformation T will be called an *inversion of S with respect to K.*

NOTATION. If T is an inversion of the plane with respect to a closed circular disc K, and M is a continuum in $S - K$, then M' will denote the closure of $T(M)$. If G is a collection of continua in $S - K$, then G' will denote the collection of all continua X' such that X is a continuum of G. This notation will be used in the statement of Theorem 1 and in the proof of Theorem 5.

THEOREM 1. *If K is a closed circular disc and G is a finite collection of mutually exclusive unbounded continua not intersecting K, then there is an inversion T of the plane with respect to K such that the continua of G' are bounded and mutually exclusive.* [1]

Indication of proof. If the plane S is inverted about the boundary of K with respect to the center o of K, then the continua of G are carried onto mutually exclusive bounded connected sets each of which has o as a limit point and is closed relative to $S - o$. Hence it will be sufficient to show that if $M_1, M_2, \cdots, M_n$ $(n > 1)$ are bounded continua such that

[1] I am indebted to the referee for some very helpful suggestions which enabled me to obtain a simplified proof of this theorem.

Received September 29, 1953. Presented to the American Mathematical Society, May 2, 1953.

Pacific J. Math. 5 (1955), 325-333

$$M_i \cdot M_j = o \text{ for each } i \text{ and } j \ (i, j \leq n) \ (i \neq j),$$

and $M_i - o$ is connected, then there is a homeomorphism Z of $S - o$ onto $S - K$ such that the closures of

$$Z(M_1 - o),\ Z(M_2 - o), \cdots, Z(M_n - o)$$

are mutually exclusive bounded continua. By using a theorem proved by Lubben [4, Theorem 18], it can be shown that there exist $n - 1$ simple closed curves $J_1, J_2, \cdots, J_{n-1}$ such that

(1) $J_i \cdot J_j = o$ for each i and j $(i, j < n)$ $(i \neq j)$ and

(2) each of the n complementary domains of $J_1 + J_2 + \cdots + J_{n-1}$ contains one of the sets $M_1 - o$, $M_2 - o, \cdots, M_n - o$. Furthermore, it can be shown that there exist $n - 1$ simple closed curves $H_1, H_2, \cdots, H_{n-1}$ satisfying the conditions required above for $J_1, J_2, \cdots, J_{n-1}$ such that $J_i \cdot H_j = o$ for each i and j $(i, j < n)$, and the connected domain having $J_i + H_i$ as its boundary does not intersect $M_1 + M_2 + \cdots + M_n$. There is a homeomorphism T_1 of S onto itself leaving o fixed such that for each i $(i < n)$, $T_1(J_i)$ and $T_1(H_i)$ are polygons. There is no loss of generality in assuming that K has radius 1 and that $T_1(K) = K$. Consider a polar coordinate system with origin o, and let T_2 be a transformation that carries (ρ, θ) into $(\rho + 1, \theta)$, where $\rho > 0$. Then $T_2 T_1$ is the desired homeomorphism Z.

2. Some properties of continua with respect to point sets which intersect them.

Notation and definitions. If G is a collection of point sets, then G^* denotes the sum of the sets of G.

If G is a collection of point sets and M is a continuum intersecting every set of G such that no proper subcontinuum of M intersects every set of G, then M is said to be *irreducible among the sets of* G.

If G is a collection of mutually exclusive closed point sets and M is a continuum intersecting every set of G such that for some two sets K_1 and K_2 of G the set $M - M \cdot (K_1 + K_2)$ is connected, then M is said to have *property* X with respect to G.

If p is a point of the continuum M, the sum of all proper subcontinua of M containing p is called a *composant* of M.

A proper subset H of the continuum M is said to be a *set of condensation* of M if every point of H is a limit point of $M - H$.

THEOREM 2. *If M is a compact indecomposable metric continuum and G is a countable collection of closed subsets of M such that no set of G intersects every composant of M, then uncountably many composants of M lie in $M - G^*$.*

LEMMA 2.1. *If*

(1) *M is a compact indecomposable metric continuum,*

(2) *H is a closed subset of M such that some composant K of M does not intersect H,*

(3) *D is a domain intersecting K but not H, and*

(4) *W is the collection of all components of $M - M \cdot D$ that intersect H, then W^* is closed and each of its points is a limit point of $M - W^*$.*

Proof of Lemma 2.1. Suppose that W^* is not closed. Since W^* contains H, then some point p of $M - (D + H + W^*)$ is a limit point of W^*. There exists a sequence of continua $L_1, L_2, L_3, \cdots$ converging to a continuum L containing p such that for each n, L_n is an element of W. Since each L_n intersects the closed set H, then L intersects H. This means that L is a subset of some element of W. Hence W^* is closed. Since each element of W intersects H and is a proper subcontinuum of the indecomposable continuum M, then W^* does not intersect K. Since K is dense in M, then every point of W^* is a limit point of $M - W^*$.

