RELATIVIZATION AND EXTENSION OF SOLUTIONS OF IRREFLEXIVE RELATIONS

MOSES RICHARDSON

1. Introduction. Let $\succ$ be an irreflexive binary relation defined over a domain $\mathfrak{D}$ of elements $a, b, c, \cdots$. We represent the system $(\mathfrak{D}, \succ)$ by an oriented graph G by regarding the elements of $\mathfrak{D}$ as vertices of G and inserting an arc ab of the graph, oriented from a to b, if and only if $a \succ b$. The sentence "$a \succ b$" is read "a dominates b". A set V of vertices is termed *internally satisfactory*[1] if and only if $x \in V$ and $y \in V$ implies $x \not\succ y$. A set V of vertices is termed *externally satisfactory* if and only if $y \in \mathfrak{D} - V$ implies that there exists an $x \in V$ such that $x \succ y$. A set V of vertices is termed a *solution* of G, or of $(\mathfrak{D}, \succ)$, if and only if it is both internally and externally satisfactory. In [4], various sufficient conditions for the existence of solutions were established.

By a *subsystem* $(\mathfrak{D}_0, \succ)$ of the system $(\mathfrak{D}, \succ)$ is meant a system where $\mathfrak{D}_0 \subset \mathfrak{D}$ and the relation $\succ$ for the subsystem is merely the restriction of the relation $\succ$ for the supersystem $(\mathfrak{D}, \succ)$. Let G_0 be the graph of the subsystem $(\mathfrak{D}_0, \succ)$ and let V_0 be a solution of G_0. A solution V of G is termed an *extension* of V_0 if $V \cap \mathfrak{D}_0 = V_0$; in this case V_0 is also said to be *relativized* from V. In this paper, some sufficient conditions for the existence of relativizations and extensions of solutions are presented. More elegant and more effective extension theorems, especially with a view toward possible applications to the theory of n-person games, remain to be desired. It is hoped that the present paper may serve to stimulate interest in this apparently difficult problem.

2. A theorem on relativization. If H is a subgraph of the graph G, then the graph obtained by adding to H all the arcs of G which join pairs of vertices of H will be termed the *juncture* of H (relative to G) and will be denoted by $\overline{H}$.

[1] In [2], internally satisfactory is called satisfactory with respect to non-domination, and in [4] it is called $\not\succ$-satisfactory.

Received March 1, 1954. Part of the work of this paper was done at the Institute for Advanced Study in 1952-3, and part while the author was consultant to the Logistics Project sponsored by the Office of Naval Research in the Department of Mathematics at Princeton University in 1953-4. A statement of many of the results contained herein appeared without proofs in [5].

Pacific J. Math. 5 (1955), 551-584

H is termed a *conjunct* subgraph of G if and only if $H = \overline{H}$.

The graph G_0 of a subsystem $(\mathfrak{D}_0, \succ)$ of the system $(\mathfrak{D}, \succ)$ having the graph G is a conjunct closed subgraph of G. If H is any subgraph of G, proper or not, and x is any vertex of G, then $D^{-1}(x, H)$ shall denote the set of all vertices y of H such that $y \succ x$. If X is any set of vertices of G, let

$$D^{-1}(X, H) = \bigcup_{x \in X} D^{-1}(x, H),$$

and let

$$D^{-n}(X, H) = D^{-1}(D^{-n+1}(X, H), H)$$

for $n > 1$. Let $D^0(X, H) = X$ by definition.

THEOREM 1. *If G_0 is a conjunct subgraph of G and V is a solution of G, then a sufficient condition for $V \cap \mathfrak{D}_0$ to be a solution of G_0, where $\mathfrak{D}_0$ is the set of vertices of G_0, is that*

$$(1) \qquad D^{-1}(y, G) \subset \mathfrak{D}_0 \qquad \textit{for every } y \in \mathfrak{D}_0 - V \cap \mathfrak{D}_0.$$

Proof. We must prove that $V \cap \mathfrak{D}_0$ is both internally and externally satisfactory with respect to G_0. That is we must prove that

(a) $x, y \in V \cap \mathfrak{D}_0$ implies $x \not\succ y$ relative to G_0, and

(b) $y \in \mathfrak{D}_0 - V \cap \mathfrak{D}_0$ implies that there exists an $x \in V \cap \mathfrak{D}_0$ such that $x \succ y$ relative to G_0.

But (a) follows immediately from the facts that G_0 is a conjunct subgraph of G and that V is a solution of G. To prove (b), consider any $y \in \mathfrak{D}_0 - V \cap \mathfrak{D}_0$. There exists an $x \in V$ such that $x \succ y$ relative to G since V is a solution of G. Then $x \in D^{-1}(y, G) \subset \mathfrak{D}_0$ by hypothesis. Thus $x, y \in \mathfrak{D}_0$ and the oriented arc $xy \subset G$. Since G_0 is a conjunct subgraph of G, arc $xy \subset G_0$. This completes the proof.

REMARK. It would suffice to replace Condition (1) by the weaker condition: $y \in \mathfrak{D}_0 - V \cap \mathfrak{D}_0$ implies that there exists a vertex $x \in V \cap \mathfrak{D}_0$ such that $x \succ y$.

3. An extension theorem. If $X \subset \mathfrak{D}$, let the *predecessor-set* of X relative to $G - G_0$ denote the set

$$P(X, G - G_0) = \bigcup_{n=1}^{\infty} D^{-n}(X, G - G_0).$$

By a *predecessor-sequence* $p(x_0, G - G_0)$ of $x_0 \in \mathfrak{D}_0$ relative to $G - G_0$ is meant a maximal regression[2] $x_0, x_1, x_2, \cdots,$ of finite or infinite length, such that all its vertices except possibly x_0 itself are in $G - G_0$; that is, such that one vertex x_n is chosen from the set $D^{-1}(x_{n-1}, G - G_0)$ for each $n > 0$, all x_n's being distinct. Let $p^*(x_0, G - G_0)$ be the set of all vertices of the predecessor-sequence $p(x_0, G - G_0)$ other than x_0 itself. A predecessor-sequence is termed *trivial* if and only if $p^*(x_0, G - G_0)$ is empty. We have

$$P(x_0, G - G_0) = \bigcup p^*(x_0, G - G_0)$$

for all predecessor-sequences $p(x_0, G - G_0)$ of x_0 relative to $G - G_0$. Note that the elements of the predecessor-set of x_0 or of a predecessor-sequence of x_0 are not necessarily ancestors of x_0, although every ancestor of x_0 belongs to at least one predecessor-sequence of x_0 (all relative to $G - G_0$). If $\succ$ is not asymmetric then a source, which has no ancestor, may have non-trivial predecessor-sequences.

Throughout the sequel we suppose that G_0 is the graph of a subsystem $(\mathfrak{D}_0, \succ)$ of the system $(\mathfrak{D}, \succ)$ the graph of which is G, that V_0 is a given solution of G_0, and that[3]

$$W_{00} = D(V_0, G_0) = \mathfrak{D}_0 - V_0.$$

THEOREM 2. *Suppose that:*

(1) *All non-trivial predecessor-sequences* $p(x_0, G - G_0)$, $x_0 \in \mathfrak{D}_0$, *are either infinite or, if finite, of odd length if* $x_0 \in W_{00}$ *and of even length if* $x_0 \in V_0$;

(2) $D(V_0, G) \cap D^{-2n}(V_0, G - G_0) = D(V_0, G) \cap D^{-2n+1}(W_{00}, G - G_0) = 0$ *for all* $n > 0$;

(3) *If* $h > 0$ *and* $k > 0$ *are of the same parity then*

$$D^{-h}(V_0, G - G_0) \cap D^{-k}(W_{00}, G - G_0) = 0,$$

and if $h > 0$ *and* $k > 0$ *are of different parities then*

$$D^{-h}(V_0, G - G_0) \cap D^{-k}(V_0, G - G_0) = D^{-h}(W_{00}, G - G_0) \cap D^{-k}(W_{00}, G - G_0) = 0;$$

[2] See [4] for definitions omitted here.

[3] This is a slight modification of the notation of [4].

(4) $\mathfrak{D} - \mathfrak{D}_0 \subset P(\mathfrak{D}_0, G - G_0)$.

Then a solution V of G which is an extension of V_0 exists.

Proof. Let

$$V = V_0 \cup \left(\bigcup_{n=1}^{\infty} D^{-2n}(V_0, G - G_0) \right) \cup \left(\bigcup_{n=1}^{\infty} D^{-2n+1}(W_{00}, G - G_0) \right),$$

$$W = W_{00} \cup \left(\bigcup_{n=1}^{\infty} D^{-2n+1}(V_0, G - G_0) \right) \cup \left(\bigcup_{n=1}^{\infty} D^{-2n}(W_{00}, G - G_0) \right).$$

We shall show that V is a solution of G. Since G_0 is a conjunct subgraph of G and V_0 is a solution of G_0, it follows that V_0 is internally satisfactory relative to G. By (4), $\mathfrak{D} = V \cup W$. By (3), $V \cap W = 0$; hence $W = \mathfrak{D} - V$. We have only to prove:

(a) $$V \cap D(V, G) = 0;$$

(b) $$W \subset D(V, G).$$

Proof of (a). If $x \in V_0$, $y \in V_0$, then $x \not\succ y$ since V_0 is internally satisfactory relative to G.

