# AN EXTENSION OF WEYL'S ASYMPTOTIC LAW FOR EIGENVALUES

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**1. Introduction.** Let $D$ be a bounded, open, connected subset of the plane $E_2$ whose boundary $B = \overline{D} - D$ is a simple closed curve whose curvature exists everywhere and is continuous with respect to arc length; consider the eigenvalues $\lambda = \lambda_n > 0$ of the problem

$$\nabla^2 u + \lambda u = 0 \text{ on } D, \qquad u = 0 \text{ on } B, \tag{1.1}$$

where $u(\mathbf{x})$ is to be continuous over $\overline{D}$ and have continuous second partials over $D$, $\nabla^2$ being the Laplacian. It has long been known (see [7, bibliography]) that in this situation, with $0 < \lambda_n \leq \lambda_{n+1}$ repeated according to multiplicity, the asymptotic distribution of $\lambda_n$ is given by Weyl's law

$$N(t) = \sum_{\lambda_n \leq t} 1 = \frac{\mu_2(D)}{4\pi} t + o(t), \qquad t \longrightarrow +\infty, \tag{1.2}$$

where $\mu_2(D)$ is the two dimensional Lebesgue measure of $D$. This can be obtained by Tauberian theorems from the estimate

$$\sum_{n=1}^{\infty} \frac{1}{\lambda_n(\lambda_n+\omega)} = \frac{\mu_2(D)}{4\pi} \frac{\ln \omega}{\omega} + \frac{C}{\omega} + O(\omega^{-5/4}), \qquad \omega \longrightarrow +\infty, \tag{1.3}$$

(see Carleman [2] for the $E_3$ analogue). By domain comparison methods [3, p. 386] Courant has shown that $o(t)$ in (1.2) can be replaced by $O(\sqrt{t} \ln t)$.

In a recent paper [6, p. 177, equation 16] Pleijel replaces the estimate (1.3) by the very much stronger

$$\sum_{n=1}^{\infty} \frac{1}{\lambda_n(\lambda_n+\omega)} = \frac{\mu_2(D)}{4\pi} \frac{\ln \omega}{\omega} + \frac{C}{\omega} + \frac{l(B)}{8} \frac{1}{\omega^{3/2}} + O\left(\frac{1}{\omega^2}\right) \tag{1.4}$$

over $\omega \geq 1$ in case the curve $B$ is very smooth (that is, it has an infinitely

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differentiable parametric representation), where $C$ is an unknown real constant and $l(B)$ is the total length of $B$. Pleijel's estimate (1.4) follows easily from a deep investigation jointly with M. T. Ganelius on the compensating part of the Green function, as yet unpublished. This investigation uses integral equations over the boundary $B$, while estimates like (1.3) come from a simple application of the maximum principle over $D$ to the modified Green function. Pleijel suggests it should be possible to sharpen (1.2) by using his methods to investigate the analogue of (1.4) over complex $\omega$.

It is the purpose of this paper to show that from (1.4) alone we can replace (1.2) by

$$(1.5) \qquad N(t) = \frac{\mu_2(D)}{4\pi}\,t - \frac{l(B)}{4\pi}\,t^{1/2} + \widetilde{O}(\ln t), \qquad t \longrightarrow +\infty,$$

in a certain sense. Precisely our result (2.13) is that with

$$F(t) = N(t) - \left\{\frac{\mu_2(D)}{4\pi}\,t - \frac{l(B)}{4\pi}\,t^{1/2}\right\}$$

we have

$$(1.6) \qquad \left|\int_{\frac{1}{2}\lambda_1}^{\infty} \exp\left(-\frac{\rho^2}{2}\left(\ln\frac{u}{t}\right)^2\right) dF(t)\right| \leq \left[M \exp\left(\frac{\pi^2\rho^2}{2} + \frac{1}{2\rho^2}\right)\right] \ln u$$

over all real $u \geq e$ and all $\rho > 0$ for some $M < +\infty$. Moreover, if $N(t)$ has an ordinary asymptotic series in powers of $t$, it must be consistent with (1.5). We discuss briefly the possibility of sharpening (1.5) by replacing averaged $\widetilde{O}$ estimates by ordinary ones. We also note the utility of our consistency result in proving false a conjecture of Minakshisundaram [5, p. 331, no. 2] about the asymptotic behavior of $N(t)$.