Proof of Theorem 2. Let X be an element of G. By applying Lemma 2.1 to each element of a sequence of domains in $M - X$ closing down on a point of a composant of M not intersecting X, it can be seen that the sum of all composants of M intersecting X is the sum of a countable number of closed sets of condensation of M. Since G is countable, the sum of all composants of M intersecting G^* is the sum of a countable number of closed sets of condensation of M. Hence by Baire's theorem [**6**, Theorem 15, p. 11], some composant of M does not intersect G^*. Since each composant of M is the sum of a countable number of continua of condensation of M, it follows, upon again applying Baire's theorem, that uncountably may composants of M lie in $M - G^*$.

THEOREM 3. *If G is a finite collection consisting of at least two mutually exclusive closed point sets and M is a compact metric continuum irreducible among the sets of G, then there exist two sets K_1 and K_2 of G such that $M - M \cdot (K_1 + K_2)$ is connected; that is, M has property X with respect to G.*

LEMMA 3.1. *Suppose*

(1) *G is a finite collection of mutually exclusive closed point sets,*

(2) *M is a compact metric continuum irreducible among the sets of G,*

(3) *K_1, K_2, and K_3 are three sets of G, and*

(4) *M_1 and M_2 are subcontinua of M such that*

(1) *for each i ($i \leq 2$), M_i intersects every set of the collection $G - K_i$ and does not intersect K_i,*

(2) *the set $M_1 - M_1 \cdot (K_2 + K_3)$ is connected, and*

(3) *M_1 intersects M_2. Then M has property X with respect to G.*

Proof of Lemma 3.1. There is a subcontinuum M_2' of M_2 irreducible from K_1 to M_1. Since the connected sets $M_1 - M_1 \cdot K_2$ and $M_2' - M_2' \cdot K_1$ have a point in common, their sum is connected. Since $M_1 + M_2'$ is a subcontinuum of M intersecting every set of G, then $M_1 + M_2' = M$. But

$$(M_1 - M_1 \cdot K_2) + (M_2' - M_2' \cdot K_1) = (M_1 + M_2') - (M_1 + M_2') \cdot (K_1 + K_2)$$

$$= M - M \cdot (K_1 + K_2).$$

Hence M has property X with respect to G.

LEMMA 3.2. *Theorem* 3 *holds true if M is decomposable.*

Proof of Lemma 3.2. Suppose that for any nondegenerate proper subcollection G_1 of G there is a subcontinuum of M having property X with respect to G_1. It will be shown that if this condition is satisfied, then M has property X with respect to G. Then Lemma 3.2 will follow by induction from the following well known facts:

(1) Theorem 3 holds true if G consists of two closed sets and

(2) if $\mathbb{W}$ is a finite collection of mutually exclusive closed sets and K is a compact metric continuum intersecting every set of $\mathbb{W}$, then some subcontinuum of K is irreducible among the sets of $\mathbb{W}$.

In addition to the above supposition, suppose that M does not have property X with respect to G. Then there exist three subcontinua M_1, M_2, and M_3 of M and three sets K_1, K_2, and K_3 of G such that

(1) for each i ($i \leq 3$), M_i has property X with respect to the collection $G - K_i$ and does not intersect K_i and

(2) the set $M_1 - M_1 \cdot (K_2 + K_3)$ is connected. By Lemma 3.1, neither of the sets M_2 and M_3 intersects M_1. Hence $M_2 + M_3$ is a proper subset of M

intersecting every set of G. Then M_2 and M_3 have no point in common. For each i ($i \leq 3$), let p_i be a point of M_i. Since M is decomposable, some proper subcontinuum M' of M contains some two of the points p_1, p_2, and p_3 [2, Theorem 4]. As the three cases are quite similar, consider the case in which $p_1 + p_2$ is a subset of M'. Since $M' + M_1 + M_2$ is a continuum intersecting every set of G, then $M' + M_1 + M_2 = M$. Hence M_3 is a subset of M'. It follows, in the same manner, that each of the sets M_1 and M_2 is a subset of M'. Then M' intersects every set of G. This is contradiction since M is irreducible among the sets of G.

Proof of Theorem 3. If M is decomposable, then Theorem 3 follows from Lemma 3.2. Suppose that M is indecomposable. Then no composant of M intersects every set of G. Let L_1 be a set of G such that some composant of M does not intersect L_1. There is a set L_2 of G such that some composant of M intersects L_1 but not L_2. Let G' be the collection consisting of all elements of G that do not intersect every composant of M. By Theorem 2, some composant of M lies in $M - M \cdot G'^*$. Since this composant is connected and dense in M, then $M - M \cdot G'^*$ is connected. As it was shown above that G' consists of at least two closed sets, then M has property X with respect to G'. Hence M also has property X with respect to G.