If $x \in V_0$, $y \in D^{-2n}(V_0, G - G_0)$, then $x \not\succ y$ by (2).

If $x \in V_0$, $y \in D^{-2n+1}(W_{00}, G - G_0)$, then $x \not\succ y$ by (2).

If $x \in D^{-2n}(V_0, G - G_0)$, $y \in V_0$, then $x \not\succ y$; for $x \succ y$ would imply that $x \in D^{-1}(V_0, G - G_0)$ contrary to (3).

If $x \in D^{-2n}(V_0, G - G_0)$, $y \in D^{-2m}(V_0, G - G_0)$, then $x \not\succ y$; for $x \succ y$ would imply that $x \in D^{-2m-1}(V_0, G - G_0)$ contrary to (3).

If $x \in D^{-2n}(V_0, G - G_0)$, $y \in D^{-2m+1}(W_{00}, G - G_0)$, then $x \not\succ y$; for $x \succ y$ would imply that $x \in D^{-2m}(W_{00}, G - G_0)$ contrary to (3).

If $x \in D^{-2m+1}(W_{00}, G - G_0)$, $y \in V_0$, then $x \not\succ y$; for $x \succ y$ would imply that $x \in D^{-1}(V_0, G - G_0)$, contrary to (3).

If $x \in D^{-2m+1}(W_{00}, G - G_0)$, $y \in D^{-2n}(V_0, G - G_0)$, then $x \not\succ y$; for $x \succ y$ would imply that $x \in D^{-2n-1}(V_0, G - G_0)$ contrary to (3).

If $x \in D^{-2m+1}(W_{00}, G - G_0)$, $y \in D^{-2n+1}(W_{00}, G - G_0)$, then $x \not\succ y$; for $x \succ y$ would imply that $x \in D^{-2n}(W_{00}, G - G_0)$ contrary to (3).

Proof of (b). If $y \in W_{00}$, then there exists an $x \in V_0$ such that $x \succ y$.

If $y \in D^{-2n+1}(V_0, G - G_0)$, then there exists an $x \in D^{-2n}(V_0, G - G_0)$ such that $x \succ y$, since y belongs to some predecessor-sequence $p(x_0, G - G_0)$ of some $x_0 \in V_0$ and such a predecessor-sequence is infinite or of even length by (1).

If $y \in D^{-2n}(W_{00}, G - G_0)$, then there exists an $x \in D^{-2n-1}(W_{00}, G - G_0)$ such that $x \succ y$, since y belongs to some predecessor-sequence $p(x_0, G - G_0)$ of some $x_0 \in W_{00}$, and such a predecessor-sequence is infinite or of odd length by (1). This completes the proof.

COROLLARY. *Suppose Conditions* (1) *and* (4) *of the theorem above, and that:*

(a) *No vertex of any* $P(x_0, G \dot{-} G_0)$, $x_0 \in \mathfrak{D}_0$, *is adjacent to any vertex of* $\mathfrak{D}_0$ *other than* x_0; *and if* x_0 *and* x_0' *are distinct vertices of* $\mathfrak{D}_0$ *then*

$$P(x_0, G - G_0) \cap P(x_0', G - G_0) = 0;$$

(b) *No* $P(x_0, G - G_0) \cup (x_0)$, $x_0 \in \mathfrak{D}_0$, *contains an odd unoriented cycle.*

Then a solution V *of* G *which is an extension of* V_0 *exists.*

Proof. We have to show that the hypotheses of the corollary imply those of the theorem. It will suffice to show that if either (2) or (3) are false then either (a) or (b) will be violated.

If (2) were false, there would exist either a vertex

$$x \in D(V_0, G) \cap D^{-2n}(V_0, G - G_0)$$

or a vertex

$$y \in D(V_0, G) \cap D^{-2n+1}(W_{00}, G - G_0).$$

In either case, the first part of (a) or (b) is contradicted.

If (3) were false there would exist either

(i) a vertex

$$x \in D^{-h}(v_0^i, G - G_0) \cap D^{-k}(w_{00}^j, G - G_0)$$

with h and k of the same parity or

(ii) a vertex y such that either

$$y \in D^{-h}(v_0^i, G - G_0) \cap D^{-k}(v_0^j, G - G_0)$$

or

$$y \in D^{-h}(w_{00}^i, G - G_0) \cap D^{-k}(w_{00}^j, G - G_0)$$

with h and k of different parities.

In Case (i), Condition (a) would be violated. In Case (ii), (a) implies $i = j$. But then $P(v_0^i, G - G_0) \cup (v_0^i)$ or $P(w_{00}^i, G - G_0) \cup (w_{00}^i)$ would contain an unoriented cycle of odd length $h + k$ contrary to (b).

4. Sinks and inverse bases. We suppose henceforth that $\mathfrak{D} - \mathfrak{D}_0 \subset P(\mathfrak{D}_0, G - G_0)$. If H is any conjunct subgraph of G, and x is a vertex of G, let

$$C^{-1}(x, H) = \bigcup_{n=0}^{\infty} D^{-n}(x, H).$$

That is, $C^{-1}(x, H)$ denotes the set of all vertices y of H which chain-dominate x by means of a chain all the vertices of which, except possibly x, lie in H, together with x itself; in symbols

$$C^{-1}(x, H) = P(x, H) \cup (x).$$

If $y \in C^{-1}(x, H)$ and $x \in C^{-1}(y, H)$, $x \neq y$, then x and y are termed *cyclically related* relative to H. If $y \in C^{-1}(x, H)$ but $x \notin C^{-1}(y, H)$ then x is termed a *descendant* of y relative to H. A sequence $x_1, x_2, x_3, \cdots$ of vertices of H is termed a *descending sequence* of H if x_{n+1} is a descendant of x_n for all n (except the last n if the sequence is finite) and if there exists no vertex y which is a descendant of all x_n. If a vertex x of H has no descendant relative to H then $C^{-1}(x, H)$ is termed an *inverse basic set* of H and x is termed a *sink* of this inverse basic set. A subgraph H is termed *descendingly finite* if every descending sequence of H is finite. The same inverse basic set may contain more than one sink; all sinks of the same inverse basic set are cyclically related relative to H, and any vertex cyclically related to a sink is a sink of the same inverse basic set of H.

LEMMA 1.[4] *If H is descendingly finite then every vertex of H belongs to some inverse basic set of H.*

Proof. Let x_1 be any vertex of H. Each descending sequence $x_1, x_2, x_3, \cdots$ of H beginning with x_1 has a last element x_λ. Then

$$x_1 \in C^{-1}(x_2, H),\ x_2 \in C^{-1}(x_3, H), \cdots, x_{\lambda-1} \in C^{-1}(x_\lambda, H)$$

but

$$x_2 \notin C^{-1}(x_1, H),\ x_3 \notin C^{-1}(x_2, H), \cdots, x_\lambda \notin C^{-1}(x_{\lambda-1}, H).$$

Hence

$$C^{-1}(x_1, H) \subset C^{-1}(x_2, H) \subset \cdots \subset C^{-1}(x_\lambda, H)$$

and

$$C^{-1}(x_\lambda, H) = \bigcup_{i=1}^{\lambda} C^{-1}(x_i, H)$$

is an inverse basic set containing x_1 of which x_λ is a sink.

LEMMA 2. *If H is descendingly finite, no proper subset B of an inverse basic set A is an inverse basic set.*

Proof. Suppose contrarywise that B were an inverse basic set and a proper subset of A. Let b be a sink of B and a a sink of A. Then $B = C^{-1}(b, H)$ and $A = C^{-1}(a, H)$. Since B is a *proper* subset of A, $b \neq a$ and $b \in C^{-1}(a, H)$. Since the sink b can have no descendant relative to H, we have $a \in C^{-1}(b, H)$, otherwise a would be a descendant of b. Then $C^{-1}(a, H) \subset C^{-1}(b, H)$, or $A \subset B$. Therefore $A = B$ contrary to hypothesis.

By an *inverse basis* of H is meant a set S of vertices of H such that (a) $x \in S$, $y \in S$, $x \neq y$, implies that x is not chain-dominated by y relative to H,

[4]Lemmas 1-5 are duals, in an obvious sense, of Lemmas 1-5 of [**4**] which are in turn generalizations of theorems of König [**1**, pp. 88-90], for finite graphs. Lemma 2 of [**4**, p. 581] should be corrected by adding to its statement "if B has a source", and deleting from the proof all mention of Case (c); this change does not affect the rest of [**4**].

and (b) $y \in H \cap \mathfrak{D} - S$ implies that there exists a vertex x of S such that x is chain-dominated by y relative to H (that is, $y \in C^{-1}(x, H)$).

