THE SYMMETRY FUNCTION IN A CONVEX BODY

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Let K_n be an n-dimensional convex body in n-dimensional Euclidean space E_n. At each point P in K_n consider the largest subset $S(P)$ of K_n radially symmetric with respect to the point P. This set is well-defined and convex for it is simply the intersection of K_n with its radial reflection through the point P. Let $m(P)$ equal the measure of $S(P)$ and let $f(P)$ equal $m(P)V_n^{-1}$ where V_n is the measure of K_n. Clearly $0 \leq f(P) \leq 1$ for all P in K_n and $f(P)=0$ only if P is on the boundary of K_n; also f is continuous. Moreover f attains the value 1 only if K_n is radially symmetric. The object of this note is to present various properties of this function f.

THEOREM 1. (Besicovitch [1], $n=2$). *There is a point P in K_2 such that* $f(P)=2/3$. (In [3, p. 46] this theorem is ascribed to S. S. Konvyer.)

THEOREM 2. (Besicovitch [2], $n=2$). *If K_2 is of constant width then there is a point P in K_2 such that $f(P)=.840\cdots$.*

H. G. Eggleston [4] studied further the symmetric function in a body of constant width.

Using a result of P. C. Hammer [5] on the ratio which the centroid of a convex body divides the chords passing through it, F. W. Levi [6] obtained the following.

THEOREM 3. *If P is the centroid of K_n then*

$$f(P) \geq 2(1+n^n)^{-1} .$$

The following properties of f will be obtained.

THEOREM 4. $\int_{K_n} f = 2^{-n} V_n .$

COROLLARY. *There is a point P in K_n such that $f(P) > 2^{-n}$.*

THEOREM 5. *If a is a real number then the set of points P in K_n at which $f(P) \geq a$ is convex. Furthermore f attains its maximum value at precisely one point.*

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COROLLARY (to proof of Theorem 5, suggested by referee). *If* $0 \leq \lambda \leq 1$ *and* P *and* Q *are in* K_n *then*

$$f(\lambda P+(1-\lambda)Q) \geq \lambda f(P)+(1-\lambda)f(Q) .$$

THEOREM 6. *If* K_n *is an n-dimensional simplex and* P *is its centroid, then* f *attains its maximum at* P *and* $f(P)=2(n+1)^{-1}$.

Proof of Theorem 4. Consider the set of points

$$K_{2n}=\{(P, Q)|P \in K_n, Q \in S(P)\} .$$

In a straightforward manner this set can be shown to be convex and hence measurable. By Fubini's theorem on the relation between iterated and multiple integrals, the volume V_{2n} of K_{2n} is seen to equal $\int_{K_n} m$ and also $\int_{K_n} h$ where $h(Q)$ denotes the measure of the cross section of K_{2n} defined by

$$\{(P, Q)|(Q \text{ fixed}), S(P) \ni Q\} .$$

Now $S(P) \ni Q$ only if P is less than half way from Q to the boundary of K_n along the line determined by P and Q. Thus $h(Q)=2^{-n}V_n$ independently of Q [7, p. 38]. Thus

$$\int_{K_n} f=V_n^{-1}\int_{K_n} h=V_n^{-1}2^{-n}(V_n)^2=2^{-n}V_n .$$

Proof of Corollary to Th. 4. Since the average value of f on K_n is 2^{-n} and since $f(P)<2^{-n}$ on (and near) the boundary of K_n there must be a point at which f exceeds 2^{-n}.

Proof of Theorem 5. Let P and Q be distinct points of K_n such that $f(P)=f(Q)$. We shall show[1] that $f((P+Q)/2)>f(P)$. This fact, combined with the fact that $\{P|f(P)\geq a\}$ is closed, would prove the theorem. Consider the convex body $(S(P)+S(Q))/2$. This body is symmetric, and, if so translated that $(P+Q)/2$ is its center, lies within K_n. By the Brunn-Minkowski theorem [7, p. 88] the measure of this set is strictly larger than $m(P)$ if $S(P)$ is not congruent to $S(Q)$ by a translation. If $S(P)$ is congruent to $S(Q)$ by a translation, consider the convex hull of the set union of $S(P)$ and $S(Q)$. This set is clearly symmetric with respect to the point $(P+Q)/2$, lies in K_n, and has a measure greater than $m(P)$. Thus $f((P+Q)/2)>f(P)=f(Q)$.

Proof of Corollary to Th. 5. A continuous function which satisfies

[1] If P and Q are on the boundary of K_n it may happen that $f((P+Q)/2)=f(P))$.

$$f(\lambda P+(1-\lambda)Q)\geqq \lambda f(P)+(1-\lambda)f(Q)$$

for $\lambda=1/2$ and all P, Q in a line segment satisfies the inequality for all λ, $0\leqq\lambda\leqq 1$, and P, Q, in the line segment.

Proof of Theorem 6. Since affine transformations preserve symmetry, centroids, and ratio of volumes it will be sufficient to consider the case where K_n is regular.

Let Q be the point in K_n maximizing f. If T is an orthogonal transformation interchanging two of the vertices of K_n, and leaving the remaining vertices fixed then $f(Q)=f(T(Q))$. Thus, by Theorem 5, $T(Q)=Q$. Since this is true for each pair of vertices of K_n, Q must be equidistant from all the vertices of K_n. Thus $Q=P$.

Now to compute $f(P)$.

Let K_n' be the reflection of K_n through P of altitude h and volume V. The boundary of $K_n \cap K_n'$ is readily seen to be composed of $2(n+1)$ congruent $n-1$ dimensional sets B_i, $1\leqq i\leqq 2(n+1)$ each of volume V^*. Let S denote the volume of $K_n \cap K_n'$.

Considering $K_n \cap K_n'$ as being composed of $2(n+1)$ congruent joins with the common vertex P, bases B_i, and altitude $h(n+1)^{-1}$ one obtains

$$(1) \qquad S=2(n+1)h(n+1)^{-1}V^*n^{-1} .$$

On the other hand, considering $K_n \cap K_n'$ as being obtained from K_n by the removal of $n+1$ congruent sets, each of which is a join of a vertex of K_n with a B_i and has an altitude $(n-1)(n+1)^{-1}h$, one obtains

$$(2) \qquad S=V-(n+1)(n-1)(n+1)^{-1}hV^*n^{-1} .$$

Elimination of the product hV^* from (1) and (2) yields

$$S=2(n+1)^{-1}V$$

and thus

$$f(P)=2(n+1)^{-1} .$$

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