Clearly our theorems will apply to give results like (1.5) for a wide variety of more general problems than (1.1) for which estimates like (1.4) obtain; in particular such results hold for (1.1) in 3-space $E_3$.

**2. Results and proofs.** The difficulty arising in trying to get an asymptotic series like (1.5), with $\widetilde{O}$ replaced by an ordinary $O$ or $o$, is that Tauberian theorems yielding such results seem to require essential nonnegative conditions after subtracting all but the last term of the series. It is quite clear that $N(t)$ does not satisfy such a condition. For this reason we use an *indirect*

Abelian type argument [4, p. 224] to get averaged error estimates of the $\widetilde{O}$ type. The two first theorems here establish the significance of these averaged error estimates, which despite the resemblance to Gaussian summability seem to be little used for asymptotic series. Cramér [1, p. 819 and p. 823, (3)] has used Caesaro-1 type averaged error estimates on lattice point problems, but such processes do not appear strong enough for use here.

Throughout the paper all integrals are to be understood in the Lebesgue or Lebesgue-Stieltjes sense, and for the following two theorems it is understood that $F(t)$ is to be real valued of bounded variation over every finite interval of $[0,\infty)$, with positive $b$ a continuity point of $F(t)$. Also $|dF(t)|$ stands for $dV_F(t)$ where $V_F(t)$ is the total variation of $F$ over $[b,t]$.

THEOREM 1. *If*

$$\int_b^\infty t^{-r_0}\,|dF(t)| < +\infty$$

*for some* $r_0 > 0$, *if*

$$\psi(s) = \int_b^\infty t^{-s}\,dF(t),$$

*which must exist and be analytic in s over* $\Re[s] > r_0$, *also has an analytic continuation without singularities throughout* $\Re[s] > 0$, *and if*

$$|\psi(r+iv)| \le \frac{M_1}{r}\,e^{h|v|}$$

*over* $0 < r \le r_0$ *and all real v for some* $M_1 < +\infty$ *and* $h \ge 0$, *then over all real* $u \ge e$ *and* $\rho > 0$ *we have*

$$\left|\int_b^\infty e^{-(\rho^2/2)(\ln(u/t))^2}\,dF(t)\right| \le \left(2M_1 \exp\left(1+\frac{\rho^2}{2}h^2+\frac{1}{2\rho^2}\right)\right)\ln u. \tag{2.1}$$

In view of (2.1) it becomes convenient to define $F(t) = \widetilde{O}(f(t))$ over $t \ge b$ for some nonnegative $f(t)$ defined over $t \ge k > 0$ if for each $\rho > 0$ there exists some $M_\rho < +\infty$ such that the left side of (2.1) exists and is $\le M_\rho f(u)$ for all $u \ge k$. With this definition we can restate the conclusion of Theorem 1 as

$$F(t) = \widetilde{O}(\ln t)$$

over $t \geq b$ with $k = e$. Note that in (2.1) $M_\rho \longrightarrow +\infty$ as either $\rho \longrightarrow 0^+$ or $\rho \longrightarrow +\infty$, so that (2.1) becomes meaningless then. The significance of the result (2.1) is greatly increased by the following consistency theorem.

THEOREM 2. *If*

$$\int_b^\infty t^{-r_0} \, |dF(t)| < +\infty$$

*for some* $r_0 > 0$, *if*

$$F(t) = \widetilde{O}(\ln t)$$

*over* $t \geq b$, *and if*

$$F(t) = c_1 t^{r_1} + o(t^{r_1})$$

*as* $t \longrightarrow +\infty$ *for some* $r_1 > 0$, *then* $c_1 = 0$.

*Proof of Theorem* 1. Let

$$f_z(y) = \exp\left(-\frac{\rho^2}{2} y^2 - zy\right)$$

for $\rho > 0$ and any complex $z$; thus

$$\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f_z(y) e^{-ivy} \, dy = \frac{1}{\rho} \exp\left(-\frac{1}{2\rho^2}\left(v + \frac{z}{i}\right)^2\right).$$