LEMMA 3. *Every descendingly finite subgraph H has an inverse basis.*

Proof. Let the distinct inverse basic sets of H be $B_1, B_2, \cdots$, where $B_i \neq B_j$ for $i \neq j$. (The range of i and j is any lower segment of ordinal numbers, finite or not.) By Lemma 1, every vertex of H belongs to at least one B_i. Let b_i be a sink of B_i. Then no b_i chain-dominates b_j, $i \neq j$. For, if so, $b_i \in C^{-1}(b_j, H)$. Then b_i has b_j as a descendant unless $b_j \in C^{-1}(b_i, H)$; that is, unless b_i and b_j are cyclically related relative to H. In this case,

$$C^{-1}(b_i, H) \subset C^{-1}(b_j, H) \text{ and } C^{-1}(b_j, H) \subset C^{-1}(b_i, H);$$

that is, $B_i = B_j$, a contradiction. Let S be the set of b_i's just chosen, consisting of one sink from each inverse basic set B_i. It has just been shown that Condition (a) of the definition of inverse basis is satisfied by S. That Condition (b) is satisfied follows immediately from Lemma 1.

LEMMA 4. *If H has an inverse basis S and $b_i \in S$, then $C^{-1}(b_i, H)$ is an inverse basic set of which b_i is a sink.*

Proof. If not, b_i has a descendant p in H. That is,

$$b_i \in C^{-1}(p, H) \text{ but } p \notin C^{-1}(b_i, H).$$

Since $p \in H \cap \mathfrak{D}$, there exists a vertex b_j of S such that $p \in C^{-1}(b_j, H)$. Now, $b_j \neq b_i$ since $p \notin C^{-1}(b_i, H)$. Hence b_i chain-dominates p which chain-dominates b_j, so that b_i chain-dominates b_j since chain-domination is transitive. This contradicts the fact that b_i and b_j both belong to the inverse basis S.

LEMMA 5. *Every inverse basis S of a descendingly finite subgraph H consists of one sink from each inverse basic set of H.*

Proof. By Lemma 4, each vertex of S is a sink of some inverse basic set. Two distinct vertices of S cannot both be sinks of the same inverse basic set since, if so, they would be chain-dominated by each other. There remains only to show that every inverse basic set has a sink in the given basis S. Suppose B were an inverse basic set none of the sinks of which were in S. Let b be a sink of B. Since b is not in S, there exists a vertex b' of S such that b chain-dominates b'. Hence $C^{-1}(b, H) \subset C^{-1}(b', H)$. But b has no descendant relative to H since b is a sink. Therefore b' and b must be cyclically related relative to H since, if not, b' would be a descendant of b. Therefore $C^{-1}(b',H) \subset C^{-1}(b,H)$, so that $C^{-1}(b',H) = C^{-1}(b,H) = B$. Then b' is a sink of B which does lie in S.

5. Progressively finite graphs. A graph H is termed *completely descendingly finite* if and only if all its closed subgraphs are descendingly finite. A sequence $\{x_n\}$ of vertices of H is termed a *progression* of H if and only if $x_n \succ x_{n+1}$, and $\mathrm{Cl}(x_n x_{n+1}) \subset H$ for all n (except the last if the sequence is finite). H is termed *progressively finite* if and only if all the progressions of H are finite.

LEMMA 6. *A necessary and sufficient condition that H be completely descendingly finite is that H be progressively finite.*

Proof. If H is progressively finite then it is descendingly finite. If H is progressively finite then every closed subgraph of H is progressively finite. Hence if H is progressively finite then it is completely descendingly finite.

If H is completely descendingly finite, there can exist no infinite progression $x_1 \succ x_2 \succ \cdots \succ x_n \succ \cdots$. For, if so, the subgraph consisting of the vertices x_i and the oriented arcs $x_i x_{i+1}$ $(i = 1, 2, 3, \cdots)$ would constitute a closed subgraph which would not be descendingly finite. This completes the proof.

For example, the graph G of Figure 1 is descendingly finite but not completely descendingly finite since $G - \mathrm{St}(y)$ is an infinite progression.

We suppose henceforth that $\mathrm{Cl}(G - G_0)$ *is progressively finite, where G_0 is a conjunct closed subgraph of G having the solution V_0. Let*[5]

$$W_{00} = D(V_0, G_0) \quad \text{and} \quad W_0 = D(V_0, G) \cup D^{-1}(V_0, G - G_0).$$

Let

$$G_{-1} = G - \mathrm{St}(V_0 \cup W_0).$$

Let V_{-1} be an inverse basis of G_{-1} which exists by Lemma 3. For each finite ordinal number $k \geq 1$, let

$$W_{-k} = D(V_{-k}, G_{-k}) \cup D^{-1}(V_{-k}, G_{-k}),$$

[5]This is a slight modification of the notation of [4].

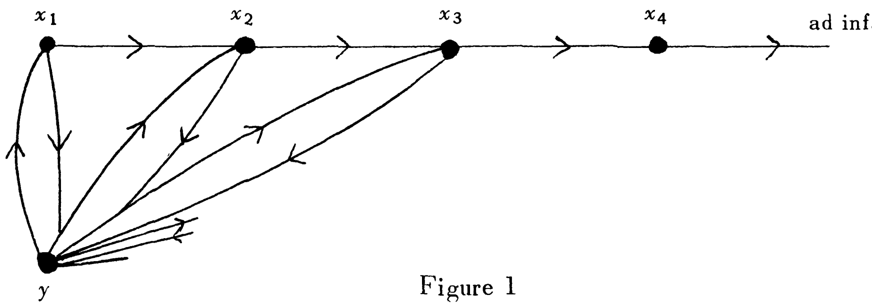


Figure 1

and

$$G_{-k-1} = G - \mathrm{St}\left[\bigcup_{0 \leq i \leq k} (V_{-i} \cup W_{-i})\right] = G_{-k} - \mathrm{St}(V_{-k} \cup W_{-k})$$

and let V_{-k-1} be an inverse basis of G_{-k-1}.

LEMMA 7. *G_{-k-1} is a conjunct subgraph of G for all $k \geq 0$.*

Proof. Any arc of G not in G_{-k-1} lies in

$$\mathrm{St}\left[\bigcup_{i \leq k} (V_{-i} \cup W_{-i})\right]$$

and hence has at least one endpoint in this star. Thus if x and y are vertices of G_{-k-1} and $x \succ y$ relative to G then $x \succ y$ relative to G_{-k-1} since arc xy cannot lie in the star while both endpoints are in G_{-k-1}.

LEMMA 8. *For all $k \geq 0$,*

$$\bigcup_{0 \leq i \leq k+1} V_{-i}$$

is internally satisfactory.

Proof. We prove the lemma by mathematical induction.

For $k = 0$, we must prove that

$$(V_0 \cup V_{-1}) \cap D(V_0 \cup V_{-1}, G) = 0.$$

(1) $x, y \in V_0$ implies $x \not\succ y$ relative to G; for $x \not\succ y$ relative to G_0 since V_0 is a solution of G_0 and G_0 is a conjunct subgraph of G.

(2) $x \in V_0$, $y \in V_{-1}$ implies $x \not\succ y$ relative to G; for $D(V_0, G - G_0) \cap G_{-1} = 0$ by definition of G_{-1} while $V_{-1} \subset G_{-1}$.

(3) $x \in V_{-1}$, $y \in V_0$ implies $x \not\succ y$ relative to G; for $D^{-1}(V_0, G - G_0) \cap G_{-1} = 0$ by definition of G_{-1} while $V_{-1} \subset G_{-1}$.

(4) $x, y \in V_{-1}$ implies $x \not\succ y$ relative to G; for V_{-1} is an inverse basis of G_{-1} which implies $x \not\succ y$ relative to G_{-1} while G_{-1} is a conjunct subgraph of G by Lemma 7.

Assuming that $\bigcup_{i \leq k} V_{-i}$ is internally satisfactory, we complete the proof by showing:

(a)
$$V_{-k-1} \cap D\left(\bigcup_{i \leq k} V_{-i}, G\right) = 0;$$

(b)
$$V_{-k-1} \cap D(V_{-k-1}, G) = 0;$$

(c)
$$\left(\bigcup_{i \leq k} V_{-i}\right) \cap D(V_{-k-1}, G) = 0.$$

If

$$x \in \bigcup_{i \leq k} V_{-i},\ y \in V_{-k-1},$$

then $x \not\succ y$; for if $x \succ y$ then

$$y \in \bigcup_{j \leq k} W_{-j}$$

and $y \notin G_{-k-1}$, while $V_{-k-1} \subset G_{-k-1}$. This proves (a). Since G_{-k-1} is a conjunct subgraph of G, $x \succ y$ relative to G, where $x, y \in V_{-k-1}$, would imply $x \succ y$ relative to G_{-k-1}, contrary to the definition of inverse basis. This proves (b). If

$$x \in V_{-k-1},\ y \in \bigcup_{i \leq k} V_{-i},$$

then $x \not\succ y$; for if $x \succ y$ then

$$x \in \bigcup_{i \leq k} D^{-1}(V_{-i}, G_{-i}) \subset \bigcup_{i \leq k} W_{-i}$$

so that $x \notin G_{-k-1}$, a contradiction. This completes the proof.