THEOREM 4. *If, in the plane, K_1 and K_2 are mutually exclusive bounded continua, K_3 is a closed circular disc not intersecting $K_1 + K_2$, and M_1, M_2, and M_3 are mutually exclusive bounded continua each intersecting each of the sets K_1, K_2, and K_3 such that $M_i - M_i \cdot (K_1 + K_2)$ is connected for each i ($i \leq 3$), then one of the continua M_1, M_2, and M_3 intersects the interior of K_3.*

Proof of Theorem 4. F. B. Jones [3, Theorem 28] has proved that if J is the boundary of a simple domain D in the plane, H is connected set lying in $D + J$, and K is a continuum lying in $(D + J) - H$, then no two points of $H \cdot J$ separate two points of $K \cdot J$ from each other on J. Repeated use of this result will be made in the following argument.

Suppose that no one of the continua M_1, M_2, and M_3 intersects the interior of K_3. Let J_3 denote the boundary of K_3. There exists an arc of J_3 which intersects two but not three of the continua M_1, M_2, and M_3. Consider the case in which such an arc intersects both M_2 and M_3. By [6, Theorem 17, p. 189], there exist two points a_1 and a_2 of J_3 and an arc $a_1 a_2$ lying, except for a_1 and a_2, in the exterior of J_3 such that $a_1 + a_2$ separates $J_3 \cdot M_1$ from $J_3 \cdot (M_2 + M_3)$ on J_3 and $a_1 a_2$ does not intersect $M_1 + M_2 + M_3$. Hence $a_1 a_2$ intersects $K_1 + K_2$.

Therefore $a_1 a_2$ contains two arcs $a_1 b_1$ and $a_2 b_2$ irreducible from J_3 to $K_1 + K_2$. Since the set

$$K_1 + a_1 b_1 + a_2 b_2 - (b_1 + b_2)$$

does not intersect the connected set

$$K_2 + M_1 + M_2 - (M_1 \cdot K_1 + M_2 \cdot K_1),$$

then one of the points b_1 and b_2 does not lie in K_1. Similarly K_2 does not contain both of these points. Hence one of the points b_1 and b_2 belongs to K_1 and the other belongs to K_2. It follows from [**6**, Theorem 17, p. 189] that there exist two points c_1 and c_2 of J_3 and an arc $c_1 c_2$ lying, except for c_1 and c_2, in the exterior of J_3 such that c_1 and c_2 belong to M_1 and M_2 respectively and $c_1 c_2$ does not intersect $K_1 + K_2 + a_1 b_1 + a_2 b_2$. Let H be an arc in $c_1 c_2$ irreducible from M_1 to M_2. Since $a_1 + a_2$ separates $J_3 \cdot M_1$ from $J_3 \cdot M_2$ on J_3, then the connected set

$$H + M_1 + M_2 - (M_1 + M_2) \cdot (K_1 + K_2)$$

intersects the continuum

$$K_1 + K_2 + M_3 + a_1 b_1 + a_2 b_2 .$$

Hence H intersects M_3. Then there is an arc H' in H intersecting both M_1 and M_3 but not M_2. The connected set

$$H' + M_1 + M_3 - (M_1 + M_3) \cdot (K_1 + K_2)$$

does not intersect the continuum

$$K_1 + K_2 + M_2 + a_1 b_1 + a_2 b_2 .$$

This is contrary to [**3**, Theorem 28].

THEOREM 5. *If, in the plane, $\mathbb{W}$ is a collection consisting of n mutually exclusive bounded continua and G is a collection consisting of $n^2 - n + 1$ mutually exclusive continua each intersecting every continuum of $\mathbb{W}$, then some continuum of G contains a bounded continuum which intersects every continuum of $\mathbb{W}$.*

Proof of Theorem 5. Use the notation described in § 1. Let K be a closed

circular disc not intersecting $G^* + W^*$. By Theorem 1, there is an inversion T of the plane with respect to K such that the continua of G' are bounded and mutually exclusive and no one of them intersects the interior of K. The continua of W' are mutually exclusive and bounded and no one of them intersects K. Since Theorem 5 obviously holds true if $n = 1$, suppose that W' consists of at least two continua. Each continuum of G' contains a continuum irreducible among the sets of W'. Hence by Theorem 3, each continuum of G' contains a continuum which has property X with respect to W'. Since the number of continua of G' is one more than twice the number of distinct pairs of elements of elements of W', then there exist three mutually exclusive continua M_1', M_2', and M_3' and two continua K_1' and K_2' of W' such that

(1) for each i ($i \leq 3$), M_i' is a subset of a continuum of G' and intersects every continuum of W' and

(2) $M_i' - M_i' \cdot (K_1' + K_2')$ is connected. Since no one of the continua M_1', M_2', M_3' intersects the interior of K, it follows from Theorem 4 that some M_j' ($j \leq 3$) does not intersect K. Then $T^{-1}(M_j')$ is a bounded subcontinuum of some continuum of G and intersects every continuum of W.