Now

$$M \geq \int_b^\tau t^{-r_0} \, |dF(t)| \geq \tau^{-r_0} V_F(\tau)$$

shows

$$V_F(t) = O(t^{r_0});$$

thus

$$g(z, \omega) = \int_{y=\ln b}^{\infty} f_z(\omega - y) e^{-zy} \, dF(e^y)$$

exists as an entire function of $z$ over all real $\omega$ and all complex $z$. The Fubini theorem also shows $g(z, \omega) \in L_1(-\infty, \infty)$ over $\omega$ with

$$(2.2) \qquad \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} g(z, \omega) e^{-iv\omega} d\omega = \frac{1}{\rho} \exp\left(-\frac{1}{2\rho^2}\left(v + \frac{z}{i}\right)^2\right) \psi(z + iv)$$

over $\Re[z + iv] = \Re[z] \geq r_0$, $v$ being real. But the right side of (2.2) is in $L_1(-\infty, \infty)$ over $v$ since

$$|\psi(s)| \leq \int_b^{\infty} t^{-r} |dF(t)|, \qquad r = \Re[s],$$

and thus the Fourier transform inverse yields

$$(2.3) \qquad \int_{y=\ln b}^{\infty} f_z(\omega - y) e^{-zy} dF(e^y) = g(z, \omega)$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \exp\left(-\frac{(v + z/i)^2}{2\rho^2}\right) \psi(z + iv) e^{iv\omega} \frac{dv}{\rho}$$

for $\Re[z] \geq r_0$. The given estimate on $\psi(s)$ actually makes the far right side of (2.3) exist and be analytic in $z$ throughout $\Re[z] > 0$, and thus by analytic continuation (2.3) holds there also. Thus with $z = r$ we have for every positive $r$ and $\rho$ and for every real $\omega$ the estimate

$$(2.4) \qquad \left|\int_{y=\ln b}^{\infty} \exp\left(-\frac{\rho^2}{2}(\omega - y)^2 - r\omega\right) dF(e^y)\right|$$

$$\leq \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \exp\left(\frac{-v^2 + r^2}{2\rho^2}\right) \frac{M_1}{r} e^{h|v|} \frac{dv}{\rho}$$

$$= \frac{M_1}{r} \exp\left(\frac{r^2}{2\rho^2} + \frac{h^2\rho^2}{2}\right)\left\{\frac{1}{\sqrt{2\pi}} \int_0^{\infty} e^{-\frac{1}{2}(v/\rho - \rho h)^2} \frac{dv}{\rho}\right.$$

$$\left. + \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{0} e^{-\frac{1}{2}(v/\rho + \rho h)^2} \frac{dv}{\rho}\right\} \leq \frac{2M_1}{r} \exp\left(\frac{r^2}{2\rho^2} + \frac{h^2\rho^2}{2}\right).$$

Multiplying (2.4) by $e^{r\omega}$ and letting $r = 1/\omega > 0$ we note for $\omega \geq 1$ that

$$r\omega + \frac{r^2}{2\rho^2} - \ln r = 1 + \frac{1}{2\rho^2\omega^2} + \ln\omega \leq 1 + \frac{1}{2\rho^2} + \ln\omega,$$

thus with $y = \ln t$ and $\omega = \ln u \geq 1$ we get the estimate (2.1) as desired.

*Proof of Theorem* 2. As before we have

$$|F(t)| \leq |F(b)| + V_F(t) = O(t^{r_0}),$$

so that we can integrate by parts in the left side of (2.1) and obtain from $F(t) = \widetilde{O}(\ln t)$ over $t \geq b$ the estimate

$$\left|\rho^2 \int_{y=\ln b}^{\infty} F(e^y)(\omega - y)e^{-\rho^2(\omega-y)^2/2}dy\right| \leq |F(b)| + M_\rho\,\omega \tag{2.5}$$

over $\omega \geq k > 0$. Now we are given

$$F(t) = c_1 t^{r_1} + f(t)t^{r_1}$$

over $t \geq b$ with $\lim_{t\to+\infty} f(t) = 0$. Thus multiplying (2.5) by $e^{-r_1\omega}$, letting $y = \omega - x$, and taking $\omega \longrightarrow +\infty$ we get

$$0 = \lim_{\omega\to+\infty}\left\{c_1\int_{-\infty}^{\omega-\ln b} x\exp\left(-r_1x - \frac{\rho^2}{2}x^2\right)dx\right.$$

$$\left.+\int_{-\infty}^{\omega-\ln b} f(e^{\omega-x})x\exp\left(-r_1x - \frac{\rho^2}{2}x^2\right)dx\right\}.$$