It may happen that $G_{-n} = 0$ for no finite ordinal n, in which case we may let

$$G_{-\omega} = G - \mathrm{St}\left[\bigcup_{i < \omega} (V_{-i} \cup W_{-i})\right],$$

$V_{-\omega}$ = any inverse basis of $G_{-\omega}$, and

$$W_{-\omega} = D(V_{-\omega}, G_{-\omega}) \cup D^{-1}(V_{-\omega}, G_{-\omega}),$$

and so on. Transfinite induction shows that if β is an ordinal number for which

$V_{-\alpha}$ is nonempty for all $\alpha < \beta$ then

$$\bigcup_{\alpha<\beta} V_{-\alpha}$$

is internally satisfactory. Let the cardinal number of the set $\mathfrak{D}$ be $\aleph_\mu$. Let λ be the next largest ordinal after those of $\mathfrak{Z}(\aleph_\mu)$ where $\mathfrak{Z}(\aleph_\mu)$ is the set of all ordinal numbers of well-ordered sets having cardinal number $\aleph_\mu$. Then no matter how we well-order the elements of $\mathfrak{D}$, its ordinal number is $< \lambda$. Well-order them as follows:

$$\overbrace{x_1, x_2, \cdots}^{V_0} \quad \overbrace{x_\alpha, x_{\alpha+1}, \cdots}^{W_0} \quad \overbrace{x_\beta, x_{\beta+1}, \cdots}^{V_{-1}} \quad \overbrace{x_\gamma, x_{\gamma+1}, \cdots}^{W_{-1}} \quad \cdots$$

$$\overbrace{x_\delta, x_{\delta+1}, \cdots}^{V_{-\omega}} \quad \overbrace{x_\epsilon, x_{\epsilon+1}, \cdots}^{W_{-\omega}} \quad \cdots.$$

Then every vertex of $\mathfrak{D}$ is in some $V_{-\zeta}$ or some $W_{-\zeta}$ with $\zeta < \lambda$. Let κ be the lowest ordinal for which $G_{-\kappa} = 0$. Then every vertex of G is ultimately used up in some $V_{-\zeta}$ or $W_{-\zeta}$, $\zeta < \kappa$. We have then the following theorems in which we let

$$V = \bigcup_{0 \le \alpha < \kappa} V_{-\alpha}:$$

THEOREM 3. *If V_0 is a solution of the subsystem $(\mathfrak{D}_0, \succ)$ of the system $(\mathfrak{D}, \succ)$, and if the graph $\mathrm{Cl}(G - G_0)$ is progressively finite, and every vertex of $G - G_0$ is in the predecessor-set $P(\mathfrak{D}_0, G - G_0)$, then V is a maximally internally satisfactory set.*

THEOREM 4. *If, in addition to the hypotheses of Theorem 3, there exist inverse bases $V_{-\alpha}$ for each α with $1 \le \alpha < \kappa$ such that*

$$D^{-1}(V_0, G - G_0) \subset D(V, G) \text{ and } D^{-1}(V_{-\alpha}, G_{-\alpha}) \subset D(V, G),$$

then V is a solution of G and an extension of V_0.

THEOREM 5. *If, in addition to the hypotheses of Theorem 3, $\succ$ is symmetric, then V is a solution of G and an extension of V_0.*

The proofs of Theorems 4 and 5 are immediate.[6]

[6] As to Theorem 5, the fact that if $\succ$ is symmetric then every maximally internally satisfactory set is a solution is established in [2].

THEOREM 6. *If the hypotheses of Theorem 2 are satisfied, then so are the hypotheses of Theorem 4.*

Proof. Let

$$W_0 = D(V_0, G) \cup D^{-1}(V_0, G - G_0),$$

$$V_{-1} = D^{-2}(V_0, G - G_0) \cup D^{-1}(W_{00}, G - G_0),$$

$$W_{-1} = D^{-3}(V_0, G - G_0) \cup D^{-2}(W_{00}, G - G_0) = D^{-1}(V_{-1}, G - G_0) \cup D(V_{-1}, G - G_0),$$

$$V_{-2} = D^{-4}(V_0, G - G_0) \cup D^{-3}(W_{00}, G - G_0),$$

and so on. Then

$$V = \bigcup_{0 \le \alpha} V_{-\alpha} = V_0 \cup \bigcup D^{-2n}(V_0, G - G_0) \cup \bigcup D^{-2n+1}(W_{00}, G - G_0)$$

and

$$W = \bigcup_{0 \le \alpha} W_{-\alpha} = W_{00} \cup \bigcup D^{-2n+1}(V_0, G - G_0) \cup \bigcup D^{-2n}(W_{00}, G - G_0),$$

so that V is a solution.

There remains to show that $V_{-\alpha}$ is an inverse basis of $G_{-\alpha}$. Clearly, neither of two distinct vertices $x, y \in V_{-\alpha}$ chain-dominates the other by virtue of the parity restrictions (2), (3) of Theorem 2. We must show now that every vertex y of $G_{-\alpha}$ chain-dominates some x of $V_{-\alpha}$. This is obvious since by (4) every y belongs to $P(\mathfrak{D}_0, G - G_0)$, that is, to some $D^{-n}(V_0, G - G_0)$ or to some $D^{-n}(W_{00}, G - G_0)$, that is, to some $V_{-\alpha}$ or $W_{-\alpha}$. By (1) it is clear that every $D^{-1}(V_{-\alpha}, G_{-\alpha}) \subset D(V, G)$. This completes the proof.

The example of Figure 2 shows that Theorem 4 is less restrictive than Theorem 2. For

$$w_0^2 \in D^{-1}(W_{00}, G - G_0) \cap D^{-2}(V_0, G - G_0) \cap D^{-1}(V_0, G - G_0)$$

but an extension exists and the hypotheses of Theorem 4 are satisfied.

6. Some extension theorems. If H is a subgraph of G, let

$$K(x, H) = D(x, H) \cup D^{-1}(x, H), \qquad x \in \mathfrak{D};$$

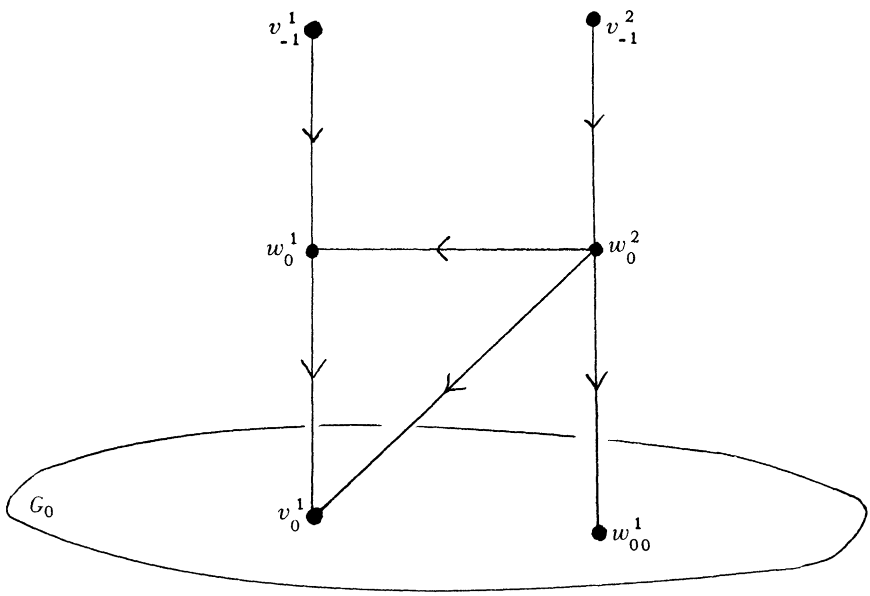


Figure 2

let

$$K(X,H) = \bigcup_{x \in X} K(x,H), \qquad X \subset \mathfrak{D};$$

let

$$K^n(X,H) = K(K^{n-1}(X,H), H) \qquad \text{for } n > 1.$$

That is, $K^n(X,H)$ denotes the set of vertices of H connected to vertices of X by unoriented one-dimensional chains of length n.

LEMMA 9. *If* $\mathfrak{D} - \mathfrak{D}_0 \subset P(V_0, G - G_0)$, *then every inverse basic set* B *of* G_{-i-1} *has a sink in* $K^2(V_{-i}, G_{-i-1})$, $i \geq 0$.

Proof. Suppose $i = 0$. Each sink y of B chain-dominates some vertex of V_0 since $\mathfrak{D} - \mathfrak{D}_0 \subset P(V_0, G - G_0)$. Consider the chains of minimum length m by which y chain-dominates vertices of V_0. Then $m \geq 2$ since $K(V_0, G - G_0) \cap G_{-1} = 0$. Suppose the lemma were false, so that $m > 2$, and let y_0 be a sink of B for which this minimum length is attained. Then there exist distinct vertices $x_1, x_2, \cdots, x_{m-1}$ of $G - G_0$ such that

$$y_0 \succ x_{m-1} \succ x_{m-2} \succ \cdots \succ x_1 \succ v_0^j$$

for some $v_0^j \in V_0$. Then either

(1) $x_{m-1} \notin G_{-1}$,

or (2) $x_{m-1} \in G_{-1}$ and is a descendant of y_0,

or (3) $x_{m-1} \in G_{-1}$ and is cyclically related to y_0 relative to G_{-1}.

