AN INEQUALITY FOR SETS OF INTEGERS

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Small italics denote nonnegative integers. Let $A = \{a\}$, $B = \{b\}, \cdots$ be sets of such integers. Define $A + B = \{a + b\}$ and put

$$A(n) = \sum_{0 < a \leq n} 1 \quad \text{and} \quad A(m, n) = \sum_{m < a \leq n} 1.$$

Thus

$$A(n) = A(0, n) \text{ and } A(m, n) = A(n) - A(m) \text{ if } m \leq n.$$

The following estimate is well known:

LEMMA. *If* $m < k < n$, $n \notin A + B$, *then*

$$(1) \qquad k - m \geq A(n - k - 1, n - m - 1) + B(m, k).$$

Proof. If $b = n - a$, then $n = a + b \in A + B$. Hence the $A(n - k - 1, n - m - 1)$ numbers $n - a$ with $m < n - a \leq k$ and the $B(m, k)$ numbers b satisfying $m < b \leq k$ are mutually distinct. The right hand term of (1) gives their total number. It is not greater than the number $k - m$ of all the integers z with $m < z \leq k$.

The most important result on $A + B$ is due to Mann [2]: Let $n \notin C = A + B$. Then there exists an m satisfying $0 \leq m < n$ and $n - m \notin C$ such that

$$C(m, n) \geq A(n - m - 1) + B(n - m - 1).$$

I wish to prove a less well known inequality which is implicitly contained in [4] and in a paper by Mann [3]. The present proof uses an idea by Besicovitch and is rather simpler than Mann's method [cf. 1].

THEOREM 1. *Let*

$$(2) \qquad x \in A \qquad (x = 0, 1, 2, \cdots, h;\ h \geq 0),$$

Received December 29, 1953.
Pacific J. Math. 5 (1955), 585-587

(3) $$0 \in B \quad or \quad 1 \in B,$$

(4) $$A \dotplus B \subset C, \ n \notin C.$$

Finally let

(5) $$C(n) < A(n-1) + B(n).$$

Then there is an m satisfying

(6) $$m \notin C, \ 0 < m < n-h-1$$

such that

(7) $$C(m,n) \geq A(n-m-1) + B(m,n).$$

We note that (7) is trivial but useless without the second half of (6). Obviously, (2)-(4) imply $m > h$ if $0 \in B$ and $m > h+1$ if $1 \in B$.

Proof. Instead of (3), we merely use the weaker assumption that B is not empty. Let b_0 denote the largest $b \leq n$. Thus $B(b_0, n) = 0$. Since C contains the integers $b_0 + a$ with $0 < a \leq n - b_0$, we have

(8) $$C(b_0,n) \geq A(n-b_0) \geq A(n-b_0-1) = A(n-b_0-1) + B(b_0,n).$$

From (5) and (8), $b_0 > 0$. By (2), the numbers $b_0, b_0+1, \cdots, b_0+h$ lie in $A+B \subset C$. Hence $n \notin C$ implies $b_0 \leq n-h-1$. Thus

(9) $$0 < b_0 \leq n-h-1.$$

By (2), $b_0 \in C$. Let m denote the greatest $z < b_0$ with $z \notin C$. If no such z exists, put $m = 0$. Applying (1) with $k = b_0$, we obtain

(10) $$C(m,b_0) = b_0 - m \geq A(n-b_0-1,\ n-m-1) + B(m,b_0).$$

Adding (8) and (10), we obtain

$$C(m,b_0) + C(b_0,n) \geq A(n-b_0-1) + A(n-b_0-1,\ n-m-1)$$
$$+ B(m,b_0) + B(b_0,n),$$

that is (7). By (7) and (5), $m > 0$. Hence $m \notin C$. Finally (9) and $m < b_0$ imply $m < n-h-1$.

The following corollary of Theorem 1 was proved in a different way by Mann.

THEOREM 2. *Suppose the sets A, B, C satisfy the assumptions* (2)-(4). *Let* $0 < \alpha_1 < 1$ *and*

$$A(x) \geq \alpha_1(x+1) \qquad (x = h+1,\ h+2, \cdots, n). \tag{11}$$

Then

$$C(n) \geq \alpha_1 n + B(n). \tag{12}$$

Proof. By (2), $0 \in A$. Furthermore, (11) and (2) imply $1 \in A$. Hence, (3) implies $1 \in C$. Thus our theorem is true for $n = 1$. Suppose it is proved up to $n - 1 \geq 1$.

If $C(n) \geq A(n-1) + B(n)$, then (11) with $x = n - 1$ yields (12). Thus we may assume (5). Choose m according to Theorem 1. By (6), $n - m - 1 \geq h + 1$. Hence, by (7), (11), and our induction assumption

$$\begin{aligned} C(n) &\geq C(m) + A(n-m-1) + B(m,n) \\ &\geq C(m) + \alpha_1(n-m) + B(m,n) \\ &\geq \alpha_1 m + B(m) + \alpha_1(n-m) + B(m,n) = \alpha_1 n + B(n). \end{aligned}$$

The case $h = 0$ of Theorem 2 is due to Besicovitch [**1**]. Obviously, this theorem can be extended to the case that $0 \notin B$, $B(n) > 0$.

A recent result by Stalley also follows readily from Theorem 1.

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