Defining $f(t) = 0$ for $t < b$ we obtain

$$0 = c_1\int_{-\infty}^{\infty} x\exp\left(-r_1x - \frac{\rho^2}{2}x^2\right)dx \tag{2.6}$$

$$+\lim_{\omega\to+\infty}\left\{\int_{-\infty}^{\infty} f(e^{\omega-x})x\exp\left(-r_1x - \frac{\rho^2}{2}x^2\right)dx\right\}.$$

$f(t)$ being bounded over all real $t$ since $\lim_{t\to+\infty} f(t) = 0$, and thus also $\lim_{\omega\to+\infty} f(e^{\omega-x}) = 0$, dominated convergence applied to (2.6) yields

$$0 = c_1 \int_{-\infty}^{\infty} x \exp\left(-r_1 x - \frac{\rho^2}{2} x^2\right) dx .$$

But

$$\int_{-\infty}^{\infty} x \exp\left(-r_1 x - \frac{\rho^2}{2} x^2\right) dx = \int_0^{\infty} x\left(e^{-r_1 x} - e^{r_1 x}\right) e^{-\rho^2 x^2/2}\, dx < 0$$

for $r_1 > 0$, so that $c_1 = 0$ follows.

To apply these two theorems we use a standard contour integral transformation on Pleijel's estimate (1.4). The contour $C_\rho$, $\rho \geq 0$, in the $z$ plane is defined to be first along the negative real axis from $-\infty$ to $-\rho$, then around the circle $z = \rho e^{i\theta}$ from $\theta = -\pi$ to $\theta = \pi$, then back along the axis to $-\infty$. On this contour we define

$$(z)_c^{s-1} = |z|^{s-1} e^{i(s-1)\theta}, \quad z = |z| e^{i\theta},$$

with $\theta = -\pi$, $-\pi < \theta < \pi$, $\theta = \pi$ on the three parts respectively. The well known results are formulated in the following two lemmas (Carleman [2]), and we sketch the proofs for the sake of completeness.

LEMMA 3. *If* $0 < \lambda_n \leq \lambda_{n+1}$, $a_n$ *real, if*

$$\sum_{n=1}^{\infty} \frac{|a_n|}{\lambda_n^2} < +\infty,$$

*and if*

$$\sum_{n=1}^{\infty} \frac{1}{\lambda_n^2} < +\infty,$$

*then*

$$h(z) = \sum_{n=1}^{\infty} \frac{a_n}{\lambda_n(\lambda_n - z)}$$

*converges absolutely and is analytic in all complex* $z$ *except for simple poles at each* $\lambda_n$*. Moreover, for* $0 < \rho < \lambda_1$ *the function*

$$\frac{1}{2\pi i}\int_{C_\rho} h(z)\,\frac{dz}{(z)_c^{s-1}}$$

*exists and is analytic in s over* $\Re[s] > 2$,

$$\sum_{n=1}^{\infty} a_n\,\lambda_n^{-s}$$

*converges absolutely and uniformly over* $\Re[s] \geq 2$, *and over* $\Re[s] > 2$ *we obtain*

$$\sum_{n=1}^{\infty} a_n\,\lambda_n^{-s} = \frac{1}{2\pi i}\int_{C_\rho} h(z)\,\frac{dz}{(z)_c^{s-1}}. \tag{2.7}$$

LEMMA 4. *If the assumptions of Lemma* 3 *are satisfied and if*

$$h(-\omega) = \sum_{p=1}^{k}\frac{m_p + l_p \ln\omega}{\omega^{2-r_p}} + O\left(\frac{1}{\omega^2}\right) \tag{2.8}$$

*holds over* $\omega \geq 1$ *with* $0 < r_k < r_{k-1} < \cdots < r_1 < 2$, *then*

$$g(s) = \sum_{k=1}^{\infty} a_n\,\lambda_n^{-s}$$

*has an analytic extension into* $\Re[s] > 0$ *except for poles at* $r_p$,

$$g(s) = g_k(s) + \sum_{p=1}^{k}\left\{ l_p\,\frac{\sin\pi(r_p-1)}{\pi}\,\frac{1}{(s-r_p)^2} + \left[m_p\,\frac{\sin\pi(r_p-1)}{\pi} + l_p\cos\pi(r_p-1)\right]\frac{1}{s-r_p}\right\} \tag{2.9}$$

*with* $g_k(s)$ *analytic in s throughout* $\Re[s] > 0$, *and*

$$|g_k(r+iv)| \leq \frac{M_2}{r}\,e^{\pi|v|}$$

*over* $0 < r \leq 2$ *and all real v for some* $M_2 < +\infty$.