In Case (1), $x_{m-1} \in V_0 \cup W_0$ so that

$$x_{m-1} \in K^1(V_0, G_{-1}) \text{ and } y_0 \in K^2(V_0, G_{-1})$$

contrary to the supposition that the lemma is false. In Case (2) y_0 is not a sink of B since a sink can have no descendant. In Case (3), m is not the minimum length since x_{m-1} would be a sink of B which chain-dominates v_0^j by means of a chain of length $m - 1$.

Now suppose $i > 0$. Let B be an inverse basic set of G_{-i-1}. Each sink y of B chain-dominates some vertex of V_{-i} since V_{-i} is an inverse basis of $G_{-i} \supset G_{-i-1}$.

Consider the chains of minimum length m by which y chain-dominates vertices of V_{-i}. Then $m \geq 2$ since $K(V_{-i}, G_{-i}) \cap G_{-i-1} = 0$. Suppose the lemma were false, so that $m > 2$, and let y_0 be a sink of B for which this minimum length is attained. Then there exist distinct vertices $x_1, x_2, \cdots, x_{m-1}$ of G_{-i} such that

$$y_0 \succ x_{m-1} \succ x_{m-2} \succ \cdots \succ x_1 \succ v^j_{-i} \text{ for some } v^j_{-i} \in V_{-i}.$$

Then either

(1) $x_{m-1} \notin G_{-i-1}$,

or (2) $x_{m-1} \in G_{-i-1}$ and is a descendant of y_0,

or (3) $x_{m-1} \in G_{-i-1}$ and is cyclically related to y_0 relative to G_{-i-1}.

In Case (1),

$$x_{m-1} \in \mathrm{St}(V_{-i} \cup W_{-i})$$

and hence $x_{m-1} \in V_{-i} \cup W_{-i}$ and hence $x_{m-1} \in K^1(V_{-i}, G_{-i})$ so that $y_0 \in K^2(V_{-i}, G_{-i-1})$ contrary to our supposition that the lemma is false. In Case (2), y_0 is not a sink of B since a sink has no descendant. In Case (3), m is not minimal since x_{m-1} would be a sink of B which chain-dominates v^j_{-i} by means of a chain of length $m-1$. This completes the proof.

The example of Figure 3 shows that we must take K^n in the unoriented sense; for here $v^1_{-1} \in P(V_0, G - G_0)$, in fact $v^1_{-1} \in D^{-4}(V_0, G - G_0)$ but $v^1_{-1} \notin D^{-2}(V_0, G - G_0)$ although $v^1_{-1} \in K^2(V_0, G - G_0)$.

A subgraph H of G is termed *progressively bounded at the vertex* y if all progressions of H beginning with y have lengths forming a bounded set of natural numbers. H is termed *progressively bounded* if it is progressively bounded at each of its vertices.

LEMMA 10. *If* $\mathfrak{D} - \mathfrak{D}_0 \subset P(V_0, G - G_0)$ *and if* $\mathrm{Cl}(G - G_0)$ *is progressively bounded then every vertex* y *of* $\mathfrak{D} - \mathfrak{D}_0$ *is an element of* V_{-i} *or* W_{-i} *for some finite ordinal* i.

Proof. Every vertex y of $\mathfrak{D} - \mathfrak{D}_0$ is an element of $C^{-1}(v^j_0, \mathrm{Cl}(G - G_0))$ for some $v^j_0 \in V_0$ by hypothesis. Consider all progressions of $\mathrm{Cl}(G - G_0)$ beginning with y and ending with elements of V_0. Their lengths have a least upper bound $M(y)$ by hypothesis. By Lemma 9, we may select inverse bases

$$V_{-i-1} \subset K^2(V_{-i}, G - G_{-i-1}), \qquad i \geq 0.$$

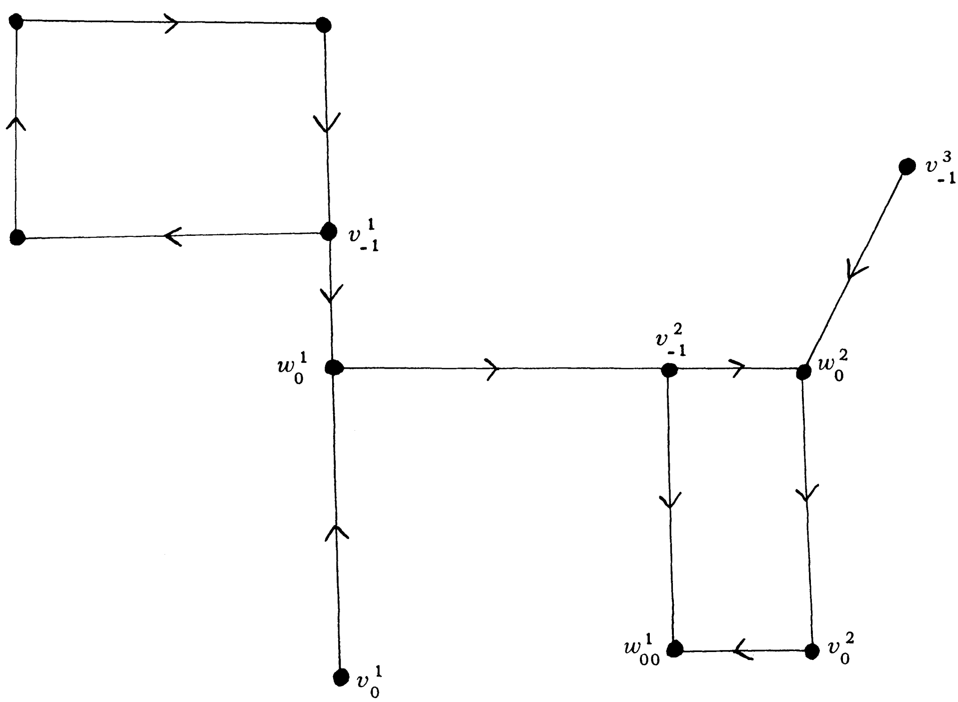


Figure 3

Since V_{-1} is an inverse basis of G_{-1}, there exist progressions starting with y and ending with elements of V_{-1} unless y is in V_{-k} or W_{-k} with $k \leq 1$. All such progressions have lengths $\leq M(y) - 2$. For if there existed a progression from y to some $v_{-1}^h \in V_{-1}$ of length $> M(y) - 2$, there would be a progression from y to some element of V_0 of length $> M(y)$ since there exists some progression from v_{-1}^h to some element of V_0 and its length must be ≥ 2 because $V_{-1} \subset G_{-1}$. Similarly the lengths of all progressions from y to elements of V_{-i} must be $\leq M(y) - 2i$. But this can be ≥ 0 for only a finite number of values of i. Hence there exists a value of i for which y chain-dominates some element of V_{-i} by means of a progression of length 0 or 1; that is, y is in either V_{-i} or W_{-i}.

By a *relative cycle* (of $\mathrm{Cl}(G - G_0)$ mod V_0 with modulo 2 coefficients) shall be meant an unoriented one-dimensional chain lying in $\mathrm{Cl}(G - G_0)$ except for its set of boundary vertices (possibly empty; that is, absolute cycles are included among the relative cycles) which lies in V_0.

THEOREM 7. *Suppose that V_0 is a solution of G_0 such that·*

(1) $\mathrm{Cl}(G - G_0)$ *is progressively bounded,*

(2) *each vertex of every $K^{2n-1}(V_0, G - G_0)$ is dominated by some element of $\mathfrak{D} - \mathfrak{D}_0$;*

(3) $\mathrm{Cl}(G - G_0)$ *contains no relative cycle of odd length;*

(4) $\mathfrak{D} - \mathfrak{D}_0 \subset P(V_0, G - G_0)$.

Then there exists a solution V of G which is an extension of V_0.

Proof. Choose V_{-i} as in Lemma 9. To show that $V = \bigcup_{0 \leq i} V_{-i}$ is a solution of G we have, by Theorem 4, only to show that

$$D^{-1}(V_0, G - G_0) \subset D(V, G) \text{ and } D^{-1}(V_{-i}, G_{-i}) \subset D(V, G) \text{ for } \qquad i \geq 1.$$

Let

$$w \in D^{-1}(V_0, G - G_0) \cup D^{-1}(V_{-i}, G_{-i}).$$

Then

$$w \in K^{2n-1}(v_0^j, G - G_0)$$

for some j and n by virtue of the way in which the V_{-i} were chosen. By (2), w is dominated by some vertex x of $\mathfrak{D} - \mathfrak{D}_0$. If $x \in V$, there is no more to prove. If $x \in \mathfrak{D} - V$, then

$$x \in K^{2m-1}(v_0^k, G - G_0)$$

for some k and m. Hence there exists a relative cycle of odd length, contrary to (3). This completes the proof.