3. Convergent sequences of continua. The author has previously shown that Theorem 6 holds true if $M_1, M_2, M_3, \cdots$ are compactly connected [**1**, Theorem 5]. As a consequence of Theorem 6, the requirement in the hypotheses of Theorems 7 and 8 of [**1**] that the continua of α be compactly connected can be omitted.

THEOREM 6. *If $M_1, M_2, M_3, \cdots$ is a sequence of mutually exclusive non-degenerate continua in the plane converging to a continuum M, then there is a sequence $T_1, T_2, T_3, \cdots$ of bounded continua converging to M such that for each n, T_n is a subset of M_n and is irreducible between some two points.*

With the aid of Theorem 5, this theorem can be proved by an argument quite similar to the argument given to prove Theorem 5 of [**1**].

4. A property of a certain type of unbounded continuum. R. L. Moore [**5**] has shown that the plane does not contain uncountably many mutually exclusive triodic continua. N. E. Rutt [**8**] has shown that if M is a continuum containing two unbounded continua neither of which contains the other, and M does not separate the plane, then the plane does not contain uncountably many mutually exclusive continua such as M. There exist continua which satisfy both the hypothesis of Rutt's theorem and the hypothesis of Theorem 7. However, in

Theorem 7 it is not required that M should not separate the plane. It should also be noted that there exist continua which satisfy the hypothesis of Rutt's theorem but do not satisfy the hypothesis of Theorem 7. (See [7, Example 1].)

THEOREM 7. *Let M be a continuum having the property that for some positive integer n there exist n domains intersecting M such that no bounded subcontinuum of M intersects each of these domains. The plane does not contain uncountably many mutually exclusive continua such as M.*

Proof of Theorem 7. Suppose that there exists a collection G consisting of uncountably many mutually exclusive such continua in the plane. There exist an uncountable subcollection G_1 of G and a positive integer r such that if Y is a continuum of G_1 then there exist r domains each intersecting Y such that no bounded subcontinuum of Y intersects each of these domains. Then there exist a sequence $M_1, M_2, M_3, \cdots$ of distinct continua of G_1 and a positive number d such that

(1) for each i, there exist r points $p_{i1}, p_{i2}, \cdots, p_{ir}$ of M_i and r mutually exclusive circular domains $D_{i1}, D_{i2}, \cdots, D_{ir}$ having the points $p_{i1}, p_{i2}, \cdots, p_{ir}$ respectively as centers and having a diameter greater than d,

(2) for each j ($j \leq r$), the sequence $p_{1j}, p_{2j}, p_{3j}, \cdots$ converges to a point p_j, and

(3) for each i, no bounded subcontinuum of M_i intersects each of the domains $D_{i1}, D_{i2}, \cdots, D_{ir}$. There exist mutually exclusive circular discs $K_1, K_2, \cdots, K_r$ such that for each j ($j \leq r$), K_j has p_j as its center and has a diameter less than $d/2$. There is a positive integer m such that for each j ($j \leq r$) and each n ($n > m$), K_j is a subset of D_{nj}. Since infinitely many of the continua $M_1, M_2, M_3, \cdots$ intersect each of the circular discs $K_1, K_2, \cdots, K_r$, then by Theorem 5, there is a positive integer n greater than m such that some bounded subcontinuum H_n of M_n intersects each of the circular discs $K_1, K_2, \cdots, K_r$. Hence H_n also intersects each of the domains $D_{n1}, D_{n2}, \cdots, D_{nr}$. This involves a contradiction.

REFERENCES

1. C. E. Burgess, *Continua and their complementary domains in the plane*, Duke Math. J. **18** (1951), 901-917.

2. S. Janiszewski and C. Kuratowski, *Sur les continus indécomposables*, Fund. Math. **1** (1920), 210-222.

3. F. Burton Jones, *Certain consequences of the Jordan curve theorem*, Amer. J. Math. **63** (1941), 531-544.

4. R. G. Lubben, *Separation theorems with applications to questions concerning accessibility and plane continua*, Trans. Amer. Math. Soc. **31** (1929), 503-522.

5. R. L. Moore, *Concerning triods in the plane and the junction points of plane continua*, Proc. Nat. Acad. Sci. **14** (1928), 85-88.

6. ———, *Foundations of point set theory*, Amer. Math. Soc. Colloquium Publications, **13**, 1932.

7. J. H. Roberts, *Concerning atriodic continua*, Monatsh. Math. **37** (1930), 223-230.

8. N. E. Rutt, *Some theorems on triodic continua*, Amer. J. Math. **56** (1934), 122-132.

THE UNIVERSITY OF UTAH