We remark that $r_1 < 2$ is no real restriction in (2.8), since the assumptions of Lemma 3 imply $\lim_{\omega \to +\infty} h(-\omega) = 0$. In demonstrating Lemma 3, the stated analyticity of $h(z)$ is clear as well as

$$|h(z)| \leq \left( \inf_{j \geq 1} \left|1 - \frac{z}{\lambda_j}\right| \right)^{-1} \left( \sum_{n=1}^{\infty} \frac{|a_n|}{\lambda_n^2} \right),$$

so

$$g(s) = \frac{1}{2\pi i} \int_{C_\rho} h(z) \frac{dz}{(z)_c^{s-1}}$$

exists and is analytic in $s$ over $\Re[s] > 2$. To show

$$g(s) = \sum_{n=1}^{\infty} a_n \lambda_n^{-s}$$

there for (2.7), let $C_m$ be the vertical line contour from $x_m - i\infty$ to $x_m + i\infty$ for $x_m$ with $\lambda_{n-1} < x_m < \lambda_n$, so that using the estimate on $h(z)$ to shift from $C_\rho$ to $C_m$ we obtain

$$g(s) - \sum_{j=1}^{n-1} a_j \lambda_j^{-s} = \frac{1}{2\pi i} \int_{C_m} h(z) \frac{dz}{z^{s-1}} \tag{2.10}$$

for $\Re[s] > 2$, $h(z)$ having the residue

$$-\lambda^{-1} \left( \sum_{\lambda_j = \lambda} a_j \right)$$

at $\lambda$.

To pass from (2.10) to (2.7), note that

$$[\limsup_{n \to \infty} \lambda_n^2 (\lambda_n - \lambda_{n-1})] = +\infty,$$

since otherwise

$$\lambda_n - \lambda_1 = \sum_{j=2}^{n} (\lambda_j - \lambda_{j-1})$$

would be bounded by

$$M\left(\sum_{j=1}^{\infty}\frac{1}{\lambda_j^2}\right),$$

which contradicts $\lambda_n \longrightarrow +\infty$ and therefore contradicts

$$\sum_{n=1}^{\infty}\frac{1}{\lambda_n^2} < +\infty.$$

Thus there exists a sequence $n_m$ such that

$$n_m < n_{m+1},\ \lambda_n - \lambda_{n-1} > 0,\ \text{ and }\ \lambda_n^2(\lambda_n - \lambda_{n-1}) \longrightarrow +\infty \text{ as } m \longrightarrow +\infty \text{ for } n = n_m.$$

We choose

$$x_m = \{\max\,(\lambda_n + \lambda_{n-1})/2,\quad (\lambda_n - 1)\}$$

for $n = n_m$, so that

$$\frac{\lambda_n}{x_m} \le 1 + \frac{1}{x_m} \longrightarrow 1$$

and

$$\frac{x_m}{\lambda_n - x_m} \le x_m + \frac{2x_m}{\lambda_n - \lambda_{n-1}} = x_m + 2\,x_m^3\left(\frac{\lambda_n}{x_m}\right)^2\frac{1}{\lambda_n^2(\lambda_n - \lambda_{n-1})} = O(x_m^3).$$

With $z = x_m + it$ and $s = r + iv$, $r > 2$, clearly

$$|h(z)| \le \frac{M}{|1 - (x_m + it)/\lambda_n|}$$

and

$$\left|\frac{1}{z^{s-1}}\right| \le \frac{\exp(\pi|v|/2)}{|z|^{r-1}}$$

with $L(v) = M\exp(\pi|v|/2)$ make

$$\left| h(z) \frac{1}{z^{s-1}} \right| \leq (x_m)^{2-r} \left( \frac{\lambda_n}{x_m} \right) \frac{L(v)}{\lambda_n - x_m} \qquad \text{over } |t| \leq \lambda_n - x_m,$$

$$\left| h(z) \frac{1}{z^{s-1}} \right| \leq (x_m)^{2-r} \left( \frac{\lambda_n}{x_m} \right) \frac{L(v)}{|t|} \qquad \text{over } \lambda_n - x_m \leq |t| \leq x_m,$$

and

$$\left| h(z) \frac{1}{z^{s-1}} \right| \leq \left( \frac{\lambda_n}{x_m} \right) \frac{L(v)}{|t|^{r-1}} \qquad \text{over } x_m \leq |t|.$$