THEOREM 8. *Let V be any maximally internally satisfactory set containing V_0 such that:*

(1) *every $v \in V$ belongs to $K^{2n}(V_0, G - G_0)$ for some $n \geq 0$;*

(2) *each element of $K^{2m-1}(V_0, G - G_0)$, for every $m > 0$, is dominated by some element of $\mathfrak{D} - \mathfrak{D}_0$;*

(3) $\mathrm{Cl}(G - G_0)$ *contains no relative cycle of odd length.*

Then V is a solution of G.

Proof. Let

$$y \in (\mathfrak{D} - \mathfrak{D}_0) \cap (\mathfrak{D} - V).$$

We shall show that there exists an $x \in V$ such that $x \succ y$. Since V is maximally internally satisfactory, $V \cup (y)$ is not internally satisfactory. Therefore either (a) some $v \succ y$, or (b) some $v \prec y$. In Case (a), there is no more to prove. In Case (b),

$$y \in K^{2n-1}(V_0, G - G_0) \qquad \text{for some } n \geq 0.$$

By (2), there exists an $x \in \mathfrak{D} - \mathfrak{D}_0$ such that $x \succ y$. If $x \in V$, there is no more to prove. If not, that is if $x \in (\mathfrak{D} - \mathfrak{D}_0) \cap (\mathfrak{D} - V)$, then $V \cup (x)$ is not internally satisfactory. Therefore there exists a $v \in V$ such that either $x \succ v$ or $x \prec v$. In either case,

$$x \in K^{2m-1}(V_0, G - G_0)$$

for some natural number m. But this together with

$$y \in K^{2n-1}(V_0, G - G_0)$$

and $x \succ y$ imply that there exists a relative cycle of odd length contrary to (3). This completes the proof.

COROLLARY. *The hypotheses of Theorem* 8 *imply that*

$$V = \bigcup K^{2n}(V_0, G - G_0) \text{ and } W = \mathfrak{D} - V = \bigcup K^{2m-1}(V_0, G - G_0).$$

Proof. We have

$$V \subset \bigcup K^{2n}(V_0, G - G_0) = E,$$

and

$$W = \mathfrak{D} - V \subset \bigcup K^{2m-1}(V_0, G - G_0) = \Omega.$$

Furthermore

$$K^{2n}(V_0, G - G_0) \cap K^{2m-1}(V_0, G - G_0) = 0,$$

for, if not, there would exist a relative cycle of odd length. Thus we have

$$E \cap \Omega = 0,\ E \cup \Omega = \mathfrak{D},\ V \subset E,\ W \subset \Omega,\ V \cup W = \mathfrak{D}, \text{ and } V \cap W = 0.$$

This implies $E = V$, $\Omega = W$ as follows. Let $e \in E$. Then $e \in \mathfrak{D}$ which implies that either $e \in V$ or $e \in W$. But $e \in W$ would imply that $e \in \Omega$ contrary to $E \cap \Omega = 0$. Therefore $e \in V$. Hence $E \subset V$ and therefore $E = V$. Similarly $\Omega \subset W$ and hence $\Omega = W$. This completes the proof.

Thus Theorem 8 resembles Theorem 2, except that now the parity restrictions are on the unoriented chains rather than on the oriented ones, and we do not restrict the sets $K^n(W_{00}, G - G_0)$.

The examples of Figures 4-6 are covered by Theorem 8 but not by Theorem 2. In Figure 4,

$$w_0^1 \in D(V_0, G) \cap D^{-1}(W_{00}, G - G_0) \neq 0$$

violating hypothesis 2b of Theorem 2, but the extension exists under Theorem 8. In Figure 5,

$$v_{-1}^1 \in D^{-2}(W_{00}, G - G_0) \cap D^{-1}(W_{00}, G - G_0) \neq 0$$

violating the second part of the hypothesis 3b of Theorem 2, but the extension

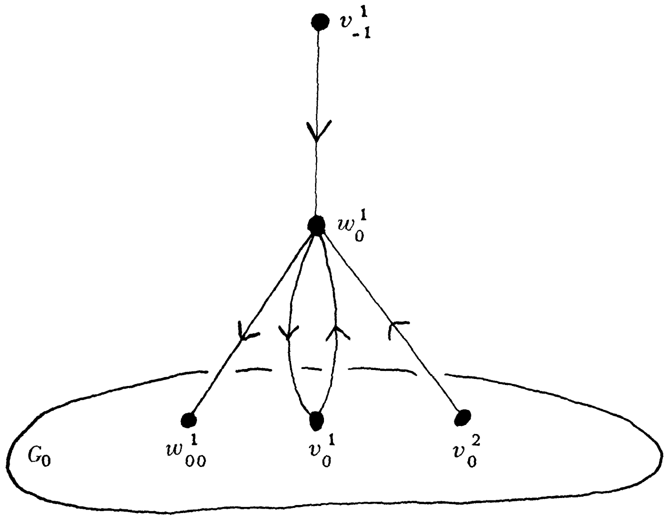


Figure 4

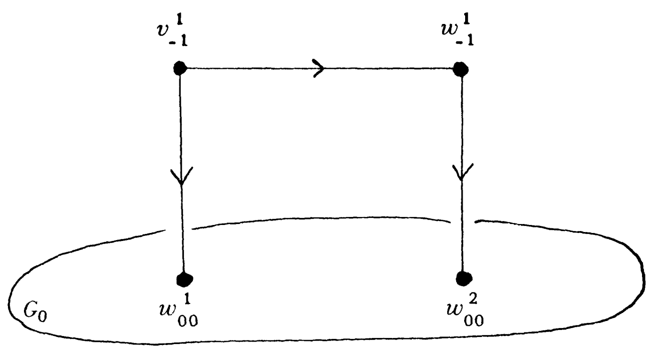


Figure 5

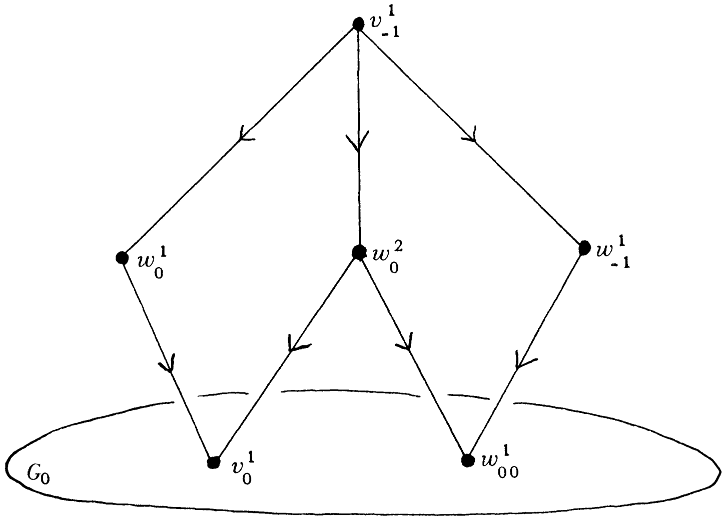


Figure 6

exists under Theorem 8. Note also that an odd relative cycle exists mod G_0 but not mod V_0. In Figure 6,

$$v_{-1}^{1} \in D^{-2}(V_0, G - G_0) \cap D^{-2}(W_{00}, G - G_0) \neq 0$$

and

$$w_0^2 \in D^{-1}(V_0, G - G_0) \cap D^{-1}(W_{00}, G - G_0) \neq 0$$

both violating hypothesis 3a of Theorem 2, but the extension exists under Theorem 8.

Let $\mu^h(X, G - G_0)$ denote the set of vertices of $G - G_0$ connected to X by an unoriented chain of minimal length h, where $X \subset \mathfrak{D}$. Then

$$\mu^h(X, G - G_0) \cap \mu^k(X, G - G_0) = 0 \qquad \text{for } h \neq k.$$

By $\mu^0(X, G - G_0)$ is meant X.

THEOREM 9. *Let V_0 be a solution of G_0 where G_0 is a conjunct subgraph of G. Let $W_{00} = \mathfrak{D}_0 - V_0$ and suppose that every vertex of $\mathfrak{D} - \mathfrak{D}_0$ is connected to $\mathfrak{D}_0$ by some unoriented chain. Let*

$$V = \bigcup_{n=0}^{\infty} \mu^{2n}(V_0, G - G_0) \cup \bigcup_{m=1}^{\infty} \mu^{2m-1}(W_{00}, G - G_0),$$

$$W = \bigcup_{n=1}^{\infty} \mu^{2n-1}(V_0, G - G_0) \cup \bigcup_{m=0}^{\infty} \mu^{2m}(W_{00}, G - G_0).$$

Suppose that:

(1) *every element of W is dominated by some element of V;*

(2) $\mu^h(V_0, G - G_0) \cap \mu^k(W_{00}, G - G_0) = 0$

if h and k have the same parity;

(3) *no two elements of the same $\mu^{2n-1}(W_{00}, G - G_0)$ are adjacent;*

(4) *no two elements of the same $\mu^{2n}(V_0, G - G_0)$ are adjacent.*

Then V is a solution of G which is an extension of V_0.

