NOTE ON THE LERCH ZETA FUNCTION

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1. Introduction. The functional equation for the Riemann zeta function and Hurwitz's series were derived by a formal application of Poisson's summation formula in some previous papers [9], [10], [12, p. 17]. The results are correct although obtained by equating two infinite series whose regions of convergence exclude each other. It was shown later that this difficulty can be overcome if a generalized form of Poisson's formula is used [6]. It is the purpose of this note to show that the application of Poisson's summation formula in its ordinary form to the more general case of Lerch's zeta function [1], [2], [3], [5, pp. 27–30], [8] does not present the difficulties with respect to convergence which arise in its special cases. Thus a very simple and direct method for developing the theory of this function can be given. It may be noted that Hurwitz's series can be obtained immediately upon specializing one of the parameters. (cf. also [1]).

2. Lerch's zeta function $\Phi(z, s, v)$ is usually defined by the power series

(1a) $$\Phi(z, s, v)=\sum_{m=0}^{\infty} z^m(v+m)^{-s} \qquad |z|<1,\ v\neq 0, -1, -2, \cdots.$$

Equivalent to this is the integral representation [5, pp. 27–30]

(1b) $$\Phi(z, s, v)=\frac{1}{2}v^{-s}+\int_0^\infty (v+t)^{-s}z^t dt$$
$$+2\int_0^\infty \sin\left(s\tan^{-1}\frac{t}{v}-t\log z\right)(v^2+t^2)^{-s/2}(e^{2\pi t}-1)^{-1}dt$$
$$|z|<1,\ \mathscr{R}v>0,$$

which can be written as

(1c) $$\Phi(z, s, v)=\frac{1}{2}v^{-s}+z^{-v}\Gamma(1-s)\left(\log\frac{1}{z}\right)^{s-1}-z^{-v}v^{1-s}\sum_{n=0}^{\infty}\frac{\left(-v\log\frac{1}{z}\right)^n}{n!\,(n+1-s)}$$
$$+2\int_0^\infty \sin\left(s\tan^{-1}\frac{t}{v}-t\log z\right)(v^2+t^2)^{-s/2}(e^{2\pi t}-1)^{-1}dt$$
$$z\neq 1,\ \mathscr{R}v>0$$

upon replacing the first integral on the right of (1b) by the second and

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third expression on the right of (1c). Hence it follows that the point $z=1$ is a singular point of Φ such that

$$(2)\quad \lim_{z\to 1}\left[\sum_{0}^{\infty} z^m(v+m)^{-s}-z^{-v}\Gamma(1-s)\left(\log\frac{1}{z}\right)^{s-1}\right]$$

$$=\frac{1}{2}v^{-s}+\frac{v^{1-s}}{s-1}+2\int_0^\infty \sin\left(s\tan^{-1}\frac{t}{v}\right)(v^2+t^2)^{-s/2}(e^{2\pi t}-1)^{-1}dt=\zeta(s,v)$$

by Hermite's formula [5 p. 26]. This relation was first proved by Hardy [7] in a different manner. For $\mathscr{R}s>1$, (2) is obviously a consequence of Abel's theorem. We apply now Poisson's summation formula [4, p. 33], [11, p. 60]

$$(3)\qquad \sum_{n=-\infty}^{\infty} e^{i2\pi nv}f(2\pi n)=\frac{1}{2\pi}\sum_{m=-\infty}^{\infty}G(v+m) \qquad 0<v\leqq 1$$

where

$$(4)\qquad G(t)=\int_{-\infty}^{\infty}f(y)e^{iyt}dy$$

to the function

$$(5)\quad f(x)=(\alpha+ix)^{-s},\ \mathscr{R}s>1,\ \mathscr{R}\alpha>0,\ x\text{ real},\ -\frac{1}{2}\pi<\arg(\alpha+ix)<\frac{1}{2}\pi\,.$$

According to (4)

$$(6)\qquad G(t)=\int_{-\infty}^{\infty}(\alpha+iy)^{-s}e^{iyt}dy=2\pi e^{-\alpha t}\frac{1}{2\pi i}\int_{\alpha-i\infty}^{\alpha+i\infty}p^{-s}e^{pt}dp\,.$$