Thus integrating over these respective parts of $C_m$, and using

$$\ln\left( \frac{x_m}{\lambda_n - x_m} \right) = O(\ln x_m),$$

the right side of (2.10) $\longrightarrow 0$ as $m \longrightarrow +\infty$ and (2.7) follows.

Passing to Lemma 4, from the estimate (2.8) it is clear that

$$g(s) = \frac{1}{2\pi i} \int_{C_\rho} h(z) \frac{dz}{(z)_c^{s-1}}$$

extends analytically from $\Re[s] > 2$ to $\Re[s] > r_1$. Also for $r_1 < \Re[s] < 2$, $C_\rho$ can be shifted to $C_0$ yielding

$$(2.11) \qquad g(s) = \frac{1}{2\pi i} \int_{C_0} h(z) \frac{dz}{(z)_c^{s-1}} = \frac{\sin \pi(s-1)}{\pi} \int_0^\infty h(-\omega) \frac{d\omega}{\omega^{s-1}}.$$

Now here

$$\frac{\sin \pi(s-1)}{\pi} \int_0^1 h(-\omega) \frac{d\omega}{\omega^{s-1}} = \frac{\sin \pi(s-1)}{-\pi(s-2)} \left\{ h(-1) + \int_0^1 h'(-\omega) \frac{d\omega}{\omega^{s-2}} \right\},$$

which is analytic in $s$ over $\Re[s] < 3$, having a removable singularity at $s = 2$. Also

$$\frac{\sin \pi(s-1)}{\pi} \int_1^\infty \omega^{r-2} \frac{d\omega}{\omega^{s-1}} = \frac{\sin \pi(s-1)}{\pi(s-r)} \qquad \text{and}$$

$$\frac{\sin \pi(s-1)}{\pi} \int_1^\infty \omega^{r-2} \ln \omega \frac{d\omega}{\omega^{s-1}} = \frac{\sin \pi(s-1)}{\pi(s-r)^2}$$

with principal parts

$$\frac{\sin \pi(r-1)}{\pi(s-r)} \quad \text{and} \quad \frac{\sin \pi(r-1)}{\pi(s-r)^2} + \frac{\cos \pi(r-1)}{s-r}$$

respectively at $s = r$. Thus (2.9) clearly follows from (2.8) and (2.11). Also from

$$|\sin \pi(s-1)| \leq 2e^{\pi|v|} \quad \text{and} \quad \int_1^\infty \frac{1}{\omega^2} \frac{d\omega}{\omega^{r-1}} = \frac{1}{r}$$

the stated estimate for $g_k(s)$ follows.

We combine Lemma 4 with our two previous theorems to obtain the following result.

THEOREM 5. *If the assumptions of Lemma* 4 *are satisfied with*

$$l_p \sin(\pi r_p) = 0$$

*in* (2.8), *then*

$$H(t) = \sum_{\lambda_n \leq t} a_n$$

*satisfies*

$$(2.12) \qquad H(t) = \left\{ \sum_{p=1}^{k} \frac{t^{r_p}}{r_p} \left( -m_p \frac{\sin \pi r_p}{\pi} - l_p \cos \pi r_p \right) \right\} + \tilde{O}(\ln t),$$

*over* $t \geq b$ *where* $0 < b < \lambda_1$. *Furthermore, if* $H(t)$ *has an ordinary asymptotic series in powers of* $t$ *as* $t \longrightarrow +\infty$, *such a series must coincide term for term as far as it goes with the terms of* (2.12).