Proof. Clearly $\mathfrak{D} = V \cup W$ and $W \subset D(V, G)$. Also (2) implies $V \cap W = 0$. There remains only to prove that no two elements of V are adjacent.

If $x, y \in \mu^{2n}(V_0, G - G_0)$ then $x \not\succ y$ by (4).

If $x, y \in \mu^{2m-1}(W_{00}, G - G_0)$ then $x \not\succ y$ by (3).

Let

$$x \in \mu^{2n}(V_0, G - G_0),\ y \in \mu^{2m}(V_0, G - G_0), \qquad m \neq n.$$

Suppose $m > n$. If x and y were adjacent then $y \in K^{2n+1}(V_0, G - G_0)$. But $2n + 1 < 2m$, contradicting the minimal property of $\mu^{2m}(V_0, G - G_0)$. A similar proof is obtained if $m < n$.

If

$$x \in \mu^{2n-1}(W_{00}, G - G_0),\ y \in \mu^{2m-1}(W_{00}, G - G_0), \qquad m \neq n,$$

then x and y are proved non-adjacent as in the preceding paragraph.

Let

$$x \in \mu^{2n}(V_0, G - G_0),\ y \in \mu^{2p-1}(W_{00}, G - G_0)$$

and suppose x were adjacent to y. Then

$$x \in K^{2p}(W_{00}, G - G_0) \text{ or } x \in K^{2p-2}(W_{00}, G - G_0).$$

Since x is connected to W_{00}, it is minimally connected to W_{00}. That is, either

(a) $$x \in \mu^{2h}(W_{00}, G - G_0)$$

or

(b) $$x \in \mu^{2h-1}(W_{00}, G - G_0)$$

for some h. In Case (a), Condition (2) would be violated. In Case (b), $h = p$ since either $h < p$ or $h > p$ would violate the minimal property of some μ. But $h = p$ contradicts Condition (3). This completes the proof.

THEOREM 10. *Let V_0 be a solution of a conjunct subgraph G_0 of G such that every vertex of $\mathfrak{D} - \mathfrak{D}_0$ is connected to V_0 by some unoriented chain. Let:*

(1) *no two elements of the same* $\mu^{2i}(V_0, G - G_0)$, $i > 0$, *be adjacent;*

(2) $x \in \mu^{2i-1}(V_0, G - G_0)$ *imply that there exists a* $j \geq 0$ *such that* $x \prec y$ *for some* $y \in \mu^{2j}(V_0, G - G_0)$.

Then

$$V = \bigcup_{i=0}^{\infty} \mu^{2i}(V_0, G - G_0)$$

is a solution of G *which is an extension of* V_0.

Proof. Every element of $\mathfrak{D} - \mathfrak{D}_0$ not in V must be in

$$W = \bigcup_{i=1}^{\infty} \mu^{2i-1}(V_0, G - G_0).$$

Clearly

$$\mathfrak{D} = V \cup W \text{ and } V \cap W = 0.$$

Also (2) implies $W \subset D(V, G)$. There remains only to prove that V is internally satisfactory.

Let

$$x \in \mu^{2i}(V_0, G - G_0),\ y \in \mu^{2j}(V_0, G - G_0), \qquad i \neq j.$$

Suppose $i < j$. If x were adjacent to y, then $y \in K^{2i+1}(V_0, G - G_0)$. But $2i + 1 < 2j$, contradicting the minimal property of $\mu^{2j}(V_0, G - G_0)$. A similar proof holds if $i > j$.

Let $x, y \in \mu^{2i}(V_0, G - G_0)$. If $i > 0$, (1) implies that x and y are non-adjacent. For $i = 0$, this follows from the facts that V_0 is a solution of G_0 and that G_0 is a conjunct subgraph of G. This completes the proof.

The conditions of Theorem 10 do not prohibit entirely the existence in $\mathrm{Cl}(G - G_0)$ of adjacent vertices of W, of odd unoriented cycles, or of transitive triples. For example, the graph in Figure 7 permits an extension by Theorem 10 and includes the three cited phenomena. Theorems 7-10 may be regarded as variants of Theorem 2.

7. Dual and alternating procedures. Let G_1 be a conjunct subgraph of G.

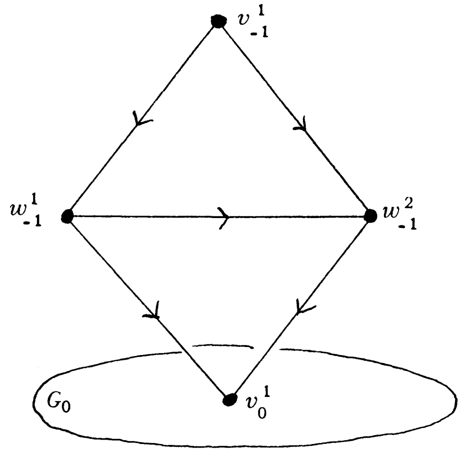


Figure 7

If $x \in \mathfrak{D}$, let $D(x, G - G_1)$ denote the set of all vertices y of $G - G_1$ such that $x \succ y$. If $X \subset \mathfrak{D}$, let

$$D(X, G - G_1) = \bigcup_{x \in X} D(x, G - G_1).$$

For $n > 1$, let

$$D^n(X, G - G_1) = D(D^{n-1}(X, G - G_1), G - G_1).$$

By the *successor-set* of X relative to $G - G_1$ is meant the set

$$S(X, G - G_1) = \bigcup_{n=1}^{\infty} D^n(X, G - G_1).$$

THEOREM 11. *Let G_1 be a conjunct subgraph of G, V_1 a solution of G_1, $W_1 = \mathfrak{D}_1 - V_1$ where $\mathfrak{D}_1 = \mathfrak{D} \cap G_1$. Suppose that:*

(1) *for every $n > 0$,*

$$D(V_1, G) \cap D^{2n}(V_1, G - G_1) = D(V_1, G) \cap D^{2n-1}(W_1, G - G_1) = 0,$$

and

$$V_1 \cap D^{2n+1}(V_1, G - G_1) = V_1 \cap D^{2n}(W_1, G - G_1) = 0;$$

(2) *if $h > 0$ and $k > 0$ are of the same parity, then*

$$D^h(V_1, G - G_1) \cap D^k(W_1, G - G_1) = 0;$$

if $h > 0$ and $k > 0$ are of different parities then

$$D^h(V_1, G - G_1) \cap D^k(V_1, G - G_1) = D^h(W_1, G - G_1) \cap D^k(W_1, G - G_1) = 0;$$

(3) $\mathfrak{D} - \mathfrak{D}_1 \subset S(\mathfrak{D}_1, G - G_1)$.

Then there exists a solution V of G which is an extension of V_1.

Proof. Let

$$V = V_1 \cup \bigcup_{n=1}^{\infty} D^{2n}(V_1, G - G_1) \cup \bigcup_{m=1} D^{2m-1}(W_1, G - G_1).$$

We must show:

(a) $$V \cap D(V, G) = 0;$$

(b) $$\mathfrak{D} - V \subset D(V, G).$$

(a) If $x \in V_1$, $y \in V_1$, then $x \not\succ y$ since V_1 is internally satisfactory relative to G.

If $x \in V_1$, $y \in D^{2n}(V_1, G - G_1)$ then $x \not\succ y$; for $x \succ y$ would imply $D(V_1, G) \cap D^{2n}(V_1, G - G_1) \neq 0$ contrary to (1).

If $x \in V_1$, $y \in D^{2m-1}(W_1, G - G_1)$ then $x \not\succ y$; for $x \succ y$ would imply $D(V_1, G) \cap D^{2m-1}(W_1, G - G_1) \neq 0$ contrary to (1).

If $x \in D^{2n}(V_1, G - G_1)$, $y \in V_1$, then $x \not\succ y$; for $x \succ y$ would imply $y \in D^{2n+1}(V_1, G - G_1)$ contrary to (1).

If $x \in D^{2n}(V_1, G - G_1)$, $y \in D^{2m}(V_1, G - G_1)$ then $x \not\succ y$; for $x \succ y$ would imply $y \in D^{2n+1}(V_1, G - G_1)$ contrary to (2).

If $x \in D^{2n}(V_1, G - G_1)$, $y \in D^{2m-1}(W_1, G - G_1)$ then $x \not\succ y$; for $x \succ y$ would imply $y \in D^{2n+1}(V_1, G - G_1)$ contrary to (2).

If $x \in D^{2m-1}(W_1, G - G_1)$, $y \in V_1$ then $x \not\succ y$; for $x \succ y$ would imply $y \in D^{2n}(W_1, G - G_1)$ contrary to (1).

If $x \in D^{2m-1}(W_1, G - G_1)$, $y \in D^{2n}(V_1, G - G_1)$ then $x \not\succ y$; for $x \succ y$ would imply $y \in D^{2m}(W_1, G - G_1)$ contrary to (2).

If $x \in D^{2m-1}(W_1, G - G_1)$, $y \in D^{2n-1}(W_1, G - G_1)$ then $x \not\succ y$; for $x \succ y$ would imply $y \in D^{2m}(W_1, G - G_1)$ contrary to (2).