The last integral multiplied by $(2\pi i)^{-1}$ is the inverse Laplace transform of p^{-s} and is equal to $\dfrac{t^{s-1}}{\Gamma(s)}$. Thus for $\mathscr{R}s>1$

$$(7)\qquad G(t)=\frac{2\pi}{\Gamma(s)}t^{s-1}e^{-\alpha t}\,,\qquad t>0\,,$$

$$=0\,,\qquad t<0\,.$$

From (3) to (7) we obtain

$$(8)\qquad \sum_{n=-\infty}^{\infty}(\alpha+i2\pi n)^{-s}e^{i2\pi nv}=\frac{1}{\Gamma(s)}\sum_{m=0}^{\infty}(v+m)^{s-1}e^{-(v+m)\alpha}$$

or, with s replaced by $1-s$

$$(9)\qquad \sum_{m=0}^{\infty}(v+m)^{-s}e^{-m\alpha}=\Gamma(1-s)e^{v\alpha}\sum_{n=-\infty}^{\infty}e^{i2\pi nv}(\alpha+i2\pi n)^{s-1}$$

$$\mathscr{R}s<0,\ 0<v\leqq 1,\ 0\leqq\mathscr{I}\alpha<2\pi\,.$$

The last restriction means no loss of generality since both sides of (9) are periodic in α with the period $2\pi i$. We write (9)

$$(10)\qquad \sum_{m=0}^{\infty}(v+m)^{-s}e^{-m\alpha}-\Gamma(1-s)e^{v\alpha}\alpha^{s-1}$$

$$=\Gamma(1-s)e^{v\alpha}\left[\sum_{n=1}^{\infty}\frac{e^{i2\pi nv}}{(\alpha+i2\pi n)^{1-s}}+\sum_{n=1}^{\infty}\frac{e^{-i2\pi nv}}{(\alpha-i2\pi n)^{1-s}}\right]$$

$$\mathscr{R}s<0,\ 0<v\leqq 1,\ 0\leqq\mathscr{I}\alpha<2\pi,$$

where according to the condition in (5) for $n=1, 2, 3, \cdots$

$$0<\arg\left(\frac{\alpha}{2\pi}+in\right)<\frac{1}{2}\pi,\qquad -\frac{1}{2}\pi<\arg\left(\frac{\alpha}{2\pi}-in\right)<0\,.$$

Hence,

$$\arg\left(\frac{\alpha}{2\pi}\pm in\right)=\pm\frac{1}{2}\pi+\arg\left(n\pm\frac{\alpha}{2\pi i}\right)$$

and consequently

$$(\alpha\pm i2\pi n)^{s-1}=(2\pi)^{s-1}e^{\pm i\pi(s-1)/2}\left(n\pm\frac{\alpha}{2\pi i}\right)^{s-1}\quad .$$

Using (1a) we obtain finally from (10)

$$(11)\qquad \Phi(e^{-\alpha}, s, v)-\Gamma(1-s)e^{v\alpha}\alpha^{s-1}$$

$$=-i(2\pi)^{s-1}\Gamma(1-s)e^{v\alpha}\left[\sum_{n=1}^{\infty}\frac{e^{i(2\pi nv+\frac{1}{2}\pi s)}}{\left(n+\dfrac{\alpha}{2\pi i}\right)^{1-s}}-\sum_{n=1}^{\infty}\frac{e^{-i(2\pi nv+\frac{1}{2}\pi s)}}{\left(n-\dfrac{\alpha}{2\pi i}\right)^{1-s}}\right]$$

$$\mathscr{R}s<0,\quad 0<v\leqq 1,\quad 0\leqq\mathscr{I}\alpha<2\pi,$$

which may be written as

$$(12)\quad \Phi(e^{-\alpha}, s, v)-\Gamma(1-s)e^{v\alpha}\alpha^{s-1}$$

$$=-i(2\pi)^{s-1}\Gamma(1-s)e^{v\alpha}\Big[e^{i(2\pi v+\frac{1}{2}\pi s)}\,\Phi\left(e^{i2\pi v}, 1-s, 1+\frac{\alpha}{2\pi i}\right)$$

$$-e^{-i(2\pi v+\frac{1}{2}\pi s)}\,\Phi\left(e^{-i2\pi v}, 1-s, 1-\frac{\alpha}{2\pi i}\right)\Big]\,.$$

This is Lerch's transformation formula and (11) corresponds to Hurwitz's series. Now it follows that the right hand side of (11) is also regular for $\alpha=0$. If in (11) we let α tend to zero, the left side becomes $\zeta(s, v)$ by (2) and the right hand side becomes Hurwitz's series. Thus

$$\zeta(s, v)=2(2\pi)^{s-1}\Gamma(1-s)\sum_{n=1}^{\infty} n^{s-1}\sin\left(2\pi nv+\frac{1}{2}\pi s\right) \qquad (13)$$

$$0<v\leqq 1, \quad \mathscr{R}s<0.$$

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