*Proof.* Let

$$F(t) = H(t) - \left\{ \sum_{p=1}^{k} \frac{t^{r_p}}{r_p} \left( -m_p \frac{\sin \pi r_p}{\pi} - l_p \cos \pi r_p \right) \right\},$$

and note that

$$\int_b^\infty t^{-s}\, d\left(\frac{t^r}{r}\right) = \int_b^\infty t^{r-s-1}\, dt = \frac{b^{r-s}}{s-r}$$

for $\Re[s] > r$ and $b > 0$. Also with $0 < b < \lambda_1$, we have

$$\int_b^\infty t^{-s}\, dH(t) = \sum_{n=1}^\infty a_n \lambda_n^{-s}$$

for $\Re[s] \geq 2$. Thus from Lemma 4 we see that

$$\psi(s) = \int_b^\infty t^{-s}\, dF(t)$$

has an analytic continuation without singularities into $\Re[s] > 0$ by the cancellation of principal parts at each $r_p = s$. Also the conditions of Theorem 1 are satisfied with $r_0 = 2$ and $h = \pi$; thus (2.1) yields (2.12). Theorem 2 gives the consistency statement obviously.

To apply this theorem to our problem (1.1), we remark that the desired condition

$$\sum_{n=1}^\infty \frac{1}{\lambda_n^2} < +\infty$$

follows from Green's function being in $L_2(D \times D)$, and thus a Hilbert-Schmidt kernel. Thus Pleijel's estimate (1.4) yields (2.12) with

$$k = 2,\ r_1 = 1,\ m_1 = C,\ l_1 = \frac{\mu_2(D)}{4\pi},\ \ \sin(\pi r_1) = 0,\ \ \cos(\pi r_1) = -1,$$

$$r_2 = \frac{1}{2},\ m_2 = \frac{l(B)}{8},\ \ l_2 = 0,\ \ \sin(\pi r_2) = 1,$$

and we can state the following.

COROLLARY 6. *Let the open, bounded, connected set $D$ in the plane $E_2$ have its boundary $B$ an infinitely differentiable Jordan curve so that Pleijel's estimate* (1.4) *holds for the problem* (1.1). *Then over $t \geq \lambda_1/2$ we have*

$$N(t) = \sum_{\lambda_n \leq t} 1 = \frac{\mu_2(D)}{4\pi}\, t - \frac{l(B)}{4\pi}\, t^{1/2} + \widetilde{O}(\ln t), \tag{2.13}$$

*and as in Theorem* 5 *any ordinary asymptotic series for* $N(t)$ *must be consistent with* (2.13).

If we consider the real valued eigenfunction $u_n(\mathbf{x})$ of problem (1.1), in place of (1.4) Pleijel gets [**6**, equation 6 and second equation of p. 177] over $\mathbf{x} \in D$ and $\omega \geq 1$

$$(2.14) \qquad \sum_{n=1}^{\infty} \frac{|u_n(\mathbf{x})|^2}{\lambda_n(\lambda_n+\omega)} = \frac{1}{4\pi}\frac{\ln\omega}{\omega} + \frac{C(\mathbf{x})}{\omega} + \frac{1}{2\pi}\frac{K_0(2r(\mathbf{x})\sqrt{w})}{\omega} + O\left(\frac{e^{-2Ar(\mathbf{x})\sqrt{w}}}{w^{3/2}}\right),$$

$A > 0$, $r(\mathbf{x})$ the distance from $\mathbf{x} \in D$ to $B$, the $O$ symbol being uniform over $\mathbf{x} \in D$ as well as $\omega \geq 1$. Now $K_0(r)$, the modified Bessel function of the second kind and zero order, has

$$K_0(r) = \sqrt{\frac{\pi}{2r}}\, e^{-r}\left(1 + o\left(\frac{1}{r}\right)\right)$$

as $r \longrightarrow +\infty$ [**8**, p. 374]. Thus for each fixed $\mathbf{x} \in D$, with $r(\mathbf{x}) > 0$, we have over $\omega \geq 1$

$$(2.15) \qquad \sum_{n=1}^{\infty} \frac{|u_n(\mathbf{x})|^2}{\lambda_n(\lambda_n+\omega)} = \frac{1}{4\pi}\frac{\ln\omega}{\omega} + \frac{C(\mathbf{x})}{\omega} + O_{\mathbf{x}}\left(\frac{1}{\omega^2}\right),$$

where the symbol $O_{\mathbf{x}}$ now depends on $\mathbf{x} \in D$. It is also easy to see that at each $\mathbf{x} \neq \mathbf{y}$ with $\mathbf{x}, \mathbf{y} \in D$ we have over $\omega \geq 1$

$$(2.16) \qquad \sum_{n=1}^{\infty} \frac{u_n(\mathbf{x})u_n(\mathbf{y})}{\lambda_n(\lambda_n+\omega)} = \frac{C(\mathbf{x},\mathbf{y})}{\omega} + O_{\mathbf{x},\mathbf{y}}\left(\frac{1}{\omega^2}\right),$$

and indeed much better estimates than $O(1/\omega^2)$ hold in (2.15) and (2.16). Also

$$\sum_{n=1}^{\infty} \frac{|u_n(\mathbf{x})|^2}{\lambda_n^2} < +\infty$$

is known at each $\mathbf{x} \in D$; thus Theorem 5 yields the following.