(b) Let

$$W = W_1 \cup \bigcup_{n=1}^{\infty} D^{2n-1}(V_1, G - G_1) \cup \bigcup_{m=1}^{\infty} D^{2m}(W_1, G - G_1).$$

By (3),

$$\mathfrak{D} = \mathfrak{D}_1 \cup S(\mathfrak{D}_1, G - G_1) = V \cup W.$$

By (1) and (2), $V \cap W = 0$. Hence $W = \mathfrak{D} - V$.

If $y \in W_1$, then there exists an $x \in V_1$ such that $x \succ y$.

If $y \in D^{2n-1}(V_1, G - G_1)$ then there exists an $x \in V_1 \cup D^{2n}(V_1, G - G_1)$ such that $x \succ y$.

If $y \in D^{2m}(W_1, G - G_1)$, then there exists an $x \in D^{2m-1}(W_1, G - G_1)$ such that $x \succ y$. This completes the proof.

COROLLARY. *Let G_1 be a conjunct subgraph of G, V_1 a solution of G_1. Suppose that:*

(a) *no vertex of any $S(x_1, G - G_1)$, $x_1 \in \mathfrak{D}_1$, is adjacent to any other vertex of $\mathfrak{D}_1$; and if x_1 and x_1' are any two distinct vertices of $\mathfrak{D}_1$ then*

$$S(x_1, G - G_1) \cap S(x_1', G - G_1) = 0;$$

(b) *no*

$$S(x_1, G - G_1) \cup (x_1), \qquad x_1 \in \mathfrak{D}_1,$$

contains an unoriented cycle of odd length;

(c) $\mathfrak{D} - \mathfrak{D}_1 \subset S(\mathfrak{D}_1, G - G_1)$.

Then there exists a solution of G which is an extension of V_1.

Proof. Condition (c) is identical with (3) of the theorem. We have only to show that (a) and (b) imply (1) and (2); that is, that if either (1) or (2) were false then (a) or (b) would be violated.

If (1) were false there would exist either

(i) a vertex $x \in D(V_1, G) \cap D^{2n}(V_1, G - G_1)$,

or (ii) a vertex $y \in D(V_1, G) \cap D^{2n-1}(W_1, G - G_1)$,

or (iii) a vertex $z \in V_1 \cap D^{2n+1}(V_1, G - G_1)$,

or (iv) a vertex $u \in V_1 \cap D^{2n}(W_1, G - G_1)$.

In Case (i)

$$x \in S(v_1^i, G - G_1) \cap S(v_1^j, G - G_1)$$

and by (a), $i = j$. But then there exists an unoriented cycle of odd length in $S(v_1^i, G - G_1) \cup (v_1^i)$ contrary to (b). In Case (ii), the second part of (a) is contradicted. In Cases (iii) and (iv), the first part of (a) is contradicted.

If (2) were false, there would exist either

(i) a vertex

$$x \in D^h(V_1, G - G_1) \cap D^k(W_1, G - G_1)$$

for some h, k of the same parity,

or (ii) a vertex

$$y \in D^h(v_1^i, G - G_1) \cap D^k(v_1^j, G - G_1)$$

for some h, k of different parities,

or (iii) a vertex

$$z \in D^h(w_1^i, G - G_1) \cap D^k(w_1^j, G - G_1)$$

for some h, k of different parities. In Case (i), the second part of (a) is contradicted. In Cases (ii) and (iii), (a) implies $i = j$ and then (b) is contradicted.

Now suppose G_0 is a nonempty conjunct subgraph of G_1 and let V_0 be a solution of G_0. For each natural number n, let G_{2n-1} be constructed by adjoining to G_{2n-2} the vertices of $P(\mathfrak{D}_{2n-2}, G - G_{2n-2})$, where $\mathfrak{D}_i = \mathfrak{D} \cap G_i$, and taking the juncture; that is,

$$G_{2n-1} = G_{2n-2} \cup P(\mathfrak{D}_{2n-2}, G - G_{2n-2}).$$

Similarly let

$$G_{2n} = G_{2n-1} \cup S(\mathfrak{D}_{2n-1}, G - G_{2n-1}).$$

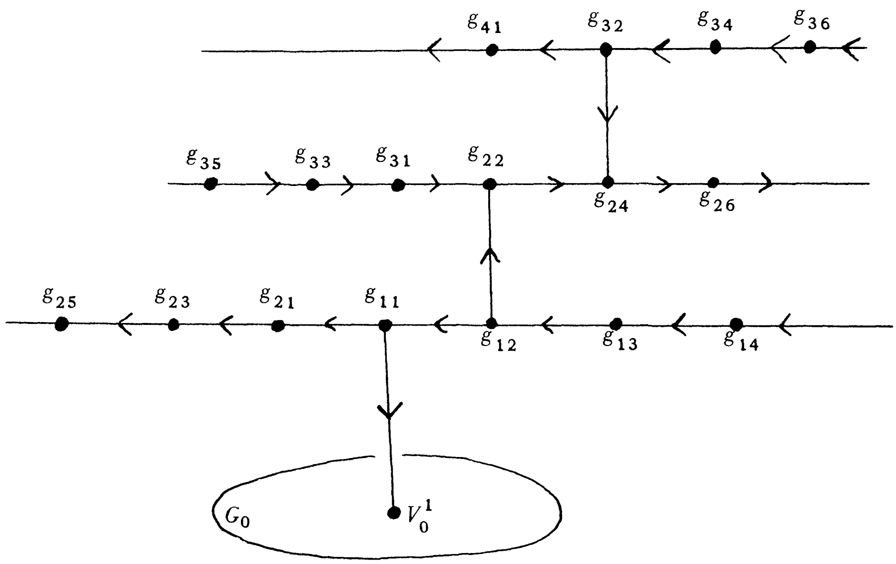
Then each G_i is a conjunct subgraph of G_{i+1}. For $x, y \in \mathfrak{D}_i$, $x \succ y$ relative to G_{i+1} implies $x \succ y$ relative to G_i since at least one endpoint of every arc in $G_{i+1} - G_i$ is not in G_i.

If G_0 intersects every component of G, then

$$\mathfrak{D} = \bigcup_{i=0}^{\infty} \mathfrak{D}_i.$$

For then every vertex of G is joined to some vertex of G_0 by a finite unoriented chain and therefore lies in some G_i. In particular, this is true if G is connected.

THEOREM 12. *Let G_0 be a conjunct subgraph of G which intersects every*



g_{41}
g_{32}
g_{34}
g_{36}
g_{35}
g_{33}
g_{31}
g_{22}
g_{24}
g_{26}
g_{25}
g_{23}
g_{21}
g_{11}
g_{12}
g_{13}
g_{14}
G_0
V_0^1

Figure 8

component of G, let V_0 be a solution of G_0, and let G_i, $i \geq 1$, be defined as above. Suppose that for every even i, G_i satisfies Conditions (1), (2), (3) *of Theorem* 2 *relative*[7] *to G_{i+1}, and that for every odd i, G_i satisfies Conditions* (1) and (2) *of Theorem* 11 *relative to G_{i+1}. Then there exists a solution of G which is an extension of V_0.*

Proof. The solution V_0 of G_0 can be extended stepwise to a solution V_1 of G_1, V_2 of $G_2, \cdots, V_i$ of $G_i, \cdots$ by Theorems 2 and 11 applied alternately. Hence $\bigcup_{i=0}^{\infty} V_i$ is a solution of G.

For example, in Figure 8, G_i has the set of vertices $\mathfrak{D}_i = [g_{i1}, g_{i2}, g_{i3}, \cdots]$. Then

$$V_1 = [g_{12}, g_{14}, \cdots] \cup V_0, \quad V_2 = [g_{21}, g_{25}, \cdots; g_{24}, g_{28}, \cdots] \cup V_1,$$

$$V_3 = [g_{31}, g_{35}, \cdots; g_{34}, g_{38}, \cdots] \cup V_2,$$

$$V_4 = [g_{41}, \cdots] \cup V_3.$$

Theorem 11 is a sort of dual to Theorem 2. Theorem 12 merely uses the procedures of Theorems 2 and 11 in alternation. Similar processess dual to those of other preceding theorems can be introduced so as to yield extensions in the direction of successor-sets rather than predecessor-sets, and similar alternating procedures can then be used.

[7] That is, with G_{i+1} in the role of G in Theorem 2.

REFERENCES

1. D. König, *Theorie der endlichen und unendlichen Graphen*, Leipzig, 1936.

2. J. von Neumann and O. Morgenstern, *Theory of games and economic behavior*, Princeton 1944; 2nd edition 1947.

3. M. Richardson, *On weakly ordered systems*, Bull. Amer. Math. Soc. **52** (1946), 113-116.

4. ———, *Solutions of irreflexive relations*, Ann. of Math. **58** (1953), 573-590.

5. ———, *Extension theorems for solutions of irreflexive relations*, Proc. Nat. Acad. Sci. **39** (1953), 649-655.

BROOKLYN COLLEGE