COROLLARY 7. *Let $D$ be as in Corollary 6, so that (2.15) and (2.16) hold at each $\mathbf{x} \neq \mathbf{y}$ with $\mathbf{x}, \mathbf{y} \in D$. Then over $t \geq \lambda_1/2$*

$$(2.17) \quad \sum_{\lambda_n \leq t} |u_n(\mathbf{x})|^2 = \frac{1}{4\pi} t + \widetilde{O}(\ln t), \quad \sum_{\lambda_n \leq t} u_n(\mathbf{x}) u_n(\mathbf{y}) = \widetilde{O}(\ln t),$$

*with consistency of these series with ordinary asymptotic series, if any, as in Theorem 5.*

**3. Discussion of results.** It is quite clear that $\widetilde{O}(\ln t)$ in (2.17) can be replaced by much stronger estimates in the $\widetilde{O}$ sense, say $\widetilde{O}(1/t)$, since much more than $O(1/\omega^2)$ holds in (2.15) and (2.16). In (2.13) additional terms enter if a stronger $\widetilde{O}$ type error estimate is required. These are due to additional terms entering Pleijel's equation (1.4), one of them involving the mean square curvature of $B$, if $O(1/\omega^2)$ is replaced by a stronger estimate.

A much more difficult and interesting question is the extent to which the averaged $\widetilde{O}$ estimates in our results may be replaced by ordinary $O$ estimates for the problem (1.1). It is clear that by improving the $O(e^{\pi|v|})$ estimate on the analytic continuation of

$$\sum_{n=1}^{\infty} \lambda_n^{-s}, \quad s = r + iv,$$

we can replace the Gauss kernel

$$\exp\left(-\frac{\rho^2}{2}(\omega - y)^2\right)$$

in our definition of $\widetilde{O}$ by less well behaved ones. We could get ordinary $O$ estimates if we could use the characteristic function kernel $\chi_{[-1,1]}(\omega - y)$, but since its Fourier transform is essentially $v^{-1} \sin v$, the analogue of the proof of Theorem 1 would then seem to require stronger conditions on

$$\sum_{n=1}^{\infty} \lambda_n^{-s}$$

than can be expected to hold.

It is known from the refined results of geometric number theory [**1**, p. 823] that $M(x)$, defined as the number of integer lattice points $(m, n)$ in the plane satisfying $m^2 + n^2 \leq x$, satisfies

$$M(x) = \pi x + O(x^{1/3}).$$

Since

$$\lambda = (n^2 + m^2)\frac{\pi^2}{b^2}, \qquad n > 0, \ m > 0$$

for the eigenvalues of (1.1) with $D$ a square of side $b$, the eigenfunctions being products of sine functions, we clearly see that

$$N(t) = \frac{1}{4}\left\{M\left(\frac{b^2 t}{\pi^2}\right) - \left(4\left[\frac{b\sqrt{t}}{\pi}\right] + 1\right)\right\} = \frac{b^2}{4\pi}\, t - \frac{4b}{4\pi}\sqrt{t} + O(t^{1/3})$$

for square $D$, $4[b\sqrt{t}/\pi] + 1$ being the number of lattice points on the axes. This asymptotic result for $N(t)$ agrees with (2.13), although the corners of a square prevent it from satisfying the smooth boundary conditions required in Corollary 6. By carelessly dropping the $\sqrt{t}$ term in going from $M(x)$ to $N(t)$, Minakshisundaram [5, p. 331, no. 2] is led to the conjecture that domain comparison methods [3, p. 386] should yield

$$N(t) = \frac{\mu_2(D)}{4\pi}\, t + O(t^{1/3})$$

for general domains $D$. Clearly the consistency statement of Corollary 6 makes such asymptotic behavior impossible for $N(t)$.

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